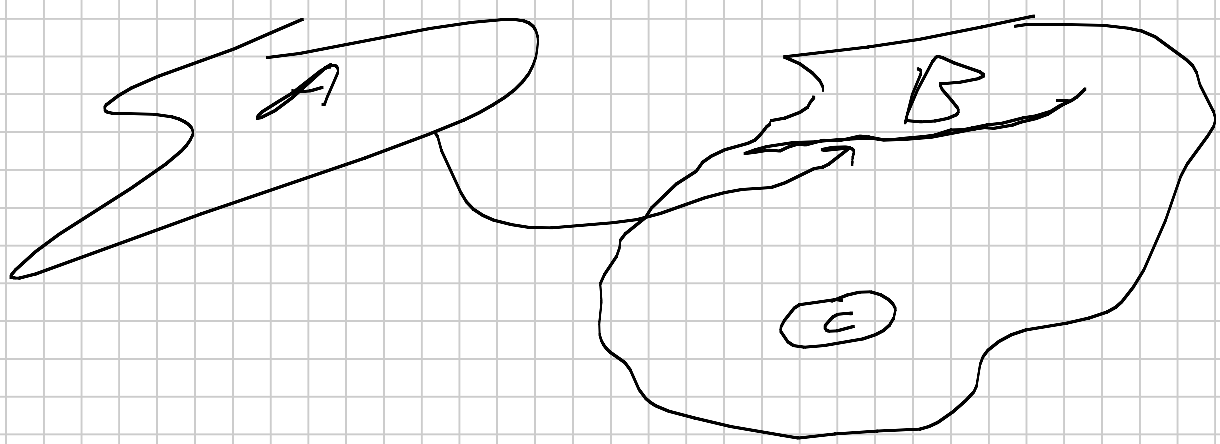
Note Title Sept 11 9/11/2009

2 other ways
to view Continuity

$f : A \to B$

$C \subseteq B$

$f^{-1}(C) = \{x \in A : f(x) \in C\}$.



$\{x : f(x) \in C\} = \emptyset$.

Thm

$f: D \to \mathbb{R}^m$

$D \subseteq \mathbb{R}^n$.

f is continuous on D.

iff $\forall U \subseteq \mathbb{R}^m$, U-open.

$f^{-1}(U)$ is open.

proof

$\Rightarrow$ f is cont.

Let U be open in $\mathbb{R}^m$.

Want to show

$f^{-1}(U)$ is open

Let $\vec{x} \in f^{-1}(U)$

i.e. $f(\vec{x}) \in U$.

so as U-open $\exists \varepsilon > 0$

$B_\varepsilon(f(\vec{x})) \subseteq U$.

So $\exists \delta > 0$ & if

$\|\vec{x} - \vec{y}\| < \delta$ then

$\|f(\vec{x}) - f(\vec{y})\| < \varepsilon$.

i.e. $\vec{y} \in B_{\vec{\delta}}(x)$

then $f(\vec{y}) \in B_{\varepsilon}(f(\vec{x})) \subseteq U$

so $\vec{y} \in f^{-1}(U)$

or $B_{\vec{\delta}}(\vec{x}) \subseteq f^{-1}(U)$.

i.e. $f^{-1}(U)$ open

$\Leftarrow$ Suppose U - open

$\Rightarrow$ $f^{-1}(U)$ - open.

Show f is cont.

$\vec{x} \in D$, $\varepsilon > 0$

+ consider

$B_{\varepsilon}(f(\vec{x}))$ $\leftarrow$ open set.

$$\underbrace{f^{-1}(B_{\varepsilon}(f(\vec{x})))}_{\downarrow} \text{ - open}$$

$\vec{x} \in V$. as

$$f(\vec{x}) \in B_\varepsilon(f(\vec{x}))$$

So $\exists \delta > 0$,

$$B_\delta(\vec{x}) \subseteq V.$$

Suppose $\|\vec{x} - \vec{y}\| < \delta$

$$\Rightarrow \vec{y} \in B_\delta(\vec{x}) \subseteq V = f^{-1}(B_\varepsilon(f(\vec{x})))$$

$$\Rightarrow f(\vec{y}) \in B_\varepsilon(f(\vec{x}))$$

$$\Rightarrow \|f(\vec{y}) - f(\vec{x})\| < \varepsilon.$$

<u>Cor</u> f is cont. iff for all closed sets C

$f^{-1}(C)$ is closed.

pf $f^{-1}(A^c) = f^{-1}(A)^c$.

Sequences

Sequence -

$\vec{a} : \mathbb{N} \to \mathbb{R}^n$

$\vec{a}_1, \vec{a}_2 \ldots$

We say

$$\lim_{i \to \infty} \vec{a}_i = \vec{V}$$

if $\forall \varepsilon > 0 \ \exists K$

+ for all $k \geq K$

$\|\vec{a}_k - \vec{V}\| < \varepsilon$.

or — $\forall \varepsilon > 0$,

$\{k : \|\vec{a}_k - \vec{V}\| \geq \varepsilon\}$

is finite.

Decimal exp.

Continuity

Suppose $f : D \to \mathbb{R}^m$

$$D \subseteq \mathbb{R}^n.$$

$$\vec{x} \in \overline{D}.$$

Suppose $\vec{a}_i \in D$,

$$\underbrace{\vec{a}_i \to \vec{x}.}_{\equiv}$$

$$\lim_{i \to \infty} \vec{a}_i = \vec{x}.$$

I can examine –

$$f(\vec{a}_i) = \vec{b}_i.$$

$f(\vec{x})$

Thm Suppose $f : D \to \mathbb{R}^m$

$D \subseteq \mathbb{R}^n$

$+ \vec{x} \in D$.

Then f is cont. at $\vec{x}$ iff for all

$\vec{a}_i \in D$, $\vec{a}_i \to \vec{x}$

we have

$$f(\vec{a}_i) \to f(\vec{x}).$$

pf: $\Rightarrow$ f - cont. at $\vec{x}$

Suppose $\vec{a}_i \in D$

$\vec{a}_i \to \vec{x}$.

Let $\varepsilon > 0 \Rightarrow \exists \delta$

+ if $\|\vec{x} - \vec{y}\| < \delta$

then $\|f(\vec{x}) - f(\vec{y})\| < \varepsilon$.

as $\vec{a}_i \to \vec{x}$, $\exists K$

+ $k \geq K$ then

$\|\vec{a}_k - \vec{x}\| < \delta$.

$= \|f(\vec{a}_k) - f(\vec{x})\| < \varepsilon$.

$\Rightarrow f(\vec{a}_k) \to f(\vec{x})$.

$\Leftarrow$ Show if
f is not cont.
at $\vec{x}$ then $\exists\ \vec{a}_i \to \vec{x}$
& $f(\vec{a}_i) \not\to f(\vec{x})$.

f not cont.
$\Rightarrow \exists\ \varepsilon_0 > 0$ with
no δ.

Means all δ's fail.

i.e. $\delta = \frac{1}{n}$

$\exists\ \bar{a}_n$ with $\|\vec{x} - \vec{a}_n\| < \frac{1}{n}$

but $\|f(\vec{x}) - f(\vec{a}_n)\| \geq \varepsilon_0$.

$\{\vec{a}_n\}$, $\|\vec{x} - \vec{a}_n\| < \frac{1}{n}$

so $\vec{a}_n \to \vec{x}$.

but $\|f(\vec{x}) - f(\vec{a}_n)\| \geq \varepsilon_0 > 0$.

So $f(\vec{a}_n) \not\to f(\vec{x})$.