Aug 26

Note Title 8/26/2009

Dot products
length in $\mathbb{R}^n$

$$\|\vec{a}\| = \underline{(a_1^2 + \cdots a_n^2)}^{1/2}$$

estimates

$\vec{a} = (a_1 \cdots a_n)$

$m = \max(|a_1|, \cdots, |a_n|)$

Then

$$m \leq \|\vec{a}\| \leq \sqrt{n} \cdot m$$

proof

$m = |a_i|$ for some i.

$$m^2 = a_i^2 \leq a_1^2 + a_2^2 + \cdots + a_n^2$$

$$m \leq \sqrt{a_1^2 + \cdots + a_n^2} = \|\vec{a}\|.$$

$$\begin{array}{ccc} m^2 & \geq & |a_1|^2 \\ + & & + \\ m^2 & \geq & |a_2|^2 \\ + & \vdots & + \\ + & & + \\ m^2 & \geq & |a_n|^2 \end{array}$$

$$n \cdot m^2 \geq (a_1^2 + \cdots + a_n^2)$$

$$\sqrt{n} \cdot m \geq \|\vec{a}\|.$$

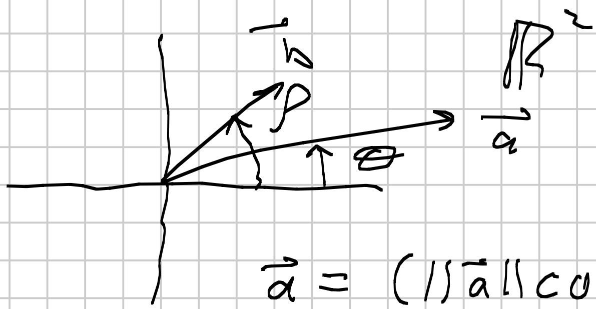
C. S. Inequ. $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$

assume $\|\vec{a}\|, \|\vec{b}\| \neq 0$.

$$-\|\vec{a}\| \|\vec{b}\| \leq \vec{a} \cdot \vec{b} \leq \|\vec{a}\| \|\vec{b}\|$$

$$-1 \leq \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \leq 1$$

$\Rightarrow \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$ is cos of an $\angle$ between 0 & π.



$$\vec{a} = (\|\vec{a}\|\cos\theta, \|\vec{a}\|\sin\theta)$$

$$\vec{b} = (\|\vec{b}\|\cos\rho, \|\vec{b}\|\sin\rho)$$

$$\vec{a}\cdot\vec{b} = (\|\vec{a}\|\|\vec{b}\|\cos\theta\cos\rho + \|\vec{a}\|\|\vec{b}\|\sin\theta\sin\ )$$

$$= \|\vec{a}\|\|\vec{b}\|(\cos\theta\cos(-\rho) - \sin\theta\sin(-\rho))$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= \|\vec{a}\|\|\vec{b}\| \cos(\theta - \rho)$$

$\theta - \rho$ is $\angle$ from $\vec{a}$ to $\vec{b}$.

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos(\angle \text{ from } \vec{a} \text{ to } \vec{b})$$

$$\cos(\angle \text{ between } \vec{a} + \vec{b}) := \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

Cross Products

$\mathbb{R}^3$

basis $(1,0,0) = \vec{i}$

$(0,1,0) = \vec{j}$

$(0,0,1) = \vec{k}$

$\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

$\vec{a} \times \vec{b}$

$= \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$ //

$\vec{i}(a_2 b_3 - a_3 b_2) + \vec{j}(a_3 b_1 - a_1 b_3)$

$$+\vec{k}(a_1 b_2 - a_2 b_1)$$

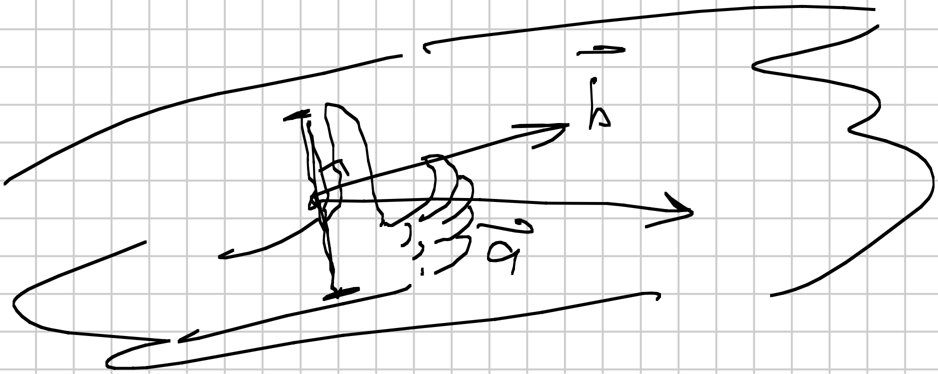
What IS it.

$\vec{a} \times \vec{b}$ is the unique vector in $\mathbb{R}^3$ with

1) orthogonal to both $\vec{a}$ + $\vec{b}$.

2) length $\|\vec{a}\|\|\vec{b}\||\sin(\angle \text{ between } \vec{a} + \vec{b})|$

3) Satisfies "right hand rule"



Topology

Sets –

- open sets
- closed sets
- interiors
- boundaries

Spheres & balls in $\mathbb{R}^n$.

Sphere of radius r centered at $\vec{a}$ is

$$\{\vec{x} : \|\vec{x}-\vec{a}\| = r\}$$

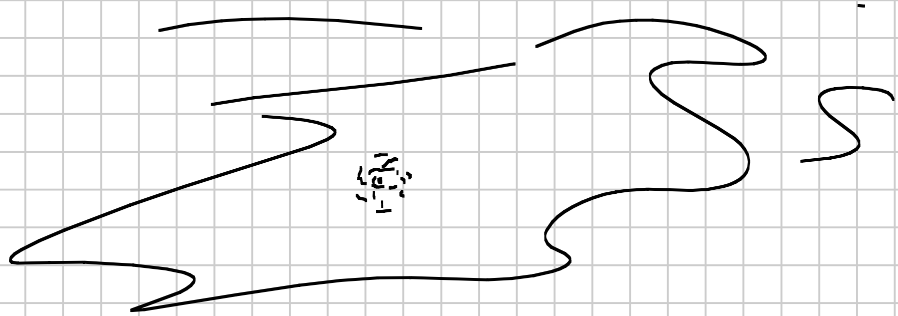
Ball (open ball) of radius r centered at $\vec{a}$ is

$$\{\vec{x} : \|\vec{x}-\vec{a}\| < r\}$$

$$= \underline{B_r(\vec{a})} = \cancel{B(r, \vec{a})}$$

Def. For $S \subseteq \mathbb{R}^n$, a point $\vec{a}$ is an "interior point" of S, if for some $r>0$,

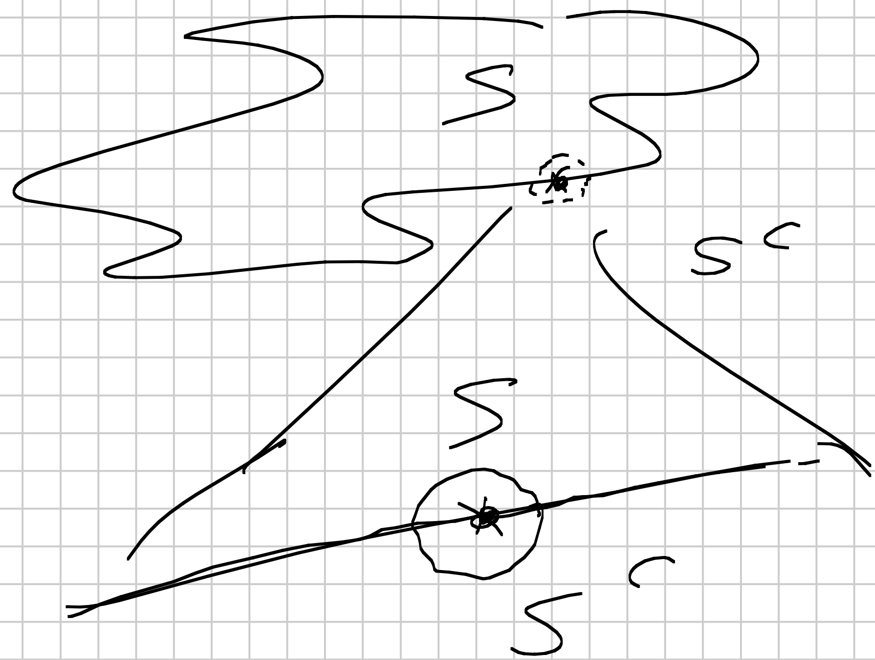
$$B_r(\vec{a}) \subseteq S.$$



A point $\vec{b}$ is a boundary pt. of S if for all $r>0$

$$B_r(\vec{b}) \cap S \neq \emptyset,$$

$B_r(\vec{p}) \cap S^c \neq \emptyset$



Interior of S, S^o

Boundary of S $\quad \partial S$.

S breaks $\mathbb{R}^n$ into 3 pieces —

$$S^\circ, \quad (S^c)^\circ, \quad \partial S = \partial S^c$$

? Does there exist a set $S \subseteq \mathbb{R}$ with $\partial S = \mathbb{R}$?

$S = \mathbb{Q}$.

$\supset \overline{\mathbb{Q}} = \mathbb{R}$.

1) Let $x \in \mathbb{R}$.

& $r > 0$. Consider

$$B_r(x) = \{y : |x-y| < r\}$$

$$= (x-r, x+r).$$

As every nonempty open interval contains rat'ls, $B_r(x) \cap \mathbb{Q} \neq \emptyset$

As every nonempty open interval contains irrationals

$$B_r(x) \cap \mathbb{Q}^c \neq \emptyset$$