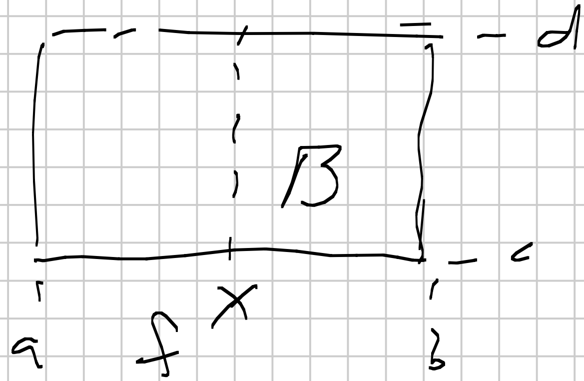
Dec 4



$$x \in [a,b]$$

$$f_x(y) = f(x,y)$$

Thm Suppose -

1) f is int. on B.

2) $\forall x \in [a,b]$, f_x is int. on $[c,d]$

Then

a) $g(x) = \int_c^d f_x(y)\,dy$

is int on $[a,b]$ &

b) $\int_a^b g(x)\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$

$$= \int_B f \, dA.$$

Proof

$P \times Q$

$$\overline{S}(f, P \times Q)$$

$$\geq$$

$$\overline{S}(\overline{S}(f_x, Q), P)$$

$$\leq \qquad \geq$$

$$\boxed{\overline{S}\left(\int_c^d f_x \, dy, P\right)} \qquad \boxed{\int_B f \, dA} \qquad \underline{S}(\overline{S}(f_x, P), Q)$$

$$\geq \qquad \qquad \geq$$

$$\overline{S}(\underline{S}(f_x, Q), P) \qquad \boxed{\int_a^b \int_c^d f_x \, dy \, dx} \qquad \underline{S}\left(\int_c^d f_x \, dy, Q\right)$$

$$\geq \qquad \qquad \leq$$

$$\underline{S}(\underline{S}(f_x, Q), P)$$

$$\geq$$

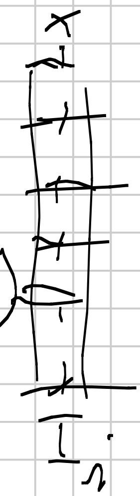
$$\underline{S}(f, P \times Q)$$

$$\overline{S}(f, P \times Q)$$

$$= \sum_{i,j} M_{i,j}\, \ell(I_i)\, \ell(J_j)$$

$$= \sup\Big(\sum_{i,j} f(x_{i,j}, y_{i,j}) : x_{i,j} \in I_i,\ y_{i,j} \in J_j \Big)$$

$$\overline{S}(f_x, Q)$$



$$\sup\Big(\sum_{j} f(x, y_{i,j})\, \ell(J_j)$$

$$x \in I_i,$$

$$y_{i,j} \in J_j \Big)$$

$$\overline{S}(\overline{S}(f_x, Q), P)$$

$$\sup\Big(\sum_i \sum_j f(x_i, y_{i,j})\, \ell(J_j)\, \ell(I_i)$$

$$: x_i \in I_i$$

$$y_{i,j} \in J_j \Big)$$

a) Show $g(x) = \int_c^d f_x(y)\,dy$

is int.

$\forall \varepsilon > 0$, $\exists$ $P \times Q$

st $\overline{S}(f, P \times Q) - \underline{S}(f, P \times Q)$

$< \varepsilon$.

$\overline{S}(g, P) - \underline{S}(g, P) < \varepsilon$.

$\Rightarrow$ integ

$$\left| \int_B f\,dA - \int_a^b \int_c^d f(x,y)\,dx\,dy \right|$$

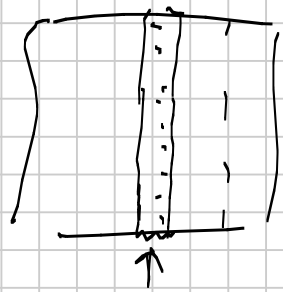
$< \varepsilon$

& so

$$\int_B f\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

Ex $[0,1]\times[0,1]$

$$f(x,y) = \begin{cases} 1, & x = \frac{1}{2},\ y \text{ is rat'l} \\ 0 - & \text{otherwise} \end{cases}$$



x, f_x is not int.

Let's improve Thm

Thm Suppose f is int over $[a,b]\times[c,d]$

Then

1) $\overline{g}(x) = \overline{I}_c^d(f_x)$ &

$\underline{g}(x) = \underline{I}_c^d(f_x)$ are int. in x &

2) $$\int_a^b \overline{I}_c^d f_x \, dx = \int_a^b \underline{I}_c^d f_x \, dx = \int_B f \, dA$$

$\triangleright$ ⌉

replace $\overline{S}\left(\int_c^d f_x\,dy, P\right)$

with $\overline{S}\left(\overline{I}_c^d(f_x), P\right) \geq \overline{S}\left(\underline{I}_c^d(f_x), P\right)$

on left

$\underline{S}\left(\int_c^d f_x\,dy, P\right)$ with

$\underline{S}\left(\overline{I}_c^d(f_x), P\right) \geq \underline{S}\left(\underline{I}_c^d(f_x), P\right)$