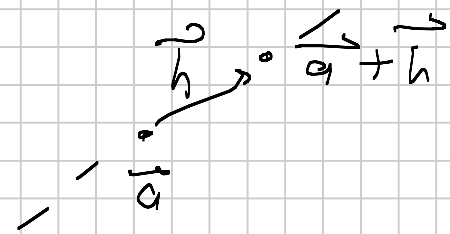
Oct 23

Note Title 10/23/2009

Taylor Series in Higher dim

S - open & convex

$\vec{a}\ ,\ \vec{a} + \vec{h} = \vec{x} \in S$



f - at least C^k some k.

$$g(t) = f(\vec{a} + t\vec{h})$$

$$f \quad \begin{array}{l} \vec{a} \longrightarrow 0 \\ \vec{x} \longrightarrow 1. \end{array} \quad g$$

$$g'(t) = \nabla f(\vec{a} + t\vec{h}) \cdot \vec{h}$$

$$= \underbrace{(h_1 \partial_{x_1} + h_2 \partial_{x_2} + \cdots + h_n \partial_{x_n})}_{\text{operator}}(f)$$

$$= (\vec{h} \cdot \nabla) f$$

$$g''(t) = (\vec{h} \cdot \nabla)(\vec{h} \cdot \nabla)(f(\vec{a} + t\vec{h}))$$

$$= (\vec{h} \cdot \nabla)^2 (f)(\vec{a} + t\vec{h})$$

$$\vdots$$

$$g^{(j)}(t) = (\vec{h} \cdot \nabla)^j (f)(\vec{a} + t\vec{h}).$$

$$(h_1 \partial_{x_1} + h_2 \partial_{x_2} + \cdots + h_n \partial_{x_n})^j$$

$$(j \leq k)$$

$$= \underbrace{\left(\sum_{|\vec{\alpha}| = j} \frac{j!}{\vec{\alpha}!} \vec{h}^{\vec{\alpha}} \partial^{\vec{\alpha}} \right)}(f)(\vec{a} + t\vec{h})$$

$$f(\vec{a} + \vec{h}) = \sum_{j=0}^{k} \frac{(\vec{h} \cdot \nabla)^j f(\vec{a})}{j!} + R_{\vec{a},k}(\vec{h}).$$

$$= \sum_{|\vec{\alpha}| \leq k} \frac{\partial^{\vec{\alpha}} f(\vec{a})}{\vec{\alpha}!} \vec{h}^{\vec{\alpha}} + R_{\vec{a},k}(\vec{h})$$

Cor $S \subseteq \mathbb{R}^n$, convex + open,

$f : S \to \mathbb{R}$ is C^k

Then
$$\lim_{\vec{h} \to 0} \frac{|R_{\vec{a},k}(\vec{h})|}{\|h\|^k} = 0.$$

If f is C^{k+1} +

$|\partial^{\vec{\alpha}} f| \leq M \quad \forall |\vec{\alpha}| = k+1$

then
$$|R_{\vec{a},k}(\vec{h})| \leq \frac{M}{(k+1)!} |\vec{h}|^{k+1}$$

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pf of 2nd.

From Lagrange

$\exists$ c bet. 0 + 1,

$$|R_{\vec{a},k}(\vec{h})| = \left| \sum_{|\vec{\alpha}|=k+1} \partial^{\alpha} f(\vec{a}+ch) \frac{\vec{h}^{\alpha}}{\alpha!} \right|$$

$$\leq M \left| \sum_{|\alpha|=k+1} \frac{\vec{h}^{\alpha}}{\alpha!} \right|$$

$$= \frac{M}{(k+1)!} \Big| \underbrace{\sum_{|\alpha| = k+1} \frac{(k+1)!}{\vec{\alpha}!} \vec{h}^{\vec{\alpha}}}_{} \Big|$$

$$\frac{M}{(k+1)!} \left| (h_1 + \cdots + h_n)^{k+1} \right|$$

$$\leq \frac{M}{(k+1)!} \big(\underbrace{|h_1| + \cdots + |h_n|}_{|\vec{h}|} \big)^{k+1}$$

Find the Taylor poly of deg 5 for $(e^x - 1)(\cos 2x - 1)^2$

$$\left(\cancel{1} + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \cancel{1}\right) \times$$

$$\underbrace{\left(\cancel{1} - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} - \cancel{1}\right)^2}_{}$$

$$x\,(4x^4)$$

$$= 4x^5$$

Higher dim.

$$f(\vec{a}+\vec{h}) \simeq f(\vec{a}) + \nabla f(\vec{a})\cdot\vec{h} + \boxed{\text{order } 2} + \underline{\underline{R}}$$

order 2

$$\underline{\underline{\sum_{|\vec{\alpha}|=2} \frac{\partial^{\vec{\alpha}} f(\vec{a})}{\vec{\alpha}!}\,\vec{h}^{\vec{\alpha}}}}$$

$$\vec{\alpha} = \begin{cases} (0\cdots 0,\underset{=}{2},0,\cdots) \\ (0\cdots \underset{i}{1}\,0\cdots \underset{j}{1}\cdots 0) \end{cases}$$

$$= \sum_{i=1}^{n} \frac{\partial_{x_i}^2 f(\vec{a})}{2} h_i^2$$

$$+ \sum_{\substack{i\neq j\\ i,j=1}}^{n} \frac{\partial_{x_i}\partial_{x_j} f(\vec{a})}{2} h_i h_j$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \underbrace{\partial_{x_i}\partial_{x_j} f(\vec{a})}_{a} h_i h_j$$

$$H = \left[\partial_{x_j} \partial_{x_i} (f)(\vec{a}) \right]$$

$$\begin{bmatrix} \partial^2_{x_1} & \partial_{x_1 x_2} \\ \partial_{x_2 x_1} & \partial^2_{x_2} \end{bmatrix}$$

2nd order terms are

$$\vec{h}^T H \vec{h}$$

$$(h_1, h_2, \dots, h_n) \, H \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

Hessian H

$$\left[\partial_{x_j} \partial_{x_i} (f) \right] \qquad f \text{ is } C^2$$

Symmetric

$$\partial_{x_j} \partial_{x_i} f = \partial_{x_i} \partial_{x_j} f$$

Hessian can be diagonalized by an orthonormal change of <u>basis</u>.