Sept 14



HW due 18th

p23 #2, 6

p29 #5, 8

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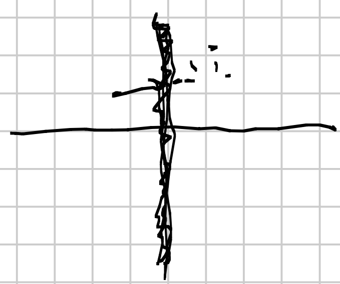
#3 $\frac{\sin xy}{x}$, $x \neq 0$



pt $(0,y)$

sequences $(x_i, y_i) \to (0,y)$

$x_i \neq 0$,

$$f(x_i, y_i) = \frac{\sin(x_i y_i)}{x_i} = a_i$$

? $\lim_{i \to \infty} a_i$ ?

$$\left(\frac{\sin(x_i y_i) - \sin(0)}{x_i y_i - 0}\right) y_i = a_i$$

$$\underline{\qquad\qquad}$$

$$\left\|\ \frac{f(b) - f(a)}{b - a}\right.$$

Mean Value Thm

$$= f'(c)$$

c is between

$\underline{a + b.}$

$$\underbrace{\left(\frac{\sin(x_i y_i) - \sin(0)}{(x_i y_i) - 0}\right)} y_i$$

deriv of sin
at some ~~#~~ $c_i$
between $x_i y_i$ + 0.

$$= \cos(c_i) y_i$$

$$\lim_{i\to\infty} \cos(c_i) y_i \qquad ||$$

$$y_i \to y$$

$$x_i \to 0$$

$$x_i y_i \to 0$$

$0 < c_i < x_i y_i$

$c_i$ is between $x_i y_i$ & $0$.

$x_i y_i < c_i < 0$

$$\Rightarrow \lim_{i\to\infty} c_i = 0$$

$$\Rightarrow \lim_{i\to\infty} \cos(c_i) = 1$$

$$\& \lim_{i\to\infty} a_i = y$$

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Three C's

Completeness ←
Compactness
Connectedness

Real #'s
3- kinds of properties

1) arithmetic
2) order
3) <u>Completeness</u> —

Classical picture —

$S$ ~ set $\subseteq \mathbb{R}$.

It might have upper bounds.

$$\mathcal{U}(S) = \{b \in \mathbb{R} : \forall s \in S, \ b \geq s\}.$$

If $\mathcal{U}(S)$ is not $\emptyset$ we say $S$ is bounded <u>above</u>.

If $b \in \mathcal{U}(S)$ & $b' \geq b$ then $b' \in \mathcal{U}(S)$

A "least upper bound" is $b_0 \in \mathcal{U}(S)$ such that for all $b \in \mathcal{U}(S)$,

$$b \geq b_0.$$

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Completeness prop. of $\mathbb{R}$ says any nonempty, bounded subset $S \subseteq \mathbb{R}$ has a <u>least</u> upper bound.

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If $S$ is not bounded then we set its l.u.b. $= \infty$.

If $S = \emptyset$, l.u.b. $= -\infty$

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$$\sup(S) = \text{l.u.b.}(S)$$

& $\inf(S) = -\text{l.u.b.}(-S)$

if $\sup(S) \in S$
then call it $\max(S)$.

if $\inf(S) \in S$ then
call it <u>$\min(S)$</u>,

Alternative-

Cauchy Sequences.

Def. A sequence
$\vec{a}_n$ is called Cauchy
if $\forall \varepsilon > 0$ $\exists N$, ~~&~~ for
all $n, m \geq N$
$$\|\vec{a}_{\underline{\underline{n}}} - \vec{a}_{\underline{\underline{m}}}\| < \varepsilon.$$

<u>Lemma</u> If $a_n$ converges
it is Cauchy.

Lemma If $a_n$ is Cauchy then it converges.

proof Assume $a_n \in \mathbb{R}$.

a) As $a_n$ are Cauchy they are bounded.

Let $\varepsilon = 1$, $\exists N$, $n, m \geq N$, $|a_n - a_m| < 1$

$$b = \max(a_1, a_2, \ldots, a_N, a_N + 1)$$

$n \leq N$, $a_n \leq b$

$n \geq N$, $|a_N - a_n| < 1$

so $a_n < a_N + 1$

Let $b_n = \sup(a_n, a_{n+1}, a_{n+2} \ldots)$

$b_{n+1} \leq b_n$

Values $b_n$ are bounded below.

Let $L = \inf(b_n)$

$\forall \varepsilon > 0 \quad \exists N, \; n, m \geq N$

$|a_n - a_m| < \varepsilon$.

$$b_N = \sup(\{a_N, a_{N+1}, \dots\})$$

so $\forall n \geq N$,

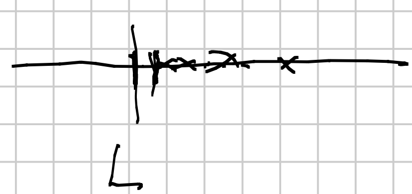
$|b_N - a_n| \leq \varepsilon$

The $b_n$'s decrease

+ $\inf b_n = L$.

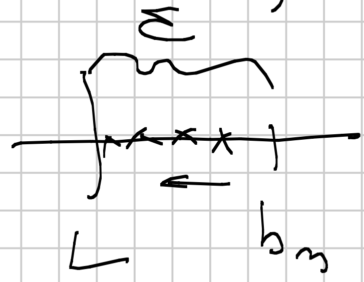
Since $\inf b_n = L$,

$\forall \varepsilon > 0, \; \exists n, \; 0 \leq b_n - L \leq \varepsilon$




if $b_n - L \leq \varepsilon$

$\forall\, N > n,\ b_N - L \leq \varepsilon,$




$\forall N$ large enough

$0 \leq b_N - L \leq \varepsilon$

& moreover

if $N$ is large enough

$\forall\, n \geq N$

$|a_n - b_N| < \varepsilon$

$|a_n - L| < 2\varepsilon$

Convergence of Cauchy sequences follows from completeness.

More general def.

of completeness is that Cauchy sequences converge.

Thm In $\mathbb{R}^n$, Cauchy sequences converge & so $\mathbb{R}^n$ is complete.

pf $(x_i^1, x_i^2, \cdots, x_i^n) = \vec{x}_i$

is Cauchy then each coord is Cauchy

$x_i^j$ — & so conv.

& each coord. converges to some $a^j$ //

$\Rightarrow \vec{x}_i \to \vec{a}$.