Advanced Analysis (Math 417) **Name:** __________________
Exam 1 **March 14, 2008**

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} x^2 \sin(\frac{1}{x}) + y^2 \sin(\frac{1}{y}) & \text{if } xy \neq 0; \\ x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \text{ and } y = 0; \\ y^2 \sin(\frac{1}{y}) & \text{if } x = 0 \text{ and } y \neq 0; \\ 0 & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

(a) Prove or disprove: f is differentiable at $\mathbf{0} = (0,0)$.

(b) Prove or disprove: f is of class C^1 on an open neighborhood of $\mathbf{0}$.

2. Consider the following assertion:

> Suppose f is of class C^2 on an open subset of $\mathbb{R}^2$ containing $\mathbf{0}$, and that $\nabla f(\mathbf{0}) = \mathbf{0}$. Then f has either a local maximum or a local minimum at $\mathbf{0}$.

(a) Disprove the assertion with a counterexample.

(b) Add one more hypothesis to make the assertion true (no proof required).

3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be of class C^{k+1} on $\mathbb{R}^n$, and $\mathbf{a} \in \mathbb{R}^n$. Suppose $\partial^\alpha f(\mathbf{a}) = 0$ for all $|\alpha| = k+1$. Use the fact that

$$R_{\mathbf{a},k}(\mathbf{h}) = (k+1) \sum_{|\alpha|=k+1} \frac{\mathbf{h}^\alpha}{\alpha!} \int_0^1 (1-t)^k \partial^\alpha f(\mathbf{a} + t\mathbf{h})\, dt$$

to prove that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{R_{\mathbf{a},k}(\mathbf{h})}{\|\mathbf{h}\|^{k+1}} = 0.$$