Oct 16

Note Title 10/16/2009

$$F(x_1, \dots, x_n)$$

$$= f(r)$$

$$r = \sqrt{x_1^2 + \dots + x_n^2}$$

$$\partial_{x_1}^2 F + \partial_{x_2}^2 F + \dots + \partial_{x_n}^2 F$$

$$\partial_{x_1} F = \partial_{x_1}(f(r(\underline{x})))$$

$$= f'(r)\frac{\partial r}{\partial x_1}$$

$$\frac{\partial r}{\partial x_1} = \frac{1}{\cancel{2}}(x_1^2 + \dots x_n^2)^{-1/2} \cdot \cancel{2} x_1$$

$$= \frac{x_1}{r}$$

$$f'(r)\frac{x_1}{r}$$

$$\partial_{x_1}^2 F = \partial_{x_1}\left(f'(r)\frac{x_1}{r}\right)$$

$$= \underline{f''(r)}\frac{x_1}{r}\cdot\frac{x_1}{r}$$

$$+$$

$$f'(r)\cdot\left(\frac{\cancel{r} - x_1\frac{x_1}{r}}{r^2}\right)$$

$$= f''(r)\frac{x_1^2}{r^2} + \frac{f'(r)}{r} - f'(r)\frac{x_1^2}{r^3}$$

$$\partial_{x_i}^2 F = \underbrace{f''(r)\frac{x_i^2}{r^2}} + \frac{f'(r)}{r} - f'(r)\frac{x_i^2}{r^3}$$

$$\sum_{i=1}^{n}\partial_{x_i}^2 F = \frac{f''(r)}{\cancel{r^2}}\cdot\cancel{r^2} + n\frac{f'(r)}{r} - f'(r)\frac{\cancel{r^2}}{r^{\cancel{3}}}$$

$$\underbrace{\partial_{x_{i_k}}\cdots\partial_{x_{i_2}}\partial_{x_{i_1}}}_{n} f$$

a partial deriv. of

order $\underline{\underline{k}}$

∂_x^2 - order 2

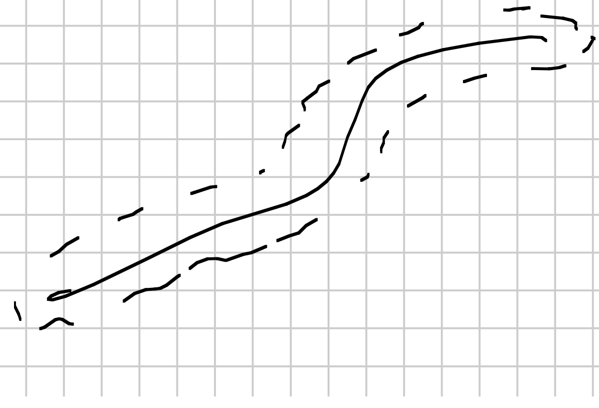
$\partial_x \partial_y$ " "

Def $D \subseteq \mathbb{R}^n$ - open

& all deriv. of order $\underline{\le} k$ are cont. Then we say f is of order C^k.

If all orders are cont then C^∞.

If D is not open to be C^k, f must be C^k in some open set cont. D.



Thm In C^k the order of diff. of any partial of order $\le k$ does not matter.

proof

$$\partial_{x_{i_k}} \cdots \underbrace{\partial_{x_{i_j}} \partial_{x_{i_{j+1}}}}_{} \cdots \partial f_{x_{i_1}}$$

can interchange

$$\partial_{x_{i_j}} \partial_{x_{i_{j+1}}}$$

If the rearrangement I want can be obtained by a sequence of such interchanges then it will be allowed & give same deriv.

All are obtained this way.

Notation —

x^2 , $\cancel{x_2}$, $\cancel{{}^2x}$

Multi-indices

$x_1^2 x_2^3 x_4$ in $\mathbb{R}^4$

$(x_1, x_2, x_3, x_4)^{(2,3,0,1)}$

$\parallel$

$\vec{x}^{\vec{\alpha}}$

$\vec{x} \neq \vec{x}_{1}^{\vec{1}}$ "

$(x_1 \cdots x_n) \neq x_1 x_2 \cdots x_n$

$\vec{x}$ – var

$\vec{\alpha}$ – multi-index

$$\|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

$$|\vec{\alpha}| = |\alpha_1| + |\alpha_2| \cdots |\alpha_n|$$

$$\left[\|\vec{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}\right]$$

$$\vec{x}^{\vec{\alpha}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$$\deg(\vec{x}^{\vec{\alpha}}) = |\vec{\alpha}|$$

$$\alpha_i \in \{0, 1, 2 \cdots\}$$

Poly of deg $\underline{\underline{k}}$

$$P(\vec{x}) = \sum_{|\vec{\alpha}| \leq k} c_{\vec{\alpha}} \vec{x}^{\vec{\alpha}}$$

Can use multindices for derivaties

$$\partial^{\vec{\alpha}} f = \underline{\partial_{x_n}^{\alpha_n} \cdots \partial_{x_2}^{\alpha_2} \partial_{x_1}^{\alpha_1} f}$$

Assumes C^k

k - order of $\vec{\alpha}$

$$\underline{\vec{\alpha}! = \alpha_n! \cdots \alpha_2! \alpha_1!}$$

__Thm__ (Multinomial Thm)

$$(x_1 + \cdots + x_n)^k = \sum_{|\vec{\alpha}| = k} \frac{k!}{\vec{\alpha}!} \vec{x}^{\vec{\alpha}}.$$

pf Induction –

$n = 2$

$$(x+y)^k = \sum_{j=0}^{k} \left(\frac{k!}{j!(k-j)!} \right) x^j y^{k-j}$$

$$\frac{k!}{j!(k-j)!} = \binom{k}{j}$$

$$\vec{\alpha} = (j, k-j)$$

$$|\vec{\alpha}| = k$$

$$\vec{\alpha}! = j!(k-j)!$$

$$(x,y)^{\vec{\alpha}} = x^j y^{k-j}$$

$n+1$

$$(\underbrace{(x_1 + \cdots + x_n)}_{y} + x_{n+1})^k$$

$$= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \left(\overbrace{(x_1 + \cdots + x_n)^j}\, x_{n+1}^{k-j} \right)$$

Bin. Thm

$$= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \sum_{|\vec{\alpha}|=j} \frac{j!}{\vec{\alpha}!} \underbrace{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} x_{n+1}^{k-j}}_{\vec{x}^{\vec{\alpha}}}$$

$$\alpha_{n+1} = k - j$$

Fix $\vec{\alpha}$, def. $\alpha_1, \ldots, \alpha_n$ & j,

Coeff

$$\frac{k!}{\cancel{j!}\,\underbrace{(k-j)}_{\alpha_{n+1}}!} \cdot \frac{\cancel{j!}}{\alpha_1! \cdots \alpha_n!}$$

$$= \frac{k!}{\vec{\alpha}!}$$ Hurray