Oct 19

Note Title 10/19/2009

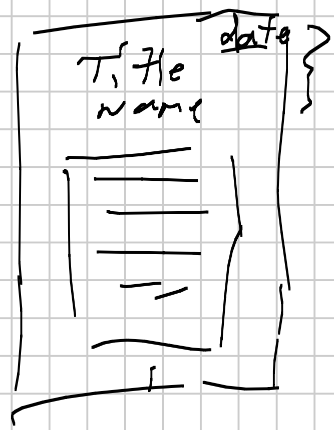
Paper due Fri.

3 ref.

1 - paper -

Biblio -

5 - 7 p.



Lemma $P(\vec{x})$, $\vec{x} \in \mathbb{R}^n$

is a polynomial —

$$\sum_{|\vec{\alpha}| \le k} C_{\vec{\alpha}} \vec{x}^{\vec{\alpha}}.$$

k-order —

Then $$\frac{\partial^{\vec{\alpha}} P(\vec{0})}{\alpha!} = c_\alpha$$

proof $$x_n^{\alpha_n} \cdots x_2^{\alpha_2} x_1^{\alpha_1} = \vec{x}^{\vec{\alpha}}$$

$$\partial_{x_1}(\vec{x}^{\vec{\alpha}}) = \begin{cases} 0, & \alpha_1 = 0 \\ \alpha_1 x_n^{\alpha_n} \cdots x_1^{\alpha_1 - 1} \end{cases}$$

$$\partial_{x_1}^{\alpha_1}(\vec{x}^{\vec{\alpha}})$$

$$= \alpha_1! \underbrace{x_n^{\alpha_n} \cdots x_2^{\alpha_2}}$$

$$\partial_{x_1}^{\alpha_1 + 1}(\vec{x}^{\vec{\alpha}}) = 0$$

$$\partial_{x_2}^{\alpha_2} \partial_{x_1}^{\alpha_1}(\vec{x}^{\vec{\alpha}})$$

$$= \alpha_2! \, \alpha_1! \, x_n^{\alpha_n} \cdots x_3^{\alpha_3}$$

$$\partial^{\vec{\alpha}} \vec{x}^{\vec{\alpha}} = \alpha!$$

$\vec{\beta} \neq \vec{\alpha}$.

$$\partial^{\vec{\alpha}} \vec{x}^{\vec{\beta}}$$

1) if some $\alpha_i > \beta_i$

$= 0$

2) if some $\alpha_i < \beta_i$.

still be an x_i.

so $\partial^{\vec{\alpha}} \vec{x}^{\vec{\beta}} \Big|_{\vec{0}} = 0$

$$\partial^{\vec{\alpha}} \left(\sum_{|\vec{\beta}| \le k} C_{\vec{\beta}} \vec{x}^{\vec{\beta}} \right) \Big|_{\vec{0}}$$

$$= C_{\vec{\alpha}} \cdot \alpha!$$

Suppose

$$P(\vec{x}) = \sum_{|\vec{\alpha}| \le k} d_{\vec{\alpha}} \left(\frac{\vec{x} - \vec{a}}{r} \right)^{\vec{\alpha}}$$

$$d_{\vec{\alpha}} = \frac{\partial^{\vec{\alpha}} (P(\vec{a}))}{\vec{\alpha}!}$$

2nd proof of Multinomial thm.

$$(x_1 + \cdots + x_n)^k$$

$$= \sum_{|\vec{\alpha}| = k} \frac{k!}{\vec{\alpha}!} \vec{x}^{\vec{\alpha}}.$$

Coeff. of $\vec{x}^{\vec{\alpha}}$?

$$\frac{\partial^{\vec{\alpha}}\left((x_1 + \cdots + x_n)^k\right)}{\vec{\alpha}!}\Bigg|_{\vec{0}}$$

$$\partial^{\vec{\alpha}}\left((x_1 + \cdots + x_n)^k\right) ?$$

$$u = x_1 + \cdots + x_n$$

$$\partial_{x_i}(u) = 1$$

$$\partial_{x_1}(u^k) = k u^{k-1} \cdot 1$$

$$\partial_{x_1}^{\alpha_1}(u^k) = k(k-1)\cdots(k-\alpha_1+1) u^{k-\alpha_1}$$

$$\partial_{x_n}^{\alpha_n} \partial_{x_{n-1}}^{\alpha_{n-1}} \cdots \partial_{x_1}^{\alpha}(u^k)$$

$$= k!$$

if $|\vec{\alpha}| = k$.

if $|\vec{\alpha}| > k$, or

$|\vec{\alpha}| < k$, still have

power of u left

$u(\vec{0}) = 0$.

Coeff of $\vec{x}^{\vec{\alpha}}$

$$= \begin{cases} 0, & \text{if } |\vec{\alpha}| \neq k \\ \frac{k!}{\vec{\alpha}!} & \text{if } |\vec{\alpha}| = k \end{cases}$$

Taylor Polynomials

f : class C^k on an interval I, $a \in I$.

$$P_{a,k}(\underbrace{x-a}_{h}) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j$$

Error or Remainder

$$R_{a,k}(h) = f(a+h) - P_{a,k}(h).$$

$$= f(a+h) - \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^{j},$$

Thm (Integral form)

Suppose f is of class C^{k+1} on I, $a \in I$.

Then

$$R_{a,k}(h) = \frac{h^{k+1}}{k!} \int_0^1 (1-t)^{k} f^{(k+1)}(a+th)\,dt$$

proof $k = 0$

$$f(a+h) - f(a) = h \int_0^1 f'(a+th)\,dt$$

$u = a + th$

$du = h\,dt$

$$= \int_a^{a+h} f'(u)\,du$$

$$= f(a+h) - f(a)$$

$$h \int_0^1 1\, f'(a+ht)\,dt$$

$$du = 1\,dt$$

$$v = f'(a+ht)$$

$$dv = h f''(a+ht)\,dt$$

$$u = (t-1)$$

$$= h\left(uv\Big|_0^1 - \int_0^1 u\,dv\right)$$

$$= h\left[(t-1)f'(a+ht)\Big|_0^1 + h\int_0^1 (1-t) f''(a+ht)\,dt\right.$$

$$= \cancel{t}\; h f'(a) + h^2 \int_0^1 (1-t) f''(a+ht)\,dt$$

$$f(a+h) - f(a)$$

$$f(a+h) - \left(f(a) + h f'(a)\right)$$

$$= h^2 \int_0^1 (1-t) f''(a+ht)\,dt$$

Int by parts –

$$du = (1-t)\,dt$$
$$v = f''(a+ht)$$

$$u = -\frac{(1-t)^2}{2}$$
$$dv = h f^{(3)}(a+ht)\,dt$$

etc

Thm If $f: I \to \mathbb{R}$ is C^k & $a \in I$ then

$$R_{a,k}(h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} \left(f^{(k)}(a+ht) - f^{(k)}(a)\right) dt$$

proof

$R_{a,k-1}$

$$f(a+h) - \sum_{j=0}^{k-1} \frac{f^{(j)}(a)}{j!} h^j$$

$$= \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+ht)\,dt$$

Next term is

$$\underline{\underline{\frac{f^{(k)}(a)}{k!}h^k}}$$

$$= \frac{h^k}{(k-1)!}\int_0^1 \underbrace{(1-t)^{k-1}}_{\frac{1}{k}} \underline{f^{(k)}(a)}\,dt$$

$$\int_0^1 (1-t)^{k-1}\,dt = -\frac{(1-t)^k}{k}\Bigg|_0^1 = \frac{1}{k}$$

Subtract $\frac{f^{(k)}(a)h^k}{k!}$ from

$R_{a,k-1}(h)$ & you get

$R_{a,k}(h)$.

Subtract $\frac{h^k}{(k-1)!}\int_0^1 (1-t)^{k-1} f^{(k)}(a)\,dt$

from integral formula for $R_{a,k-1}(h)$ & get

the new <u>integral formula</u>.