M417, Fall 2008, Final Exam

This is a closed book, closed notes two hour exam. Proofs are expected for all results. You may quote or state standard theorems from the text or lecture without proof. Each problem is worth 40 points toward a total of 200 points.

Prob. 1

Suppose $g : [0,1] \to \mathbb{R}$ is continuous. Prove that the graph of g, that is to say $\{(x, g(x)) \subseteq \mathbb{R}^2 : x \in [0,1]\}$, is a compact set.

Prob. 2

Suppose $F : \mathbb{R}^2 \to \mathbb{R}$, is of class C^1 and $F(0,0) = 0$. Suppose the function G defined by

$$G(x,y) = F(2F(x,y), F(x,y))$$

has a critical point at the origin and $\partial_x(F)(0,0) = 1$. What is $\partial_y(F)(0,0)$?

Prob. 3

Suppose $f : (a,b) \to \mathbb{R}$ has derivatives of all orders and moreover there is a constant M so that for all n and $x_0 \in (a,b)$, $f^{(n)}(x_0) \leq M^n$. Write down the Taylor series expansion for f based at x_0 and prove that it converges for all $x \in \mathbb{R}$.

Prob. 4

Define a function $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$f(x,y,z) = (f_1(x,y,z), f_2(x,y,z)) = (e^{x+y+z}, x^2 + y^2 + z^2)$$

a) For what points $(a,b,c) \in \mathbb{R}^3$ does the Implicit Function Theorem guarantee that there is a neighborhood N of (a,b,c) in which the equation

$$f(x,y,z) - f(a,b,c) = 0$$

can be solved for y and z as C^1 functions of x?

b) If we write these solutions as $y = u(x)$ and $z = v(x)$ what are $u'(a)$ and $v'(a)$?

Prob. 5

Define a function $g : [0,1] \times [0,1] \to \mathbb{R}$ by

$$g(x,y) = \begin{cases} 1 \text{ if } x \leq y \\ 0 \text{ otherwise} \end{cases}$$

Prove that g is Riemann integrable.