Nov 16

Note Title 11/16/2009

$f: [a,b]$, bounded

$|f(x)| \le B$

Then $\overline{I}_a^b(f) - \underline{I}_a^b(f)$

$\le 2B(b-a)$.

If $P = \{a = x_0 < x_1 \dots < x_n = b\}$.

$$\overline{I}_a^b(f) = \sum_{i=1}^{n} \overline{I}_{x_{i-1}}^{x_i}(f).$$

& $$\underline{I}_a^b(f) = \sum_{i=1}^{n} \underline{I}_{x_{i-1}}^{x_i}(f)$$

so

$$\overline{I}_a^b(f) - \underline{I}_a^b(f) = \sum_{i=1}^{n} \overline{I}_{x_{i-1}}^{x_i}(f) - \underline{I}_{x_{i-1}}^{x_i}(f).$$

<u>Thm</u> $f : [a,b] \to \mathbb{R}$

& f is <u>cont</u>.

Then f is integrable

pf Let $\varepsilon > 0$.

f is unif. cont

so $\exists\ \delta > 0$ & if

$|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$

Choose n so that $\underline{\frac{b-a}{n} < \delta}$.

Look at

$$\overline{S}(f, P_n) - \underline{S}(f, P_n).$$

$$= \sum_{i=1}^{n} (M_i - m_i)\left(\frac{b-a}{n}\right)$$

$$M_i = \sup\{f(x) : x \in I_i\}$$
$$= f(a_i)$$

$$m_i = \inf(f(x) : x \in I_i)$$
$$= f(b_i)$$
$$a_i, b_i \in I_i$$
$$|a_i - b_i| < \delta.$$

$$\bar{S}(f, P_n) - \underline{S}(f, P_n)$$
$$= \sum_{i=1}^{n} (f(a_i) - f(b_i)) \frac{b-a}{n}$$
$$\leq \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \frac{b-a}{n} = \varepsilon.$$

Cor If $f : [a, b] \to \mathbb{R}$ is cont. then
$$\bar{I}_a^b(f) - \underline{I}_a^b(f) = 0$$

Thm Suppose $f : [a, b] \to \mathbb{R}$ is cont at all but a finite # of pts & is bounded. Then f is R. int.

Pf. Let $c_1 = a < c_2 \cdots < c_k = b$ include all disc. of f.

$$|f(x)| \le B$$

Let $\varepsilon > 0$.

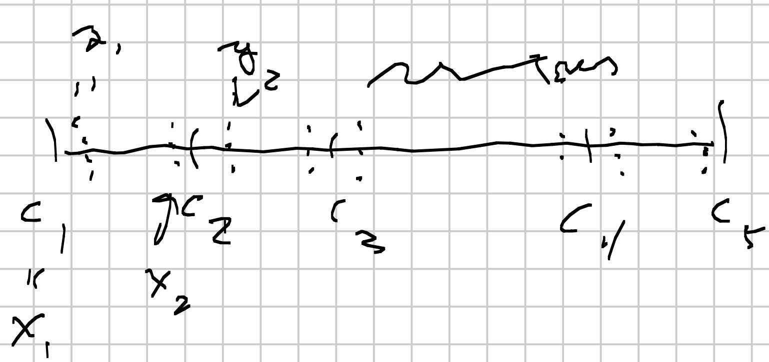
Be sure

$$\frac{\varepsilon}{2kB} < \frac{c_{i+1} - c_i}{2}$$

$\forall i$.

Consider the partition

$$P = \left\{ a,\ a + \frac{\varepsilon}{4kB},\ c_2 - \frac{\varepsilon}{4kB},\ c_2 + \frac{\varepsilon}{4kB},\ c_3 - \frac{\varepsilon}{4kB},\ c_3 + \frac{\varepsilon}{4kB},\ \ldots\ b - \frac{\varepsilon}{4kB},\ b \right\}$$



$$\{x_1 < y_1 < x_2 < y_2 \cdots x_k < y_k\}$$

$$\overline{I}_a^b(f) - \underline{I}_a^b(f)$$

$$= \sum_{i=1}^{k} \overline{I}_{x_i}^{y_i}(f) - \underline{I}_{x_i}^{y_i}(f)$$

$$+ \sum_{i=1}^{k-1} \overline{I}_{y_i}^{x_{i+1}}(f) - \underline{I}_{y_i}^{x_{i+1}}(f)$$

$$0$$

$$\leq \sum_{i=1}^{k} 2B \frac{\varepsilon}{2kB} = \varepsilon.$$

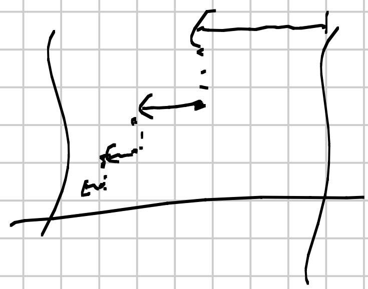
So long as f is bounded & we can sequester the disc. of f in a finite union of intervals whose total length is $< \frac{\varepsilon}{2B}$

f will be int.

Ex

$$f : [0,1] \to \mathbb{R}$$

$$f(x) = \begin{cases} 1 & , x = 1 \\ \frac{1}{n} & \frac{1}{n+1} < x \leq \frac{1}{n} \\ 0 & \text{at } x = 0. \end{cases}$$



disc accum at 0.

Ex

$$f(x) = \begin{cases} 0, & x\text{-irrat'l} \\ \frac{1}{q} & \text{if } x = p/q, \text{ in least terms.} \end{cases}$$

Fact: f is cont at x iff x is irrat'l.

pf. Suppose x is irrat'l & $a_i \to x$.

Show $f(a_i) \to f(x)$.

Enough to show this for $a_i \in \mathbb{Q}$

$$a_i = \frac{p_i}{q_i}$$

& $f(a_i) = \dfrac{1}{q_i}$.

If $q_i \to \infty$ done

Suppose not - i.e.

$\exists\ q_{n_j} \leq B$

& $\exists\ q_{k_j} = q_0$

$\Rightarrow x$ is rat'l

of denom $q_0 \Rightarrow \Leftarrow$

x - rat'l

then $\exists\ a_n$ - irrat'l

$a_n \to x$

$f(a_n) = 0 \not\to f(x) > 0.$