Mathematical induction and the natural numbers

For this work we assume $\mathcal{F}$ is an ordered field, that is to say it satisfies all the axioms listed in Ross except completeness.

Definition 1 *A subset $S \subseteq \mathcal{F}$ is called inductive if for all $x \in S$ we must have $x + 1 \in S$ as well.*

Definition 2 *Set $\mathcal{I} = \{S \subseteq F | 1 \in S \text{ and } S \text{ is inductive}\}$. Now set $S_0 = \bigcap_{S \in \mathcal{I}} S$, the intersection of all elements in $\mathcal{I}$.*

Problem 1 *Show that $1 \in S_0$ and that if $x \in S_0$ then $x + 1 \in S_0$ and hence that $S_0 \in \mathcal{I}$.*

Definition 3 *We call S_0 the "natural numbers" in $\mathcal{F}$.*

The set S_0 is the smallest set for which "mathematical induction" works. To explain this, suppose P is some formula or property that a value $x \in \mathcal{F}$ might have. That is to say, $P(x)$ is either a true or a false statement. Now suppose you show $P(1)$ is true and further that whenever $P(x)$ is true, you also must have that $P(x + 1)$ is true. You probably recall that these are the two hypotheses of mathematical induction. In our situation what this means is that the set $S = \{x | P(x) \text{ is true}\}$ must belong to $\mathcal{S}$. Since S_0 is the intersection of all such sets this implies P is true for all elements of S_0. Thus by defining the "natural numbers" as the intersection S_0 we find we can use mathematical induction to prove properties hold on the set S_0.

Problem 2 *Show that if a and b belong to S_0 so does $a + b$.*

hint: Fix $a \in S_0$ and set $S_1 = \{b \in S_0 | a + b \in S_0\}$. Now show $S_0 \subseteq S_1$ by mathematical induction.

In $\mathcal{F}$ we can define open and closed intervals $(a, b) = \{x \in \mathcal{F} | a < x < b\}$ and $[a, b] = \{x \in \mathcal{F} | a \leq x \leq b\}$.

Problem 3 *Show that if $b \in S_0$ then $(b, b + 1) \cap S_0 = \emptyset$.*

hint: First show that $S_0 \cap (-\infty, 1) = \emptyset$. Now show that for all $b \in S_0$ we have $S_0 \cap (b, b + 1) = \emptyset$ by induction on b.

Problem 4 *Show that S_0 is complete in that for any Cauchy sequence $\{s_i\}$ of terms from S_0, there is a I and for $i \geq I$, $s_i = s_I$, i.e. the sequence is constant once i is large enough.*

Problem 5 *Now assume also that $\mathcal{F}$ satisfies the least upper bound property in that any bounded subset has a least upper bound and show that S_0 cannot be bounded above.*

hint: Show that if S_0 is bounded above, then the supremum is in S_0 and this is impossible.

Problem 6 *Show that if $\mathcal{F}$ satisfies the least upper bound property then for all $x \in \mathcal{F}$ there is a unique $b \in S_0$ with $x \in [b, b + 1)$.*