Nov 4

Note Title 11/4/2009

Smooth Curves in $\mathbb{R}^2$

1) $C \subseteq \mathbb{R}^2$ -

a) Connected

b) For any $c \in C$, in some nghd of c - C is a C^1 graph of, y as a fctn of x or x " " y

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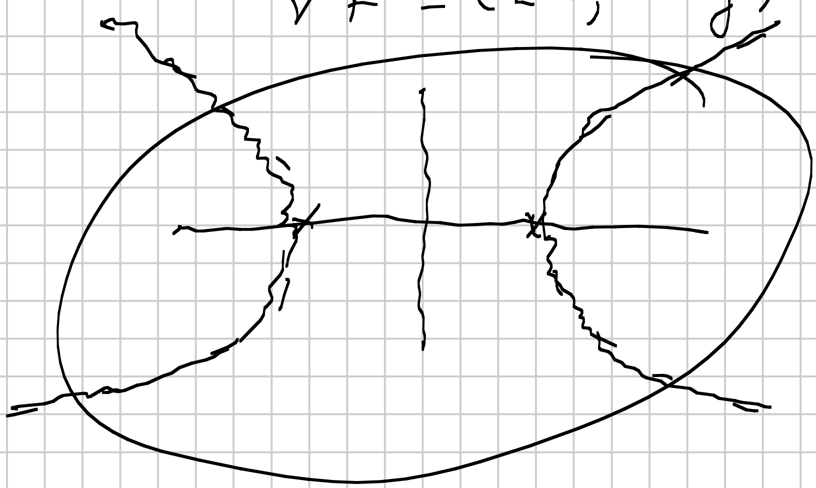
A level set of a C^1 fctn F, where ∇F nevers vanishes on the level set - is "almost" a smooth curve - It is if its connected.

Ex $\quad x^2 - y^2 = 1$

$\nabla F = (2x, -2y)$



Parametrized curves -

Thm If $g(t)$ $t \in I$ - open int.

is a C^1 param set in $\mathbb{R}^2$ + $\|Dg\| \neq 0$.

then for any $t_0 \in I$ there is a subinterval $t_0 \in J \subseteq I$ (J. open)

So that $g(J)$ is a smooth interval. —

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proof

Define F

$$F_1(x,y,t) = x - g_1(t)$$
$$F_2(x,y,t) = y - g_2(t)$$

$F(x,y,t) = 0$
iff $g(t) = (x,y)$.

Examine

$$DF = \begin{bmatrix} 1 & 0 & -g_1'(t) \\ 0 & 1 & -g_2'(t) \end{bmatrix}$$

Note $Dg(t_0) \neq \vec{0}$.

so 3rd column
is not $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

So 3rd col is lin
ind of either col. 1
or col. 2

Assume col 1

$$\begin{bmatrix} 1 & 0 & -g_1'(t_0) \\ 0 & 1 & -g_2'(t_0 \end{bmatrix}$$

By IFT, in some nghd of $x_0 = g_1(t_0)$, $y_0 = g_2(t_0)$

(x_0, t_0),

$x + t$ are C^1 fctns of $\underline{y}$.

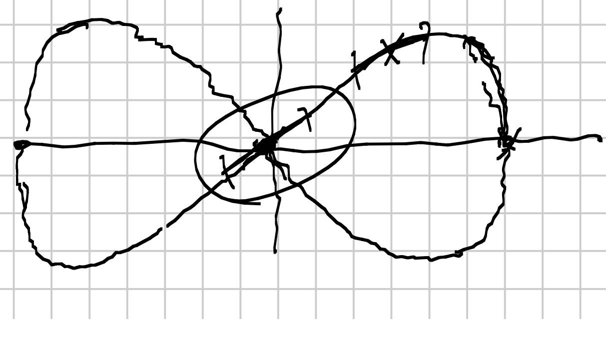
Let the nghd of $\underline{t_0}$ be J.

Rest. g to J – & now here x is a C^1 fctn $\underline{\text{of}}$ y.

$\underline{\text{Ex}}$ $x = \sin(t + \pi/2)$

$y = \sin(2t)$

Lissajous $\underline{\text{Curves}}$



$$\overline{Dg} = (\cos(t+\pi/2), 2\cos(2t))$$

$$? = (0,0)$$

$$\cos(t + \pi/2) = 0$$

$$t = n\pi$$

$$2\cos(2n\pi) = \underline{\underline{2}}$$

Sfces $\underline{\text{in}}$ $\mathbb{R}^3$

1) Σ is the locus of

$z = f(x, y)$ or

$y = f(x, z)$ or $\qquad \underline{\underline{f - C^1}}$

$x = f(y, z)$

near any $s \in \Sigma$.

2) Σ - level sfc of some mapping

$F(x, y, z) = 0$

$F - C^1$

on an open set

3) Σ is the image of some

$g : S \to \mathbb{R}^3,$

S - open in $\mathbb{R}^2$.

Defn A smooth sfc. $\Sigma \subseteq \mathbb{R}^3$ is a conn. set which in a nghd of every $s \in \Sigma$ is one of: graph

i) $z = f(x,y)$

$y = f(x,z)$

$\cancel{x} = f(y,z)$, $f \in C^1$.

Thm Suppose $F: S \to \mathbb{R}$

S - open in $\mathbb{R}^3$,

& F is C^1.

Let $\Sigma = \{(x,y,z) \in S : F(x,y,z) = 0\}$

& assume $\nabla F \neq 0$ on Σ.

Then for every $s \in \Sigma$ there is a nghd N of s

& $N \cap \Sigma$

is a smooth sfc.

Just. IFT

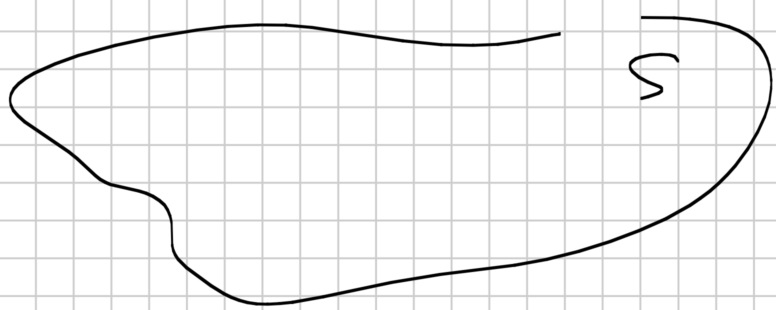
Thm Suppose $S \subseteq \mathbb{R}^2$, open, $g : S \to \mathbb{R}^3$, g is C^1

& $Dg(s)$ has rank 2. for all $s \in S$.

Then for any $s \in S$, there is a nghd N

$$s \in N \subseteq S$$

& $g(N)$ is a smooth sfc.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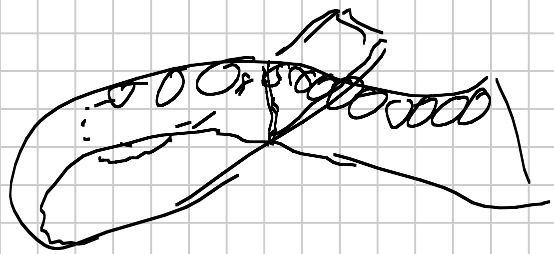

Pf

Def.

$$F = \begin{bmatrix} u - g_1(x,y) \\ v - g_2(x,y) \\ w - g_3(x,y) \end{bmatrix}$$

$$F : \mathbb{R}^5 \to \mathbb{R}^3$$

$$DF = \begin{array}{c} \begin{array}{ccccc} u & v & w & x & y \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -\partial_x g_1 & -\partial_y g_1 \\ 0 & 1 & 0 & -\partial_x g_2 & -\partial_y g_2 \\ 0 & 0 & 1 & -\partial_x g_3 & -\partial_y g_3 \end{bmatrix} \end{array}$$

These col. are lin ind

One of first 3 is lin ind of last 2.

$$\begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \boxed{Dg} \\ 0 & 0 & 1 & \end{bmatrix}$$

So in nbhd of (w_0, x_0, y_0)

w, x, y are C^1
fctns of $\underline{u \,\&\, v}$,
Restrict the values
of (x, y) to this
$\underline{\text{nghd}}$ - N

& Now $\underline{\underline{w}}$ is a
C^1 fctn of $\underline{\underline{u, v}}$.