Nov 13

Note Title 11/13/2009

$$a \le b$$

$$-a \ge -b$$

$$-\sup(S) = \inf(-S)$$

$$-\overline{S}(f,P) = \underline{S}(-f,P)$$

$$\overline{I}_a^b(f) = \lim_{n\to\infty} \overline{S}(f,P_n)$$

$$-\overline{I}_a^b(f) = \underline{I}_a^b(-f)$$

Thm $f,g : [a,b] \to \mathbb{R}$, bdd.

a) $\overline{I}_a^b(f+g)$

$$\le \overline{I}_a^b(f) + \overline{I}_a^b(g)$$

$$\underline{I}_a^b(f+g) \ge \underline{I}_a^b(f) + \underline{I}_a^b(g)$$

b) If $c > 0$,

$$\overline{I}_a^b(cf) = c\,\overline{I}_a^b(f)$$

If $c < 0$

$$\underline{I}_a^b(cf) = c\,\overline{I}_a^b(f).$$

pf a) $S \subseteq [a,b]$

$$\sup(f(x)+g(x) : x \in S)$$

$$\le \sup(f(x) : x \in S)$$

$$+$$

$$\sup(g(x) : x \in S).$$

$\Rightarrow \forall P$

$$\overline{S}(f+g, P) \le \overline{S}(f,P) + \overline{S}(g,P).$$

$$\overline{I}_a^b(f+g) = \lim_{n\to\infty} \overline{S}(f+g, P_n)$$

$$\le \lim_{n\to\infty}\left(\overline{S}(f,P_n) + \overline{S}(g,P_n)\right)$$

$$= \overline{I}_a^b(f) + \overline{I}_a^b(g).$$

b) $c > 0$

$$\sup(cS) = c\sup(S).$$

$\Rightarrow$ $\forall P$

$$\overline{S}(cf, P) = c\,\overline{S}(f, P).$$

$$\overline{I}_a^b(cf) = \lim_{n\to\infty} \overline{S}(cf, P_n)$$

$$= c\lim_{n\to\infty} \overline{S}(f, P_n)$$

$$= c\,\overline{I}_a^b(f)$$

Cor If $f, g : [a,b] \to \mathbb{R}$

& are $\mathbb{R}$. int. then

so is $f+g$, $\int_a^b f+g = \int_a^b f + \int_a^b g$

& cf, $\forall c \in \mathbb{R}$.

$$\int_a^b cf = c\int_a^b f.$$

Pf

$$\overline{I}(f+g) \leq \overline{I}(f) + \overline{I}(g)$$

$$\underline{I}(f+g) \geq \underline{I}(f) + \underline{I}(g)$$

$$\Rightarrow \overline{I}(f+g) = \underline{I}(f+g).$$

as $\overline{I} \geq \underline{I}$ always.

& $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

$c > 0$

$$\overline{I}(cf) = c\overline{I}(f)$$

$$\underline{I}(cf) = -\overline{I}(c(-f))$$

$$= -c\overline{I}(-f) = c\underline{I}(f)$$

$$\overline{I}(cf) = \underline{I}(cf)$$

& cf is int.

$$\int_a^b cf = \overline{I}(cf) = c\overline{I}(f)$$

$$= c\int_a^b f.$$

$c < 0$

$$\underline{I}(cf) = c\underline{I}(f).$$

$$\overline{I}(cf) = c\overline{I}(f)$$

so cf is int &

$$\int_a^b cf = c\int_a^b f.$$

$c = 0$ $\quad$ $\int_a^b cf = \int_a^b 0 = 0$

$$= 0 \cdot \int_a^b f = c\int_a^b f.$$

$$\int_a^b f(t)\,dt$$

Thm If $f \geq g$ on $[a,b]$

& both are bounded

then $\overline{I}_a^b(f) \geq \overline{I}_a^b(g)$

& $\underline{I}_a^b(f) \geq \underline{I}_a^b(g)$

& if f, g int. then

$$\int_a^b f \geq \int_a^b g.$$

pf if $f \geq g$ on $[a,b]$

& $S \subseteq [a,b]$

then $\sup(f(x) : x \in S)$

$\geq \sup(g(x) : x \in S)$

& $\inf(f(x) : x \in S)$

$\geq \inf(g(x) : x \in S)$.

$\Rightarrow \forall P$

$\overline{S}(f,P) \geq \overline{S}(g,P)$

$\underline{S}(f,P) \geq \underline{S}(g,P)$.

Take limits for

P_n.

Defn: $f : [a,b] \to \mathbb{R}$.

$$f^+(x) = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

Note:

$$f(x) = f^+(x) - f^-(x)$$

$$|f(x)| = f^+(x) + f^-(x)$$

Lemma For $f: [a,b]$ - bded + P a part,

$$\overline{S}(f,P) - \underline{S}(f,P)$$

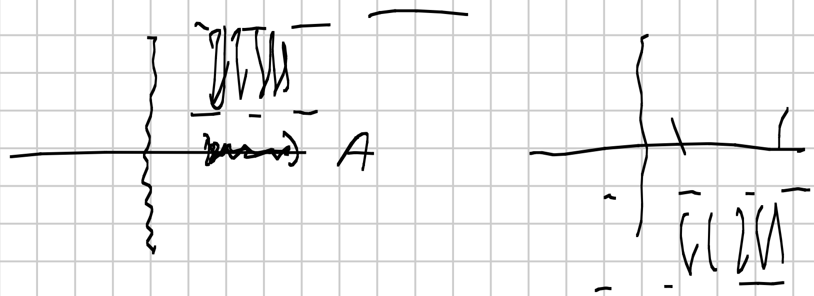
$$\geq \overline{S}(f^+,P) - \underline{S}(f^+,P)$$

+ similarly for f^-.

proof Suppose $A \subseteq [a,b]$

$$\sup(f(x) : x \in A) - \inf(f(x) : x \in A)$$

$$\geq \sup(f^+(x) : x \in A) - \inf(f^+(x) : x \in A).$$



$$\Rightarrow \quad \overline{S}(f,P) - \underline{S}(f,P)$$
$$\geq \overline{S}(f^+,P) - \underline{S}(f^+,P).$$

<u>Cor</u>. $\overline{I}(f) - \underline{I}(f)$
$$\geq \overline{I}(f^+) - \underline{I}(f^+).$$

& if f is R. int. so is f^+.

$$f^- = (-f)^+$$

& so if f is R-int so is f^-.

<u>Cor</u> If f is R. int. so is $|f|$ &

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

pf. $|f| = f^+ + f^-$.

$$\int |f| = \int f^+ + \int f^-$$

$$= \left|\int f^+\right| + \left|\int f^-\right|$$

$$= \left|\int f^+\right| + \left|-\int f^-\right|$$

$$= \left|\int f^+\right| + \left|\int -f^-\right|$$

$$\geq \left|\int f^+ + \int -f^-\right|$$

$$= \left|\int (f^+ - f^-)\right|$$

$$= \left|\int f\right|$$