LECTURE PLAN FOR MATH 672
TORIC GEOMETRY
Fall 2014

1 Aug 25

Introduction

2 Aug 27-29: read 1.0

Principal concepts:

1. The Coordinate Ring of an Affine Variety. Define the function field.
2. $\mathbb{C}[X]$ integral domain iff $I(X)$ prime iff X irreducible.
3. functoriality of Spec.
4. Affine varieties are isomorphic if so are their coordinate rings.
5. Points correspond to Maximal ideals.
6. Zariski topology.
7. Localization. Do example of line minus a point.
8. Introduce the affine torus via localization.
9. Normality: do the node as another example of non-normal variety (Book does cusp). Do normalization map for the node.
10. Local rings. Maximal Ideals. Zariski Tangent space. Smoothness. Smooth implies normal - viceversa is not true.
11. Products of affine varieties.

Questions that arose:

1. Discussion of Spec as a functor. Difference between Spec and Specmax. (related to spectrum of a matrix?) Points = maximal ideals. Mention relevance of Nullstellensatz, and Notherian condition giving equivalence to solution sets of finite systems of equations. Contravariance.
   - Classical affine AG. Bijection between varieties and solution sets to finite lists of polynomials.
   - Because polynomial ring is Notherian can replace that with ideals.

- Natural functions between varities and ideals. To make it bijective, require ideal to be radical and the field to be algebraically closed (Nullstellensatz). ($V(I) = \phi$ iff $I = (1)$)
- Spec shifts attention to coordinate ring. Remember to always think of R as the ring of functions for the space $Spec(R)$.
- Calculus definition of function. Show how it essentially it is defined by pullbacks of functions on the space.
- Define a little bit what it means to be a functor.

2. Zariski topology: give some intuition and difference wrt ordinary topology.
   - why axioms of a topology are satisfied.
   - open sets are dense. Very coarse topology.
   - for curves it is the finite complement topology. Do example of cutting down a point in a conic.
3. Open sets defined by localization.
   - start with example of $\mathbb{C}$ minus a point.
   - why the complement of a hypersurface is itself an affine variety, and what is its ring of regular functions.
4. Discuss torus. Point out difference between algebraic torus and topological (complex) torus. Homotopy equivalent.
5. Various notions of functions on an algebraic variety:
   - Regular functions;
   - Sheaf of regular functions;
   - Rational functions;
   - Functions regular at a given point (local ring).
   - Schemely: residue field.
6. Zariski Tangent Space.
   - Recall abstract definition from differential geometry: tangent space is space of linear functionals that satisfy Leibnitz.
   - Show that satisfying Leibnitz is equal to vanishing on elements of M^2.
   - Do parallel with tangent vectors as velocity vectors for curves.
   - Talk about maps from spec of the dual numbers. In order to do so, must also show how to think of pullback of functions gives $f(x)$.

- Do example of tangent plane at the origin to $\mathbb{C}^2$.
- Example of tangent at a node, or a cusp. (Do one example with the M/M^2, another with the maps from spec of the dual numbers.)

7. Small discussion about dimension: geometric definition= longest chain of nested irreducible subvarieties.

8. Normality and Normalization.

   - We like smooth. But we settle for not too singular.
   - Normal implies smooth in codimension 1.
   - Normalization is canonical.
   - X is normal iff the coordinate ring is integrally closed. This means that if a rational function is a solution to a monic polynomial equation with regular coefficients, it is actually regular. How to think of this. If you have a function that actually has poles, you can't cancel the poles by taking regular linear combinations of powers of it where the leading coefficient is monic (and hence achieves the highest order of poles). Normal is almost equivalent to smooth in codimension one. In fact Serre tells us:

     R1 singular in codimension 2 or higher

     S2 if a rational function has no poles in codimenision one, then it is regular. (this is true for lci in smooth variety, etc...)
   - Normalization is obtained by taking the integral closure of the coordinate ring inside the ring of rational functions. The inclusion of rings gives a birational surjection from the normalization to the original variety.
   - Do explicitly example of node or cusp.

9. Product of affine varieties is affine. Show by illustrating universal property and see it gives rise to tensor product of coordinate rings.

3 Sep 3-5: read 1.1,1.2

General discussion:

- Introduce one more sheaf of functions: $\mathcal{O}^*$, where the operation is multiplication. In the case of a torus these functions are **monomial functions** or characters.

- Note that characters here are the same things as **linear** characters in representation theory. Because tori are commutative, all irreducible representations are one-dimensional, and therefore the corresponding characters are linear. The statement on page 11: assume that a torus acts linearly on V, then the action maps can be simultaneously diagonalized etc. is precisely the statement that every representation of the torus decomposes in irreducible representations.

- Note that characters of a torus are isomorphic to $(\mathbb{Z}^n, +)$, but the isomorphism is non-canonical, just how there isn't in general a canonical isomorphism of the torus T with $(\mathbb{C}^*)^n$.

- Group homomorphism between tori are given by (collection of characters). Do a couple example to see that Images are tori, and Kernels are unions of tori (subgroups).

- Do an example to see that the closure of a subgroup of a torus inside affine space can be different depending on:
  - what is the subgroup.
  - what is the choice of affine space that the torus lives in.

- Interpret the classical sequence in affine algebraic geometry in this context:

$$\begin{array}{ccccccccc}
0 & \to & I_x & \to & \mathbb{C}[x_1, \ldots, x_n] & \to & \mathbb{C}[X] = \mathbb{C}[S] & \to & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] & \to & \mathbb{C}[M] & & \\
& & & & \updownarrow & & \updownarrow & & \\
& & & & (\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])^* & \to & \mathbb{C}[M]^* & & \\
& & & & \updownarrow\cong & & \updownarrow\cong & & \\
& & L & \to & \mathbb{Z}^n & \to & S \subset M & &
\end{array}$$

In toric varieties the KEY point is that characters go to characters. x_i can be thought as special characters of the torus in the ambient space, that simultaneously: give a basis for the character lattice that identifies an isomorphism of the torus with a product of $\mathbb{C}^*$'s AND choose an affine space for the torus to sit in. We ask that the affine algebraic variety X has a torus inside! Then naturally the characters of the torus are regular functions on the torus and rational functions on X. $\mathbb{C}[X]$ is generated by those regular functions on the torus that extend to regular functions on all of X. Now if the above picture is clear, the three alternative constructions of a toric variety fall out of it immediately:

1. The assignment $x_i \mapsto \chi^{m_i}$ is equivalent to giving a map of tori.
2. The toric ideal point of view comes from looking at the generators of I_X under the additive isomorphism. There a little bit

of weirdness because I_x is "functions that are 0 on X, whereas the sub lattice L gives characters that restrict to 1 on the toric variety.

3. The semigroup point of view comes from looking at the characters of M that extend to regular functions of X.

- Do example of rational normal cone of degree 2 carefully and extensively. Discuss rational normal cones: examples of varieties which are isomorphic but not affine linear equivalent. Preview projective space, and rational normal curves.

People seemed to collectively focus on the importance and build up to Theorem 1.1.17 (the four equivalent characterizations of affine toric varieties). Also of note as perceived important topics: affine semigroups, toric ideals, the connection to linear algebra and the potential computing/intuition power that has. Here are some questions (in no particular order of importance):

1. In general, several people were interested in seeing counterexamples– the text provides lots of illustrative examples, but few/no examples of failures. Specific suggestions include Lemma 1.1.16 not splitting into a direct sum (i.e., does anything interesting happen if the subspace is not stable's

2. Following Theorem 1.1.17, and the statement V is a toric variety precisely when the functions that extend are determined by the characters that extend, what does it look like when this isn't happening? Are there good examples of functions which clearly do not extend? **This first two questions have to do with the philosophy that monomials should go to monomials. We can interpret this in yet another way... there is an action of the torus (of V) on itself that gives dually an actions on functions. Then V is obtained by putting the torus in some ambient space and closing it up, and the functions on V are the functions that extend to regular functions on the closure. But the functions f and tf "go to infinity in the same places, and those places should be "at infinity in the big ambient space we are putting the torus. In other words, f is regular for V iff tf is. This is equivalent to asking that the set of regular functions for V is closed under the action of T, and geometrically that the action of T extends to the closure of T (=V) in the ambient space.**

3. An easy to describe variety which is not a toric variety (deleting the affine is easy by going projective, but I think this is geared more towards how do we know a variety isn't toric?–I think theorem 1.1.17 answers

this decently well, but it might be a useful application to mention about 1.1.17?). **An easy example of a variety which is not toric but it is isomorphic to a toric variety is a line not through the origin in the plane. Things that cannot be isomorphic to toric varieties are for example elliptic curves or higher genus curves in general. Non rational surfaces etc**

4. Are characters the same as characters from number theory/representation theory?

5. The rational normal cone of degree d: several people asked its significance and requested a bit of a discussion about its construction. **See above**

6. Proposition 1.1.8 had a couple detail-related issues in it.
   - It follows that the image ($\phi_A(T_N)$) is Zariski open in Y_A. Why? **talk a little bit about Zariski closure**
   - The diagram conclusion at the very end.

7. Right before Proposition 1.1.9, they say It follows easily that the binomial...vanishes on the image of ϕ_A. Why? **This should be addressed when talking about toric ideals, and same with the point below**

8. Is there anything to the L_+ and L_- beyond rewriting things to be in $\mathbb{N}$?

9. Some more discussion on the tensor product paragraph (near the top of page 12). I think these questions were just due to some lack of comfort with tensor products? That, and why we write T_N (where N is the one-parameter subgroup) instead of T_M (with M the characters), when we later focus so much on M. **Point out that $N \otimes \mathbb{C}^* \cong T$ whereas $M \otimes \mathbb{C}^* \cong T^*$**

These are the most common/biggest things people were asking about. On a side note, I found our discussion on the way to Boulder–about how focusing on functions leads naturally to toric varities as useful examples/objects–really useful in tying together motivations.

4 Sep 15: Read 1.3

Begin class by reviewing the main ideas of the previous week: an affine toric variety is an affine variety with a notion of "monomial functions, such that monomial functions from the ambient space restricts to monomial functions on the variety.

Here's the summary of questions about section 1.2:

1. Relative interior vs interior. **Relative interior is what you would call the interior of a cone as an abstract object. The interior obviously depends on how the cone lives in some ambient space**

2. How to think about dual cones.

3. Why we start with cones in one-parameter families and eventually get to characters. **These two questions are crucial, and must be illustrated both "philosophically and with examples.**

   Philosophically: you start with a torus, and you are seeking to embed it in some affine space. The cone σ selects a bunch of one parameter subgroups in the torus. Each of these subgroups has a limit at 0, and you WANT your embedding into affine space to be such that the image of such limits is at a finite spot, so that such limits appear as points in the Zariski closure of the image. The cone $\sigma^\vee$ gives all characters that do in fact send such zeroes to zero. Such characters are therefore promoted to become the regular monomial functions on the toric variety, and provide the desired embedding. (This is so cool!)

   Examples: useful to begin with one dimensional examples, which are almost trivial, but do illustrate the above philosophy. Then example of an affine chart of the blowup of the plane at a point.

4. Difference between convex polyhedral cones and rational polyhedral cones. **as far as I understand, the only difference is the interplay with the integral structure: the finite set of generators are required to be lattice points for a rational cone. Note that a non rational cone gives rise to a not finitely generated semigroup. Give a two dimensional example.**

5. For the latter, do we just need the relevant set S to be contained in a lattice? **Yes. Not a lattice, but the lattice**.

6. Some of the Figures are unclear about why they are/are not rational (specifically requested were why 3 and 4 were not) **I don't think the book claims 3 and 4 are not rational, it only does not supply sufficient information to deduce that they are.**

7. What's the motivation behind cones? Do they have relevance outside of toric varieties? Note: our discussion 30 minutes ago was extremely instructive on answering this question and the how do think about dual cones one

8. Do simplicial complexes provide a good intuition, so far as building blocks and gluing ideas go? **Yes and no. The basic idea of constructing interesting space from simple ones is there, but while simplicial complexes are assembled by things glued along closed subvarieties (hence "small with respect to the building blocks), algebraic varieties are glued along big open dense sets. It is somehow the combinatioralization of their one parameter subgroups with limits that takes a shape similar to CW complexes.**

9. A couple people asked for something like a walk through of example 1.2.22.

10. What falls apart when we don't require strong convexity? **The variety contains a torus factor. This is further explored in section** 1.3**. Leave this for last.**

11. At least half of the emails asked about if the Separating Lemma has further uses **It does, but we will see later more precisely why. Intuitively, the Separating Lemma means that given two adjacent cones sharing a face, there exist some monomials that are regular for one cone but not for the other, and such that the inverse monomial reverses the situation.**

12. There's some general confusion about exactly what the pairing between a cone and its dual is/should be thought of. **What it really is is log-evaluation of functions. The log part just means to transfer the multiplicative structure of the semigroup of monomial functions into an additive structure.**

5 Sep 17, Sep 19: read 2.0, Sep 22

Highlights:

1. The correspondences regarding points on an affine variety.

2. The link between saturation and normality (and by extension strongly convex cones).

3. Fixed points of torus actions.

4. Equivalence of toric varieties (ties in to the last homework problem).

Problems:

- Overall, there was some frustration with the large amount of notation/terminology, as well as the extensive referencing back to earlier things.

- Several people seem to want to see (some of the) the same examples given in the section, but with you walking through them. (Examples of requested examples include 1.3.11 and 1.3.19.)
- The book mentions that the correspondence between points of V and maximal ideals in $\mathbb{C}[S]$ is standard. A few requests to go over this correspondence.
- Trouble understanding saturation.
- Some confusion about the monic polynomial in the proof of 1.3.5 (a) to (b).
- Why in Example 1.3.9 can they use 1/2, when 1/2 is not a natural number?
- Theorem 1.3.12 seems important; some requests to discuss the proof. (Including, but not limited to, how all smooth affine toric varieties arise in this way?)
- How does this all fit into a bigger geometric picture–a specific question: why do we care about equivariance in relation to the torus action?

Overall, I think some/much of the big picture was lost in the terminology and notational slog. If the section amounts to listing Properties of Affine Toric Varieties, then it seems like many are wondering about the motivation for it all.

Big picture

Toric geometry is a "special version of algebraic geometry, characterized by this additional structure coming from a torus action with nice properties. This allows, instead of focusing on all regular functions like in general algebraic geometry, to single out a class of special regular functions (the monomial functions) which turn out to contain sufficient information to describe the geometry of the toric variety. Focusing on such functions - which naturally form a semigroup under multiplication- allows to combinatorialize the algebraic geometry of toric variety. The whole game of toric geometry is to take algebro-geometric concepts, which typically are pretty hard to study, and show how for toric varieties such concept translate to combinatorial concepts. Typically combinatorics is notiationally intensive, but a lot more friendly for computations that algebra or geometry. Part of the reason that for us this picture may be a little elusive, is that for many of you, you are at the same time learning the concepts and combinatorializing them... but my hope is that by doing this you will have a large collection of examples where you can do computations with these concepts.

Points and Semi Group Homomorphisms

The correspondence between points of an affine algebraic variety and maximal ideals is classical, and it relies on the fact that on well enough behaved spaces you can find functions that “separate points (i.e. that vanish at a given point and not at another).

Points and semigroup homomorphism is just “evaulation: we are familiar with points corresponding to their coordinates - but notice that this is just the selection of a special class of regular functions that “pin down the point uniquely. If we evaluate the points on ALL monomial functions, we have enough information to recover the points uniquely. Note that if you choose a set of generators for S, you get an affine embedding of your toric variety and the values of those functions ARE the affine coordinates for that embedding. The “semi-group homomorphism point of view is non-committal, in that it does not want to pick any particular embedding. Do as an example the rational normal cone of degree 2 with two different embeddings.

Torus Action

Similarly, the way the torus T_N acts on X is constructed in such a way that if you choose an affine embedding, it becomes the restriction of the natural linear action of the torus in the ambient space. That saying, it follows that the unique fixed point in affine space is the origin, and therefore an affine toric variety can have at most one fixed point, and what one has to look at is whether the origin belongs to the closure of the maps ϕ_A. Again, the semi-group homomorphism allows us to be non-committal and just identify the fixed point with the very special homomorphism (when it exists) sending the vertex of the cone to 1 and everything else to 0.

Also it gives a tautological point of view on the action. Do one example (on whiteboard) .

Strong Convexity

The key point here is that strong convexity in one of the vector spaces corresponds to full dimensionality in the dual vector space.

Now if $\sigma^\vee$ is strongly convex, then the toric variety has a torus fixed point, corresponding to sending all points of S but the origin to 0. If $\sigma^\vee$ is not strongly convex, then the toric variety has a torus factor.

If σ is strongly convex, then $\sigma^v ee$ is full dimensional, implying that the torus T_N acts with a finite kernel on X. The quotient torus, which acts effectively on an open dense set of X, has the same dimension. If σ is not strongly convex, then $\sigma^v ee$ is not full dimensional, implying that the torus T_N acts with a torus kernel on X, and hence that the effective torus has lower dimension. (Not enough regular functions to give a variety of the appropriate dimension).

Saturation and Normality

The condition of S being saturated is definitely necessary for normality. Otherwise you have a monomial strictly rational function whose power is regular, and that violates regular functions being integrally closed. It is a bit surprising that it is also sufficient, but it follows from the fact that characters in S are a linear basis for $\mathbb{C}[S]$.

Do example of the cusp and the line with lattice $2\mathbb{Z}$.

Smoothness

Key point is that smoothness can be (1) reduced to strongly convex $\sigma^\vee$ (aka maximal dimensional cones in N), and then the analysis is done just by observing the tangent space at the fixed point. Here the dimension of the Zariski tangent space is easily seen to be the dimension of a Hilbert basis.

Then the important fact to remember about polyhedral geometry is that a strongly convex cone is generated by the ray generators. Then when such generators actually generate the lattice over the integers is precisely when the Hilbert basis of the dual cone is also given by the ray generators.

Toric morphisms The torus acting on a toric variety - or equivalently, the concept of monomial functions- being the additional structures crucial to toric varieties, it makes sense that we ask morphism of toric varieties to preserve such structure. This takes various incarnations, from the more geometric to the more combinatorial:

1. the morphism preserves the torus action. This is the concept of a group-equivariant function.

2. the morphism preserves monomial functions. This amounts to saying that the pullback of an element of S_2 should be an element of S_1. Obviously, when this happens, then you get a semi-group homomorphism (because the pullback of functions is an algebra homomorphism and we are here just focusing on the multiplicative structure).

3. the morphisms maps the torus into the torus as a group homomorphims.

4. the last point also shows that you should get naturally a group homomorphims on the one parameter subgroup lattices N. Finally one can show that such a homomorphism defines a map of affine toric varieties if and only if it sends σ_1 inside σ_2. Good examples to do: projection of the plane to the line. Map from $\mathbb{C}^2$ to the quotient by μ_2.

Faces of cones

A special case of toric morphisms are given by inclusions of the faces of σ. These define a toric morphism which is the inclusion of a Zariski open in the affine toric variety. This is because a face τ is defined by intersection with a hyperplane in N, which corresponds to a character m in M. It follows that S_τ is generated by S and $\pm m$. Which means that the corresponding coordinate ring is a localization and therefore the toric variety is an open subset. Point out importance in terms of patching together affine varieties to get something more interesting!

Blowup of the plane at a point Should work out carefully this example, both from a classical and a toric perspective.

Quotients by finite groups Should talk a little bit about quotients by finite groups, orbit spaces, and invariant functions. A sub lattice of a lattice can be thought as selecting some functions that are invariant under some group - what group? The kernel of the induced homomorphism of tori. Do A_n singularities as an example.

6 Sep 24: read 2.1, 2.2

General

- There were some requests to duplicate the relationship between classical algebraic geometry and toric algebraic geometry discussion in the projective case (if there's anything worth noting of difference). This may feed into the next point:
- There's some lack of familiarity/experience with projective spaces in general and functions on them. I think this could easily be a figure this out outside of class situation (the first sentence in the chapter provides a reference), but there were a few people who haven't dealt with them as much as the text might assume.

Section 2.0
Highlights:

1. Projective spaces in general, including homogeneous coordinates and rational functions.
2. Affine pieces of projective varieties.
3. The fact that projective varieties are unions of affine open sets.

Questions:

1. There's some concern about the section's initial claim that we haven't defined a toric variety yet.
2. Why is the affine variety defined by the homogeneous coordinate ring called the affine cone of V?
3. Can the grading of the homogeneous coordinate ring give information about the affine variety that it defines?
4. State ment 2.0.8 can generalize to any finite intersection, right?
5. Explicitly writing out the second half of 2.0.8 (i.e., as the set of $f \in C[V]$ s.t. blah blah blah).
6. When does Proposition 2.0.4 have a converse?
7. Parallels and differences between affine localization and projective localization.

8. Several people asked about weighted projective spaces, and why/when they're relevant.

9. There's a general lack of familiarity with the Segre embedding.

10. Lack of general familiarity with graded rings.

Crash course in Projective Geometry

1. Projective Space: compactification of affine space. Space of linear subspaces (or linear forms) of a vector space. In some sense it is a quotient space.

2. Issue: invariant functions are only the constants, hence very little information is contained in the ring of regular functions for the quotient.

3. Solutions:

   (a) rational functions/ sheaf of functions.

   (b) look at $\mathbb{C}^*$ equivariant functions.

4. The second point of view gives rise to the homogeneous coordinate ring. The key difference between the homogeneous coordinate ring and the regular ring of V is the grading.

5. Now the key point about equivariant functions is that they can be considered as equations, in the sense that their vanishing locus in V is a cone with vertex the origin (i other words it is ruled by linear subspaces). Hence the terminology of affine cones.

6. Ideals generated by equivariant functions (called homogeneous ideals), give rise to projective varieties. Again there is a graded ring (namely the quotient ring of the homogeneous coordinate ring of projective space by the ideal) that controls all functions of the variety, and there is an algebraic procedure to extract this information (the functor Proj).

7. Bijection between homogeneous ideals and projective varieties needs a new adjective - saturated on the ideal side, because the maximal ideal of the origin in V defines the empty set in projective space.

8. Affine pieces in projective space induce affine pieces on any projective variety embedded in a given projective space. Localization works exactly the same as before. The only thing is when you localize a homog coord ring you only want to take the degree 0 part of the localized ring - these are functions that are invariant on the orbit.

9. Weighted projective spaces arise when you change the representation on the arrival vector space (when constructing equivariant functions) to a non-diagonal representation. You may consider it these as parameter spaces for other types of curves (other than lines through the origin). From the algebraic point of view this amounts to altering the grading in the homogeneous coordinate ring. The relevance of WPS is that they provide simple examples of proper spaces with mild (quotient) singularities at some points (orbifolds) - but this is another story...

10. Segre embeddings. Should talk about maps between projective spaces (and projective varieties in general). Then get embedding of a product of projective spaces into a large projective space.

11. If time permits talk about line bundles.

7 Sep 26

Section 2.1
Highlights: Finding the torus of a projective toric variety. Proposition 2.1.4. Projective normality vs. affine normality. Affine pieces of a projective toric variety can give different semigroups. Proposition 2.1.8.
Questions:

1. Requests to see some examples in class, in particular: 2.1.3

2. At the end of 2.1.5, how does $x_1^2 - x_2$ vanish at that point?

3. 2.1.7: how to get $\mathbb{Z}'A$.

4. Compare/Contrast with $\mathbb{Z}'A$ (introduced on page 58) and $\mathbb{Z}A$.

5. Can we run through the proof (or idea thereof) of Proposition 2.1.6?

6. Why look at affine pieces of projective varieties? Is it to get more info on the affine side, or the projective side? Both? Neither?

7. Projective normality is mysterious. Should we just wait for chapter 3, like this section suggests?

8. In that vein, just a grand overview of the difference between projective and affine toric varieties. I think this fits in with the not quite comfortable with projective things comment above.

9. Are pages 59 and 60 saying that a projective toric variety comes from a collection of semigroups instead of a single one?

10. Is there an analogue to Prop 1.3.2 (about the uniqueness of a torus fixed point)?

Lecture

1. The torus of projective space is naturally a quotient torus of the torus of affine space. Correspondingly the character lattice is a sublattice in the character lattice of affine space, dictated by the condition that the sum of the coordinates is equal to 0. This amounts to saying that the RATIONAL monomial functions that are invariant under the linear diagonal scaling are those of homoegenous degree 0.

2. Since we have a natural projection function from affine $(n+1)$ dimensional space to projective space, any affine toric variety in $\mathbb{A}^{n+1}$ can define a projective variety just by projection. As far as I can tell though it is not common to take a general affine variety and just projectivize it (in the sense that it is for example not easy to determine the ideal of one in term of the other), unless it is contained already in some hyperplane...instead it is nice to study affine varieties that are cones through the origin already, because in a sense they already define varieties in projective space in a nonambiguous way.

3. Given a finite set $\mathcal{A}$ which defines an affine toric variety, how to recognize it is a cone? Various points of view lead to the same conclusion: $\mathcal{A}$ must be contained in an affine but not linear subspace of $M_{\mathbb{R}}$:

   - algebraically: think of the elements of $\mathcal{A}$ as the hyperplane sections of your projective toric variety, which therefore should count as "monomials of degree one. Then you want the relations defining the ideal to be homogeneous. Perform a change of lattice basis so that the affine space containing $\mathcal{A}$ is parallel to a coordinate hyperplane and notice how now a necessary condition to have a relation is that the sum of the coefficients is zero, which is precisely what we want.
   - geometrically: we want a one parameter subtorus to scale the image of $\phi_{\mathcal{A}}$ linearly. If there is an affine hyperplane containing $\mathcal{A}$, it is defined by an element u of the dual lattice. Then THAT one parameter subgroup of the torus scales the image linearly. This is extremely evident if you again perform a change of lattice basis so as to translate the hyperplane to a coordinate hyperplane.

4. Go through the sequence of tori, and associated character lattices. Tell them it will become clear in an example!

5. Affine charts for the toric variety are obtained via intersection with the affine charts of the ambient projective space. If you take the affine

hyperplane containing $\mathcal{A}$ to be parallel to a coordinate hyperplane, then you get a cone whose section with the translate of the coordinate hyperplane is a polytope. From the polytope you can easily read the cones of the affine charts of the projective variety, and also that you really only need the cones corresponding to the vertices of the polytope to cover all of the projective variety.

6. Work out slowly and carefullty the example of

$$(1,0,1),(2,2,1),(0,1,1),(1,1,1).$$

7. Projective normality: for now note only that it is an invariant of the projective embedding of a projective variety: i.e. a normal projective variety can be embedded in a non projectively normal way. The standard example is the degree 4 rational normal curve projected down to $\mathbb{P}^3$.

8 Sep 29: read 2.3,2.4

Section 2.2

Highlights: Polytopes, polytopes, and more polytopes. All the types of polytopes. Polytoooooopes.

Various incarnations of a polytope having enough lattice points:

- A normal polynomial P is such that all lattice points of all integral sums of P's come from lattice points of P.
- This is equivalent to the cone of P being normal in the sense that the associated affine variety is: this in particular means that the Hilbert basis for the semigroup is at height one, and the projective variety is projectively normal.
- If P is a polytope of dimension n, then $n - 1$P is certainly normal.
- Normal implies very ample. Very ampleness is a notion of line bundles, and here there is the combinatorial version: for every vertex m, the semigroup generated by the lattice points of P minus that vertex is saturated.
- Discuss first example of non-normal polytope (in 3d): $0, e_1, e_2, e_1 + e_2 + 3e_3$. Can show that it is not normal and that it is not ample.

Questions:

1. Can we go through an example of a polytope and its dual, including using the $<,>$ pairing notation? Why do we need 0 to be in the interior?

Do example of family of triangles with vertices $(0,2),(\pm 1,\alpha)$. Note what happens when α transitions across 0. Generalize the idea to answer why the condition on 0 being contained in the polytope.

2. Can we see an example of constructing C(P) for a lattice polytope P, and finding the Hilbert Basis thereof (or is it just important to know of the existence of such a basis?)? **Yes, we can, issue is we run out of dimensions to visualise things very quickly. Because any $2d$ polytope is normal, the Hilbert basis for the cone is given by the lattice points of the polytope at height one. Otherwise we can (and should) use the theorem that says that the Hilbert basis must be found between heights 1 and $n-1$. This reduces the problem to a finite problem.**

3. Can we see an example of calculating the dimension of a given projective variety?

4. If P = Conv(S), it would be nice to say that S is the set of vertices of P, but this is not true. If we make sure we choose S minimally, can we make the correspondence between S and the vertices of P a bijection? **Yes.**

5. Some coverage of Definition 2.2.10 (seems similar to smooth/regular in section 1.2, but on the other lattice). **Simplicity for a simplex is a criterion for normality. Note this is saying that the localization of the projective variety at any given vertex is smooth.**

6. A couple people were reminded of simplicial complexes in topology. Not really a question, but just an intuition a few are having.

7. Is there an intuitive idea for very ample? Is there a relationship to (very) ample line bundles? Discuss a bit.

8. End of page 65 seems to imply that $(P^{dual})^{dual}$ is not always P; is this only the case when we don't have 0 in the first place (and hence don't have a well-defined dual)? **It is definitely easy to show that the dual of a lattice polytope needs not be a lattice polytope...this because integer lengths become 1 over such lengths and typically this makes the dual polytope have a**

9. In example 2.2.6, we have that$P = P^{dual} = Conv(e_1, e_2, -e_1, -e_2)$, but the pictures of them are very different. Can this be explained a bit?

9 Oct 1

General

Questions:

Our analysis of affine toric varieties used the inherent torus action. When we're in projective space, this torus action becomes trivial. Does this mean we've lost something? Is it regained by looking at the affine pieces that come together to form the projective variety?

Section 2.3

Questions:

1. A few requests to work through Example 2.3.15 (the one with the Veronese embedding).If we were given a projective toric variety without an embedding:Would we want to embed, find the lattice polytope and use the normal fan for computations?Or will we later see a better way to work with abstract toric varieties of polytopes?When the book says that in chapter 3 we'll look at abstract varieties does it just mean varieties without an embedding?

2. Why do we care about reflexive polytopes? How does it relate to the varieties produced?

3. There was some confusion about the discussion included in and immediately following Example 2.3.16, and how the parametersaandb have importance some places but not others.

Lecture:

1. Discuss polytope being full dimensional versus inside a hyperplane. Do example of $\mathbb{P}^2$ with triangle either in a three dimensional or in a two dimensional vector space M. Point out how in one case there is a one dimensional torus subgroup acting trivially, and in the other case the two dimensional torus acts effectively. The map $\Phi_{\mathcal{A}}$ associated to a full dimensional polytope is "weird, but one can easily recover a map that allows you to read the homogeneous equation by putting the polytope at height one in a one dimensional bigger. This amounts to considering a one dimensional larger torus acting but with a one dimensional kernel.

2. The map associated to a given polytope is translation invariant. This also justifies, if a polytope lives in some affine subspace of M, to just restrict to that subspace (and the corresponding sublattice.

3. The map is NOT scaling invariant. Do example of various Veronese embeddings of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. However the normal fans are invariant, which tells us that the varieties are isomorphic. Discuss a bit embedded versus abstract.

4. There is a full on inclusion reversing correspondence between normal fan and polytope. Because there is also an inclusion reversing correspondence between cones in M, the correspondence between normal fan and the pieces of the variety is covariant again.

5. Examples: $\mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$, Hirzebruch surfaces.

10 Oct 3

Section 2.4

Questions:

1. A couple requests for example walkthroughs, including 2.4.5 and 2.5.6Clarification in what smoothness means in terms of polytopes, i.e., examples/counterexamples of smooth/not-smooth polytopes.

2. Is it always true that weighted projective space embeds as a subvariety in projective space (a la 2.4.6), and is the way presented in that example how it is always realized?

3. Throughout this section we assume our polytopes are full dimensional; what goes wrong if they are not?

Lecture:

1. A projective variety defined by a polytope is automatically normal just how it was an affine variety defined by a cone!

2. Projective normality of the embedding IS exactly the condition of normality of the polytope.

3. About smoothness of the polytope, the key point is that a cone σ is smooth iff the dual cone is smooth. And vertices of the polytope give the local picture for the dual cones of the affine charts (and finally notice that smoothness is a local property).

4. Talk a little bit more again about Segre embedding, and motivate why it makes sense that product of toric varieties corresponds to product of polytopes, and similarly to product of fans.

5. Examples: Do the mirror of $\mathbb{P}^2$ example. Do weighted projective planes.

11 Oct 6: read3.0, 3.1,3.2

Lecture on abstract varieties. Draw at all moments connections with manifold theory.

1. Structure sheaf. Sheaf of functions is a way to control the local structure of your algebraic variety. If variety has a finite cover by open affine subsets, can generate the whole sheaf of functions from the regular functions of the open affine pieces. This gives a variety the structure of a ringed space. Also, you can endow both open and closed subsets the structure of an algebraic variety in a natural way (this is called the reduced induced structure).

2. Local properties then are completely understood in terms of affine pieces. Smoothness, normality, are properties that can be checked "point by point, and therefore there is no particular new feature in the general theory.

3. Morphisms of algebraic varieties can be understood locally, as Zariski continuous functions, such that on the appropriate restrictions they induce morphisms of algebras of functions.

4. Isomorphisms can be defined categorically, as morphisms that admit inverses.

5. Once you do that, you can reverse the procedure, and start from affine charts and define an abstract algebraic variety by gluing them together. This essentially means two things: (1) identifying points (2) defining a sheaf of functions. Both things are done using isomorphisms between any pair of affine charts, that satisfy triple intersection conditions. NOTE: while conceptually this is natural and easy, in practical terms this is complicated and messy etc, and often we rely on either some global structure, or on some way to organize this information (see combinatorial structure) to actually be able to handle the complexity.

6. Separatedness is the AG geometric version of Haussdorff. Diagonal is closed. Show how this fails in affine line with double origin.

7. With separatedness is relatively easy to construct fiber products. Fiber products are handy because they give you intersection of open sets, fibers of morphisms, products of spaces. They satisfy a natural universal property. And the algebraic counterpart in the affine case is tensor product of rings.

12 Oct 8-10

Dhruv's giving special lectures on non archimedean geometry. Tropicalization of the moment map.

13 Oct 13: read 3.3,3.4

Section 3.0

Questions:

1. Couple of requests for examples, including 3.0.15.

   This is $\mathbb{P}^1$. That should be done together with the affine line with a double point.

2. If we rely heavily on the analogy with manifolds and atlases, are we sweeping too much under the rug?

   No, the analogy is really quite good. The one difference is that local models are more interesting in algebraic geometry, and also they are "larger

3. Notation confusion on equation 3.0.4 (page 96).

   $$(U_i)_{x_j/x_i}$$

4. General confusion with the idea of being separated, and how it is something we want.

   Discuss analogy with Hausdorffness

5. What algebraic properties do we get from separation?

   See next question

6. Proposition 3.0.18(b). Why is this statement true, via proof or nice example. X **separated. Then:**

   (a) the equalizer of $f, g : Y \to X$ is Zariski closed in Y.

   (b) the intersection of affines in X is affine.

7. Why does the fiber product exist on any variety that is not necessarily separated?

   I don't know.

8. The support of a fan seems totally different from the support of a function in analysis. Is there any common ground between these terms, or is it just an overuse of vocabulary?

in certain cases a fan can be thought as the image of a piecewise linear function, in which case the support is indeed the support of the function

9. Can something be said about the equalizer on page 95 (and notational confusion about the double arrows right above it). **equalizer is a scary word for a not so scary concept. Talk a bit about it.**

10. Direct limit = limit (as typically thought of)? **talk a little bit about direct limit**

14 Oct 15

Section 3.1

Questions:

1. Why should we believe Theorem 3.1.7?

   I don't exactly know why we should believe it, but here is the relevance of this theorem. Given any point, the union of all orbits whose closure contain that point is affine. This is (will) be very natural for toric varieties coming from a fan once we have the orbit cone correspondence. Sumihiro's theorem tell us that in fact all toric varieties come from a fan. Of course what I just said is just a "sanity check, since as an argument it is rather circular. It is unclear to me whether "separatedness should or should not be made part of the definition of a toric variety. But what is clear is that separatedness is connected to a toric variety coming from a fan.

2. Is the notion of separation trying to capture Hausdorff without worrying about the ambient space? **No. Just giving an algebro-geometric version of Hausdorffness**

3. Requests to see an example of the process of constructing X_Sigma from some fan Σ, as well as any number of random examples from the section. **Do $\mathbb{A}^2 - (0,0)$ and line bundles on $\mathbb{P}^1$.**

4. Why is compactness in the classical topology important (and hence in relation to Theorem 3.1.19, why do we care about a fan's completeness)? **Essentially existence of limits. And invariance under deformations.**

5. In Theorem 3.1.19, part (b), what happens if we don't have a simplicial fan? How ugly does it get? Manifolds are nice; are orbifolds nice? **Surfaces are always ok - since 2d cones are always simplicial. For uglier things, I don't know an answer in terms**

of singularity classification language, but one thing that can be said is that all cones can be subdivided into simplicial cones, which means that any singularity of a toric variety can be "partially resolved via a sequence of toric blow-ups to a bunch of quotient singularities.

15 Oct 17

Section 3.2

Questions:

1. Can we go through Lemma 3.2.4?I'm just generally confused in this section.
2. Discuss looking at limit points of one-parameter subgroups (including motivation and perhaps an example?).
3. Highlight the importance/use of Theorem 3.2.6 (Orbit-Cone Correspondence)
4. Why is it significant that the closure of an orbit is the same in the classical and Zariski topologies?
5. What is this Star() thing that comes up a few times? Is it actually star-shaped (I believe this is covered more completely in the next section).
6. Confusion about the distinguished point on page 116 (second bullet under Points and Semigroup Homomorphisms)
7. Direct quote: Given the 'Final Comments,' Chapters 5 and 14 sound bomb.

Lecture:

I will try to address all of the above questions, using as a running example the line bundle on $\mathbb{P}^1$ which we have been looking at.

1. Given a toric variety X_Σ, every cone of Σ gives an affine chart. If you choose a set of generators for the semigroup, then you also have affine coordinates. In this chart there are some "special points, corresponding to where the coordinates are 0 (number that is fixed by any multiplicative action of a torus) or 1 (which is the unit of the torus). In particular, for each σ, there is precisely one point where you have as many 0's as possible. This is called the distinguished point associated to the cone σ.

2. If you don't want to choose affine coordinates, the distinguished point can be identified with the semigroup homomorphism that assigns 1 to every dot in the largest sublattice that lives in σ^v, and 0 everywhere else. Note that the largest sublattice that lives in σ^v is the intersection of the lattice M with $\sigma^\perp$.

3. Now we seek to give a meaning to this distinguished point corresponding to σ. And it is the limit point as $t \to 0$ for the one parameter subgroups in σ. This gives a nice characterization for these distinguished points in terms of the action of the torus on the toric variety.

4. Next, when we have a group action on a space, we know that get a disjoint decomposition of the space into G orbits. So the first thing we notice is that distinguished points belongs to distinct orbits. If they live in the same affine chart, it is because they have "0s in different places. And open affine charts are T invariant (this was easily seen by thinking of the points of the affine chart as semigroup homomorphism and how the torus acted on the semigroup homomorphism simply by multiplication by the corresponding characters), so if they live in different charts then they certainly don't belong to the same orbit.

5. Now we notice that each pont of the toric variety belongs to an orbit of some distinguished point: a point must belong to some affine chart, and there it has some 0 and some non-zero coordinates: the non-zero coordinates might vary depending on choices of generators for S, but the 0 coordinates are independent of these choices. And now you can use the torus to make sure that the non-zero coordinates become 1. This is a distinguished point.

6. We have thus obtained a bijection between cones and orbits of X_Σ, by showing that both sets have bijections with the distinguished points that we analyzed.

7. Now we want to study the structure of orbits and orbit closures. Orbits are tori, and in fact we can specifically identify:

$$O_\sigma = Spec(\mathbb{C}[\sigma^\perp \cap M) = T_{N/<\sigma>}$$

   You can see this by looking at the torus action on points as semigroups homomorphisms: the one parameter subgroups of σ act trivially on lattice points in $\sigma^\perp$ by definition!

8. Finally we are concerned about orbit closures. Here we note that by thinking of orbit closures by adding limits of one parameter subgroups we are really thinking of closures in the complex analytic topology, which a priori could be different from the closures in the Zariski topology. However we see that the notions coincide. The way we prove

this is by first looking at the analytic closed orbits (which technically could be smaller) and see that they are Zariski closed.

9. Start with an orbit O_τ corresponding to a face τ: the affine charts that contain such orbit are all cones which contain τ as a face. Then you can see that you can use the torus action to flow to the distinguished point of that orbit, and then again the torus action to flow along the orbit corresponding to that cone. So the closure of $O(\tau)$ is the union of all orbits corresponding to cones that contain τ. That this is Zariski closed is the fact that the interiors of τ^v is an ideal and it defines an ideal sheaf on X_Σ by giving naturally an ideal in all affine charts that contain U_τ.

10. Finally we see that the closure of orbits are toric varieties themselves. Notice first off that the natural torus acting on the closure of an orbit O_τ is $T_{N/<\tau>}$. The dual of $N/ < \tau >$ is naturally identified with $\tau^\perp \subseteq M$, and for any cone σ containing τ, the dual to the cone $\sigma/ < \tau >$ is precisely $\sigma^v \cap \tau^\perp$. But these are precisely the cones giving rise to the limit points of the orbit O_τ, therefore showing that $\overline{O}_\tau$ is the toric variety corresponding to the fan given by the star of τ.

16 Oct 20 - 22 - 24 : read 4.0

Section 3.3

Questions:

1. There's some lack of familiarity with fiber bundles and their importance, and one request to do chapter 7 and study sheaves rigorously (I vote against this option).

2. Request to see Example 3.3.8 (I'm 98% sure we've already done this though).

3. Requests to walk through Example 3.3.16.

4. What is the Γ in Corollary 3.3.10? (A couple people asked about this.) **That's "global sections. In this case it means globally defined nowhere vanishing functions.**

5. Several requests to rigorously discuss blowups (and blowdowns).

6. Definition 3.3.17 is really complicated-looking, and relates directly to the Star construction, which we use the rest of the section. Requests for some help interpreting these two definitions.

7. How important is the Star construction beyond this chapter?

8. Are extra torus factors used for anything, or at they just ways of saying that we could be looking at a smaller lattice?

9. Can we get a visual on what splitting of fans means (page 133 onward). I think this could tie in nicely with any fiber bundle discussion.

Lecture:

I want todays lecture to be devoted to finite group quotients and their resolutions, and I wish to illustrate that through the resolution of the A_2 singularity. But a little bit of general discussion here:

1. If you start from a sequence of lattices:

$$0 \to N_1 \to N_2 \to G,$$

with G a finite group, then if you dualize you get:

$$0 \to M_2 \to M_1 \to \tilde{G},$$

But now hit this with $Hom(-, \mathbb{C}^*)$ and you get:

$$0 \to G \to T_{N_1} \to T_{N_2} \to 0$$

This shows that the G is the Kernel of the action of the torus of X_1 on the torus of X_2 and essentially exhibits X_2 as the quotient of X_1 by G

17 Oct 27: read 4.1

Discussed blow-ups. Did quite carefully the example of blowing up $\mathbb{C}^2$ at one torus fixed point, both classically and toric.

18 Oct 29

Section 3.4

Questions:

1. How easy is it (generally) to prove if something is universally closed? Is this a common feature? **Not.**

2. Example 3.4.5 mentions the projective extension theorem. What is this, and how important is it to personally know? **It's important to know what it means. I don't remember how to prove it.**

3. In Example 3.4.4, what is the projection of $V(xy-1)$ and why is it not closed? **Missing** 0

4. Does the log map in Theorem 3.4.1 relate to the valuation maps from 4.0 (and to Dhruv's valuation map)?

5. Are all varieties locally compact (classically)? What's this condition about? **lacking a convergent sequence. Can't be easily arranged in AG, so the condition is pretty pathological for us**

6. Super long proof of Theorem 3.4.11 was super long. Is there a shorter intuitive way to think about it (at an acceptable loss of rigor)? **Yes. The lecture will attempt to do that.**

7. Example of a proper morphism of a variety that is not necessarily toric? **Family of elliptic curves.**

8. Is it hard to come up with exact criteria for completness and properness for general varieities? It seems like it just needs some scheme theory not covered by this book explicitly. **Valuative criterion is the standard example. Playing with base changes is also important.**

Lecture:

1. Goal of the lecture is to understantd completeness and properness in algebraic geometry, and the translation of such concepts in the toric language. First off completeness wants to be an algebraic version of compactness - and properness wants to be a relative version of completeness.

2. Definition of completeness essentially means: you can't miss limit points. Except in general a space itself would not know if it is missing limit points unless you put the space into something else - but then there is the ambiguity of how do you know if you have put the space into something large enough to see the missing limit points?. So instead we take a different point of view and come up with the definition: X is complete iff for any Z the projection map $\pi : X \times Z \to Z$ is closed. In other word - allow for an arbitrary Z to tell X that it is missing limit points by having something with a limit in Z coming from something in $X \times Z$ without a limit.

3. The definition admittedly is a pain to check. The projective extension theorem checks it for projective space. The projective extension theorem (of which I don't remember the proof) essentially says that if you have a closed set inside $\mathbb{C}^n \times \mathbb{C}^m$ such that the projection to $\mathbb{C}^m$ is not closed, then the issue is that some of the points are "running off to infinity in the $\mathbb{C}^n$ direction, and therefore get filled in if you close

up $\mathbb{C}^n$ to $\mathbb{P}^n$. This checks the definition for $\mathbb{P}^n \times \mathbb{C}^m$. From here you extend to $\mathbb{P}^n\times$ any affine variety and then by gluing to any variety.

4. Once you have proven that projctive space is complete, you get that for example all projective varieties are complete.

5. The issue is that checking completeness with the definition is kind of a royal pain - but essentially in what is an analogy with the classical equivalence of compactness with sequential compactness, one can check completeness with one parameter families. This gives a great advantage in toric geometry, since we have a special class of one parameter families, namely the orbits of one parameter subgroups of the torus. It turns out that it suffices to check that every such orbit has a limit point as $t \to 0$ to check completeness for a toric variety. This, together with the interpretation of the support of the fan that we gave in chapter $3.1, 3.2$ gives immediately that a toric variety is complete if and only is the fan is complete.

6. Properness is a way to generalize completeness to a relative notion. A natural way would be to request that the fibers of a morphism be complete - but that would allow for fibers to disappear and we don't like that. Heuristically what we want to allow is for both X and Y to possibly have holes (the definition of a hole here is where compactness fails, so either a missing limit point that may very well be an infinite direction), but for all the holes of X to be stacked over the holes of Y. That's why in classical topology the notion of relative compactness is given by properness, and in algebraic geometry the notion is given by the requirement of a morphism being universally closed.

7. Important things to note about properness:
   (a) X complete iff the structure morphism to a point is proper.
   (b) Fibers of a proper morphism are complete.
   (c) Inverse images of complete subvarieties of Y are complete.
   (d) Any morhpism obtained from base change from a proper morphism is automatically proper

8. Again what saves the day is that properness can be checked via one dimensional subvarieties and in the case of toric varieties the one dimensional subvarieties given by orbits of one parameter subgroups suffice! Hence the toric criterion for properness is as simple as it is, that the support of the inverse image of the fan of Y is the support of the fan of X.

9. If there is time do a brief discussion of the valuative criterions for separatedness and properness in AG.

19 Oct 31

Section 4.0

Questions:

1. Distinction between the divisor of f and a principal divisor? **None. Principal divisors are divisors of rational functions.**

2. Some more concrete examples of sheaves of modules. **The examples to keep in mind are ideal sheaves, sheaves of sections of line or vector bundles, then there are the weirdos like skyscraper sheaves**

3. Lack of comfortability with sheaves, and why we care about the sheaf of O_X modules. **Because we care about functions and function-like things!**

4. Some additional intuition on O_X, D (fits in with the prior bullet point). **Hopefully addressed in lecture**

5. Request to talk about Example 4.0.6.

6. Is the divisor of f a discrete valuation as well? **Discrete valuations are associated to prime/irriducible divisors, and typically principal divisors are not prime**

7. Why is the divisor class group of a scheme an interesting invariant? Page 161 (referencing Hartshorne)

8. The quick reference to singularities seemed out of place (short and seemingly disconnected to the rest of the section). **At this point it is only useful to point out that Cartier and Weil are not the same concept.**

9. In Theorem 4.0.18, why is the notation $Cl(X) = 0$ instead of $Cl(X) = \{0\}$? **Huh?**

10. Will it be easier to find divisors on toric varieties from combinatorial information, as is true of pretty much everything else toric? **Absolutely, and the key idea for that is contained in Thm 4.0.20**

11. How does Corollary 4.0.15 fail for codimension 1 in X? **Think of the function y/x restricted to the nodal cubic $y^2 - x^2 = x^3$.**

Lecture:

Assume X is a **smooth** algebraic variety. Then there are three notions that are closely related:

1. Line bundles on X.

2. Sections of line bundles (regular, or meromorphic).

3. Divisors

Recall first all that we are generalizing a very natural idea. Functions on X naturally form a sheaf. Functions are naturally associated with their graphs, which in turn are sections of the trivial line bundle. Finally any function has a "locus of zeroes and of poles and associated to each irreducible component we have a valuation on functions that then allows us to have a notion of order zeroes or poles, and to define the divisor of a function (**principal divisor**). Since all divisors of functions arise as divisors of sections of the same bundle, we decide it is a dandy idea to consider them equivalent.

Now for a general line bundle. You get in a natural way sheaf of sections which, in turn, identifies the line bundle (in the sense that if two line bundles have isomorphic sheaves of sections, then they are isomorphic line bundles).

$$\begin{tikzcd}
L \text{ a line bundle over } X \arrow[r, leftrightarrow] & \text{Sheaf of Regular sections of } L \\
+ & \text{Subsheaf of } \mathcal{O}_{X \setminus supp(D)} \arrow[u] \\
\sigma \text{ a meromorphic section} \arrow[r] & D = div(\sigma) \arrow[u]
\end{tikzcd}$$

An important thing to note in understanding the above diagram is that the ratio of two sections of a given line bundle actually give a meromorphic function, i.e. a section of the trivial line bundle. In other words the equivalence relation generated by principal divisors identifies isomorphism classes of line bundles.

20 Nov 3: read 4.2, 4.3

Section 4.1

Questions:

1. Requests to walk through the bit in Example 4.1.4 where we get $\mathbb{Z}/d\mathbb{Z}$ from just the matrix A.

2. Requests for more examples just to be familiar with computing the class group.

3. Request to walk through the exact sequences in 4.1.3.

4. What's Smith Normal Form?

It's a diagonal expression corresponding to a linear transormation upon appropriate choices of bases. It is defined when the coefficients are a PID and apparently it is well tuned to describing the cokernel of the linear map.

5. Some confusion over what they mean in, for example, Example 4.1.4 when they write $0 \sim div(character)$.

   This refers to linear equivalence, and any principal divisor is equivalent to 0**.**

6. Why do we choose the map we do from $\mathbb{Z}^{n+1}$ to $\mathbb{Z}$ in example 4.1.6? Just to set up the kernel correctly?

   Yes. This essentially amounts to a choice of an isomorphism of the quotient with $\mathbb{Z}$**.**

7. Is the dimension of the class group of a toric variety always less than the dimension of the M? (This is probably just a lack of familiarity with short exact sequences?)

   These are abelian groups so the better notion is rank. The answer is no. There is no particular direct relation among the ranks.

8. Why do we care about the class group in general? Can we tell apart the difference between toric varieties by looking at their toric-invariant Weil divisors? **Yes. It is a pretty good invariant.**

Lecture. Things to still mention from 4.0**:**

- When the variety is singular, this equivalence breaks down. Hence the different notion of Weil and Cartier divisors.
- The prototypical example is the ruling of a cone. By itself it is Weil but not Cartier.
- Cartier divisors are again associated to line bundles. Weil...I don't know, but somehow the difference between the two is one way to measure how bad a singularity is.

From 4.1**:**

1. The class group of affine space and of the torus are 0. The book proves this for any spec of a UFD, but for affine space this is in a sense an "atomic notion. The way we define codimension one subvarieties IS as divisors of functions.

2. Also note the short exact sequence:

$$\oplus \mathbb{Z} D_i \to Cl(X) \to Cl(X \setminus \cup D_i) \to 0$$

For a toric variety if you take as D_i's all closures of codimension one orbits (aka all torus invariant divisors), then the last term of this sequence is the class group of a torus, which is 0 by the previous bullet point. Which means that the class group of a toric variety is generated by the torus invariant divisors, which are in turn naturally indexed by the rays of the fan.

3. In order to actually decribe the class group one has to figure out the kernel of the first map above. In order to do so, we notice that characters are naturally rational functions, and hence natually give us divisors. Further, since characters are maps from the torus to $\mathbb{C}^*$, they only vanish/diverge along the boundary of the toric variety. This means the divisor of a character is naturally supported on the boundary. The valuation of a character associated to any prime torus invariant divisors is the most natural thing it could possibly be: just the pairing of the character with the primitive generator of the ray corresponding to said divisor.

Examples:

1. $Cl(\mathbb{C}^n) \cong 0$
2. $Cl(\mathbb{P}^n) \cong \mathbb{Z}$
3. $Cl(\mathbb{P}^1 \times \mathbb{C}^*) \cong \mathbb{Z}$
4. $Cl(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \times \mathbb{Z}$
5. $Cl(Bl_p\mathbb{C}^2) \cong \mathbb{Z}$
6. $Cl(Bl_p\mathbb{P}^2) \cong \mathbb{Z} \times \mathbb{Z}$
7. $Cl(H_r) \cong \mathbb{Z} \times \mathbb{Z}$ Coker matrix:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & r & 0 & 1 \end{bmatrix}$$

21 Nov 5

Section 4.2

Questions:

1. In the affine case, is $Pic(X) = 0$ because every divisor is already principal and is therefore clearly locally principal?

   No. There are example of (singular) affine varieties - eg. the cone over a conic, where you have divisors which are not Cartier. The statement is that for an affine variety if a torus invariant divisor IS locally principal, then it is the divisor of a character and in particular it is principal.

2. Request to go over the relationship developed with the facet presentation of polytopes (starts on page 181ish) and the very satisfying negative signs.

3. Are support functions common computational tools in AG?

   I believe so but am not the best person to ask.

4. A handful of questions on support functions in general; I think this was a sketchy area? Questions include:

   - Having a hard time seeing how the support function encodes the Cartier data.
   - Request to go over Example 4.2.13. Why does showing all support functions are linear give that $Pic(X_\Sigma) = 0$?

5. Request to do an example of computing the Picard group.

6. Is there an intuitive interpretation for when the Picard group has torsion?

   Yes - essentially if the space admits an etale cyclic cover. Explain a bit in class.

7. The text tangentially mentions inverse limits. Are these something we should learn about now, or can they wait?

   What you should really keep in mind when thinking about inverse limits is that power series are inverse limits of the inverse systems of polynomials of any bounded degree (which you should think of as truncations of power series). I will try to highlight in the lecture the parallel with the Cartier data.

Lecture:

1. Every Cartier divisor is Weil - and torus invariant Cartier divisors still generate Pic so the goal is to identify what Weil divisors are Cartier.

2. One key fact is for an affine toric variety, any torus invariant Cartier divisor is the divisor of a character, and therefore principal. The idea

here is that any locally principal divisor must have in its support the smallest torus orbit - corresponding to the unique maximal dimensional cone. And this orbit IS in the closure of every torus invariant irreducible divisor, because the rays are in the closure of the big cone. This says that even if you were to need to restrict to an open set to get your divisor to be principal, you would not be throwing away any torus invariant divisor in doing so. After restriction you identify a character whose divisor is what you want, and then the above fact shows that the divisor of that character is actually globally what you started from.

3. Another useful tool is that if a toric variety has a maximal dimensional cone in its fan, then there can be no torsion in its Picard group. This is a linear algebra fact: if you start with a Cartier divisor D with kD principal, you know that there is some character that kD is the divisor of. Restricting to one big cone, identifies uniquely the character that gives kD as a k-multiple of a character m. Then it follows that D is the divisor of m.

4. Moment to pause and do some examples:

    (a) Do the affine A_1 singularity. Show that D_1 is not Cartier but $2D_1$ is and in fact it is also principal.

    (b) Remove the 2-d cone from the previous example. Show that D_1 is Cartier now but not principal, and $2D_1$ is principal.

    (c) Do the non simplicial cone given by the rays $(0, 0, 1), (1, 0, 1), (0, 1, 0), (1, 1, 1)$. Show that for a divisor to be Cartier (and therefore principal), the condition $a + d = b + c$ must hold, where a, b, c, d are the coefficients in front of the prime divisors corresponding to those rays, in that order.

    (d) Consider the toric variety given by rays $(0, 1), (1, 2), (-1, 0)$,with the two two dimensional cones between them.

5. As the above example helps us ponder, finding a character whose divisor is a given D amounts to solve the linear system $Ax = b$, where:

    (a) A is the matrix whose rows encode the coordinates of each of the rays.

    (b) x is the unknown vector of exponents of the character we are looking for.

    (c) b are the multeplicities of D along each of the torus invariant prime divisors corresponding to the rays.

Now we can observe:

- if we have a symplicial cone which is of maximum dimension, then A is a square matrix of nonzero (integral) determinant, so a unique solution always exists - BUT! - with rational coefficients. Which means that any Weil divisor is $\mathbb{Q}$-Cartier.
- in the above case, any b is granted to provide a solution to the above system iff the determinant of A is ± 1, which is precisely the condition for the cone to be smooth.
- If the cone is not simplicial, then the linear system is overdetermined, so there are nontrivial conditions on the entries of b that are needed to grant the existence of a solution.
- If the cone is not maximal dimensional, the orthogonal space to the linear span of the cone is the kernel of the linear system with $b = 0$. Given any b, if a solution exists, then any translate by that orthogonal is also a solution.

6. The punchline is now that a divisor is Cartier if it corresponds to the divisor of a character on each maximal cone of the fan of X. Such characters are defined modulo the orthogonal to the cone. When the cone are max-dimensional the characters are unique. A character for a cone determines the corresponding character on each face of the cone. Then the Picard group can be described as an inverse limit.

22 Nov 7

Lecture:

1. A divisor is Cartier if it principal when restricted to each cone of the fan it becomes principal. If such a choice exists, then it is defined up to the orthogonal space to the cone.

2. This gives rise to an inverse limit in a directed system indexed by cones of the fan.
$$CDiv_T(X_\Sigma) := \lim(inv)_\sigma \frac{M}{\sigma^\perp}$$

3. Each character gives a linear function on N, but the way we want to think about it is that for every cone you have a corresponding function. The compatibility condition given by the inverse limit implies that these linear functions glue together continuously, to give a continuous piecewise linear function on the fan. Therefore one can think of Pic as the quotient space of piecewise linear functions (integral on the lattice!) on the fan modulo global linear functions on the fan. This is the description in terms of support functions.

4. The convenience of support functions is that they are defined by their values on the primitive vectors on the rays. What one has to check is precisely that the assignment of an integer to each primitive ray vectors defines a linear function which is integral on all lattice point inside the cone spanned by the rays!

5. When you have a polytope it naturally defines a divisor D_P on the toric variety. This is because you can think of the facets of the polytope corresponding to a ray ρ of the fan to define hyperplanes which give you characters that have a fixed $(-a_\rho)$ order of vanishing along the divisor D_ρ. Then intersection of facets give characters that work for the cones spanned by the corresponding rays. Vertices of the polytope then correspond to the characters corresponding to maximal cones. This divisor is $\sum a_\rho D_\rho$ where the coefficients a_ρ are defined by

$$F_\rho \subseteq \{< \rho, m >= -a_\rho\}.$$

   Really nice consequences:

   (a) The natural divisor D_P is not principal because... the vertices of the polytope are distinct!

   (b) Translation of the polytope corresponds precisely to adding a principal divisor to D_P.

6. Illustrate this with the following examples:

7. Fan given by $(1, 0), (1, 2), (-1, 0)$

8. Polytope whose vertices are $(0, 0), (2, 0), (0, 3)$.

23 Nov 10: Read 5.0,5.1,5.2

Section 4.3

Questions:

1. At the end of the section they reference ample and very ample Cartier divisors. Are these so labeled because they will correspond to (resp.) ample and very ample line bundles?

   (Yes.)

2. When is a polyhedron a polytope?

   (Polytope = bounded polyhedron–page 190.)

3. Requests to go through some of the examples (e.g., 4.3.5, 4.3.6, and 4.3.7).

4. Related: why in Example 4.3.6 are we homogenizing?

   No particular reason right now. Just to describe global sections in homogeneous coordinates.

5. On page 189, why is it that

   $$div(\chi^m) + D \geq 0$$

   is equivalent to

   $$< m, u_p > +a_p \geq 0?$$

   (First line of The Polyhedron of a Divisor section.)

   The second equation is nothing but the first equation restricted to the individual coefficients of D.

Lecture:

1. Recall there is a line bundle carefully hidden here. A choice of a section gives rise to the divisor, which gives a choice for a distinguished local trivialization of the bundle. In this trivialization, regular sections of the line bundle are described as meromorphic functions f on X such that $D + div(f) \geq 0$.

2. In toric geometry, we start with a torus invariant divisor and we wish to describe to torus invariant sections. This gives rise to a polyhaedron (polytope if the fan is complete), such that the lattice points are the desired global sections.

3. Operations on line bundles correspond to operations on divisors correspond to operations on the polytope:

   - tensoring line bundles- adding divisors - contains Minkowski sum of polytopes.
   - Tensoring up a line bundle by itself - multiplication of a divisor by a positive integer - scaling the polytope.
   - Changing the distinguished section/trivialization - adding a principal divisor - translation by a character.

4. Examples:

   - $\mathbb{P}^2$ and a bunch of polytopes.
   - H_2

24 Nov 12

Section 5.0

Questions:

1. Why does the action on page 195 use the inverse (why this convention)? This doesn't seem to happen at all later.

   You want to have the pairing $Fun(X) \times X \to \mathbb{C}$ G equivariant (which in this case means G invariant).

2. What's up with part (d) of Prop 5.0.8? Why is this important now?

   It is a nice geometric characterization of a geometric quotient. Essentially what it says is precisely that to every point of the quotient corresponds precisely one orbit.

3. Reductive groups: what are these? How to think about them? (maximal connected solvable subgroup is a torus; is there a tangible meaning?)

   Depends on your goals. For me, a reductive group is a group for which Prop 5.0.9 holds, and I am glad to know that most of the groups I have ever had to take quotients by have been reductive.

4. Do finite groups count because they have $(\mathbb{C}^*)^0$?

   In that case the maximally connected subgroup is the identity.

5. How do they get us R^G being a finitely generated $\mathbb{C}$-algebra?

   I don't know. Good project maybe?

6. Why is $(\mathbb{C}^*)^n$ a subgroup in $GL(n, \mathbb{C})$? (diagonal matrices)

7. Is the torus of a toric variety a reductive group (viewed via its group action on the variety)? (yes, since tori are listed as an example of a reductive group)

   And you don't really need to think about the action on the variety. Reductiveness is a property of a group, not of a group action.

8. (On page 198) they state that everything works with the right type of group action. Do we have any understanding of what's going on for the wrong actions? Clearly things work differently, but do we have a handle on how bad things get?

9. How do you build a good categorical quotient from the definition? We can check it readily (apparently), but how to actually find them? (this comes out in a later section)

10. In what ways can a quotient not preserve separatedness? Example 5.0.15 is clear in that it isn't separated, but what's the idea behind the collapse of that structure?

    See lecture

Lecture:

1. Given that we like to describe spaces via their functions, it feels natural that functions on a quotient space are functions on the original space that are invariant under the group actions. Hence, given an affine variety $Spec(R)$ acted upon by a group G, the natural candidate for the quotient variety is $Spec(R^G)$.

2. This procedure has the feature that orbits get identified with their closures. And then if two (or infinitely many orbits) share some point in their closures, then the two orbits themselves get identified.

3. If X is not affine, but it is covered by G-invariant affine charts with their intersections themselves G-invariant, then you can do the quotient chart by chart and induce gluing data on the quotient just by looking at G invariant functions.

4. Some examples:
   - $\mathbb{C}^2//\mathbb{C}^* : \lambda(x, y) = (x, \lambda y) \mapsto R^G = \mathbb{C}[x]$.
   - $\mathbb{C}^2//\mathbb{C}^* : \lambda(x, y) = (\lambda x, \lambda y) \mapsto R^G = \mathbb{C}$.
   - $\mathbb{C}^2//\mathbb{C}^* : \lambda(x, y) = (\lambda x, \lambda^{-1} y) \mapsto R^G = \mathbb{C}[xy]$.
   - In the previous example, remove the origin.
   - $\mathbb{C}^3//\mathbb{C}^* : \lambda(x, y, z) = (\lambda x, \lambda y, \lambda^{-1} z) \mapsto R^G = \mathbb{C}[xz, yz]$. Observe orbits and show that if you throw away a Zariski closed subset you get $\mathcal{O}_{\mathbb{P}^1}(-1)$ and the map from this geometric quotient to the affinization is the blowup map.

At this point it is worth recalling some terminology:

Good categorical quotient $X//G$**:** when we are quotienting by a reductive group, this is what we obtain by taking (globally or locally) invariant functions as a model for the functions of the quotient space. Points of the quotients correspond to closed orbit. Any open orbit can have at most one closed orbit in its closure, and the open orbit gets "sucked into the point in the quotient corresponding to the closed orbit. Nice properties:

- Good categorical quotients satisfy the universal property of quotients.
- Closed G invariant sets in X get mapped to closed sets in the quotient.
- Closed, disjoint G invariant sets in X remain disjoint in the quotient.

Geometric quotient X/G: here there is a bijection between points in the quotient and G-orbits. In our current way to getting at quotients, this happens precisely when all orbits are closed. A lot of times this can be achieved by throwing away a Zariski closed G-invariant subvariety. This is the notion of an **almost geometric quotient**.

25 Nov 14

Section 5.1

Questions:

1. What exactly is (c) in Lemma 5.1.1 saying?

   it is explicitly describing the elements of G in coordinates

2. What is Lemma 5.1.10 saying?

   that any point in the boundary of an affine toric variety can be "reached as a limit of the orbit of a one parameter subgroup. This will be addressed further in the lecture.

3. What is a total transform and a proper transform? (these are both mentioned in Example 5.1.16, with the blowup of C^2 at a point [again])

   Total transform just means inverse image. Proper transform of a subvariety X means take the inverse image of X minus the locus that gets blown up, and then close that inverse image in the blowup

4. Relatedly, I think seeing the end part of 5.1.16–where they show how blowing up separates tangent directions–would be good.

   Exercise!!

5. In equation (5.1.10) (on page 216), they say the last line follows from choosing a trivial action. Why is this choice allowed, or is it just another part of the non-canonicalness? Relatedly, what is the relevance of the choice of N (i.e., how does it make it non-canonical)?

   It's unclear to me why the book addresses this point here. It is really no more relevant than the fact that a torus is non-canonically isomorphic to a product of $\mathbb{C}^*$'s. In other words,

any element of $SL(r, \mathbb{Z})$ gives rise to a nontrivial automorphism of a torus. And really in this case the book is pointing out that a quotient vector space by some N' is non-canonically isomorphic to some "complementary subspace of N' in V.

Lecture:

1. This quotient construction highly resembles (to me) the concept of a group presentation. A finitely generated group can be described as a quotient group of a free group by giving a normal subgroup of relations. Here we take the fan and make it "live in a vector space generated by the rays of Σ.

2. Why do we do that? Essentially because each cone gives us some affine coordinates for the affine patch corresponding to that cone, namely given by a Hilbert basis of the semigroup in the dual cone. Some of this coordinates are non-canonical, but some are canonical, and they are in bijection with the rays of σ. Therefore the vector space $\mathbb{R}^{\Sigma(1)}$ is a vector space where all these canonical affine coordinates for all patches coexist. It is now natural, because there is redundancy in these many coordinates, that we may expect (or hope) to obtain the space as a quotient of this big vector space, in such a way that each of the coordinate hyperplanes projects to the corresponding torus invariant divisor D_ρ.

3. Again, toric geometry tells us how to do that. First of all, we have naturally a toric variety given by the fan $\tilde{\Sigma}$. By construction, the fan $\tilde{\Sigma}$ consists of a collections of coordinate cones in the totally positive orthant. This implies that the toric variety $X_{\tilde{\Sigma}}$ is obtained by $\mathbb{C}^{\Sigma(1)}$ by removing a certain number of torus orbits, i.e. of coordinate subspaces. Next, we have a natural function on lattices obtained by sending e_i to ρ_i (this is the analogous to the morphims from the free group to G), that again, pretty much by construction, defines a fan function - and therefore a morphism of toric varieties.

4. That this morphism realizes X_Σ as a categorical quotient of $X_{\tilde{\Sigma}}$, is now verified cone by cone. For every affine map $U_{\tilde{\sigma}} \to U_\sigma$, the subgroup of the torus of $U_{\tilde{\sigma}}$ acting trivially on U_σ is naturally identified with the (tensor product with $\mathbb{C}^*$ of the) kernel of the morphism of lattices mentioned in the previous point. So X_Σ is obtained by gluing together categorical quotients of affine varieties by G via compatible patching data.

5. It is interesting to note that the quotient is actually a geometric quotient when Σ is a simplicial fan. This can be explained as follows. Remember that cones of a fan of a toric variety can be identified with

distinguished points in their corresponding torus orbits, which are the limit points of orbits of one parameter subgroups corresponding to the lattice points in the cone. If a cone $\tilde{\sigma}$ is not simplicial, then the linear map on lattices has a kernel, and there is bound to be some lattice point on some face of $\tilde{\sigma}$ that lies on a coset of the kernel which also contains some points in the interior of σ. These correspond to one parameter subgroups that flow to different distinguished points in $U_{\tilde{\sigma}}$ but such that when they act on U_σ they flow to the same point. This means that the inverse image of that distinguished point is a union of orbits, and not a single one.

6. Finally, toric geometry allows us to explicitly identify both the toric variety X_Σ and the group G we are quotienting by in terms of the combinatorics of the fan.

7. First, the toric variety $X_{\tilde{\Sigma}}$: it is obtained by throwing away some closure of orbits corresponding to the faces of the totally positive orthant which don't correspond to cones of Σ. These are coordinate subspaces that can be described in two different ways:

   (a) Cone by cone: for each affine chart, throw away the product of coordinate hyperplanes corresponding to the coordinates that you are not using. This makes those directions torus directions and allows them to be "quotiented away. Then the locus that needs to be thrown away globally is the intersection of these loci.

   (b) By exceptional collections: consider the minimal (wrt inclusion) faces among the faces of the positive orthant that do not belong to $\tilde{\Sigma}$. Any cone containing them is also not in $\tilde{\Sigma}$. The minimal faces correspond to coordinate linear subspaces that need to be thrown away. And globally you need to throw away the union of such subspaces.

8. The group that we are quotienting by is the torus obtained by tensoring by $\mathbb{C}^*$ the Kernel of the map of lattices induced by $e_i \to \rho_i$. By standard nonsense, this is isomorphic to the character group of the Class group of X_Σ.

Examples:

- $\mathbb{P}^2$
- $\mathbb{P}^1 \times \mathbb{P}^1$
- $(0,0,1),(1,0,1),(0,1,0),(0,0,1)$

26 Nov 17 read 5.3,5.4

Section 5.2

Questions:

1. Requests to go over Equation (5.2.1) (on page 219) and the lines after it (before Example 5.2.1).

   Will do in examples.

2. Not entirely sure I understand the purpose behind this section.

3. Comment: Now the grading on weighted projective space makes more senses. This helps solidify that notion.

4. In Example 5.2.5, how can we have a global equation which is not a local equation? This seems backwards.

   Equations coming from functions on the space belong in the piece graded by the identity. Local equations arise there after some localization. What the book is implying is that you can't find local equations that cover the whole divisor. We will see this in an example.

5. Request to talk about Example 5.2.11, and continued questions about proper transformations (see the questions in Section 5.1).

   See lecture

Lecture:

1. Purpose of this section is to generalize "global notions in projective geometry to arbitrary toric varieties.

2. The coordinate ring of $\mathbb{C}^{\Sigma(1)}$ inherits a natural grading from the Class group of X_Σ. Remember that elements of the class group are the characters of the group G that we are quotienting by. Then each graded pieces corresponds to the subrepresentation in the polynomial ring corresponding precisely to that character.

3. Recover the subvariety/homogeneous ideal correspondence. The irrelevant ideal plays the role of the maximal ideal in projective geometry: $V(I)$ is empty if and only if I contains some power of the irrelevant ideal. The correspondence is a bijection when the quotient is geometric, otherwise the same subvariety can be obtained by honestly distinct ideals.

4. When a cone is smooth, the corresponding coordinates can be identified with affine coordinates on the corresponding chart. This really

amounts to the fact that there is a unique way to choose torus elements that bring the remaining coordinates to 1, or alternatively, that after localization, the graded piece of the coordinate ring corresponding to the identity element is generated by the coordinates of the cone multiplied by monomials in the remaining coordinates.

Examples:

1. Weighted projective space
2. $\mathbb{P}^1 \times \mathbb{P}^1$
3. $BL_p\mathbb{C}^2$
4. $[\mathbb{C}^2/\mathbb{Z}^2]$
5. $(0,0,1),(1,0,1),(0,1,0),(0,0,1)$

27 Nov 19

28 Nov 21

Section 5.4

Questions:

1. What does Thus homogenization turns multiplication of sections in ordinary multiplication mean (near the end of page 234)? Ordinary being monomial multiplication with usual exponent laws?
2. Request to go over Example 5.4.5. Personally (Vance's perspective), it was really helpful to work through Example 5.4.5 and then try to check Lemma 5.4.6 for it.
3. Relatedly, what geometric significance does the ideal of vertex monomials have?
4. Maybe a bit of a review over what a radical ideal is would be good (there were a couple conceptual questions on it).
5. Request to see an example similar to (or identical to) Example 5.4.10.
6. What's the quintic mirror family mentioned at the very end of chapter 5? Why is it astonishing?

Section 5.4 is explained through the extended example of the second Hirzebruch surface. The KEY point is that characters always give invariant rational monomial functions on the toric variety. Given a divisor D, that gives a particular monomial, and shifting all invariant rational monomials coming from characters by THAT monomial gives the list of rational monomial functions that transform (under the action of the torus on the toric variety) as prescribed by the class group element of D. The characters in the polytope given by D (shifted) correspond precisely to the rational monomial that actually are regular!

SECOND SEMESTER

29 Jan 26

Presheaves vs. Sheaves Recall that a presheaf is a functor, what makes it a sheaf are the two additional gluing and restriction axioms. Given a presheaf, there is a canonical way to "sheafify it, i.e. to add/kill only the sections that are necessary for the sheaf actions to hold.

The key idea is that the local data for sheaves is often richer, and somehow essential data that cannot be forgotten. That's why when one wants to use any attribute (injective, surjective, etc) for a map of sheaves, the correct definition is to require such attributes NOT for every induced map on sections, but for every induced map on stalks. I.e., one makes sure that around every point one can restrict enough for the attribute to hold. Sometimes the attribute will also hold on every map of sections as a consequence (i.e. injectiveness), but sometimes it won't (e.g. surjectiveness). Examples:

$$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \stackrel{(x,y)}{\to} \mathcal{O}$$

is a surjective map of sheaves on $\mathbb{P}^1$.

$$\mathcal{O} \stackrel{exp(2\pi -)}{\to} \mathcal{O}^*$$

is a surjective map of sheaves on any smooth complex X. Similarly, given any algebraic operation ($\oplus$, $\otimes$, Im, Ker, Coker...) you can make it into a notion for sheaves in two steps. First you define the presheaf by applying the operation at each section level - then, if necessary, you sheafify the resulting presheaf. Look at first example to see why "Im is not automatically a sheaf.
Exact Sequences of Sheaves Example of exact sequences. Exponential sequence:

$$0 \to \mathbb{Z} \to \mathcal{O} \stackrel{exp(2\pi -)}{\to} \mathcal{O}^* \to 0$$

Sequence associated to a divisor:

$$0 \to \mathcal{I}_D = \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D$$

Every time you have a short exact sequence of sheaves you get a long exact sequence in cohomology. This is actually one of the axioms in defining sheaf

cohomology axiomatically. Then remember that H^0 is just global sections.
Locally Free Sheaves and Vector Bundles Stalks are direct limits of sections of sheaves such that the open sets contain a point and ordering is given by inclusion. Every time you see a statement about stalks, if you are uncomfortable with direct limits, you should replace it with " for every point x, there exists an affine open $U \ni x$ such that.... Talking about stalks is just a way to not committing to making any choice of such an affine. Vector bundles are locally products $U \times \mathbb{C}^n$. This means that locally their sheaves of sections are isomorphic to polynomial functions in n-variables with coefficients $\mathbb{C}[U]$. Another way to say the same exact thing is that $\mathcal{E}(U) \cong \mathcal{O}(U)^{\oplus n}$, and finally if one wants to get rid of U one just mentions stalks $\mathcal{E}_x \cong \mathcal{O}_x^n$.

$\mathcal{O}$ is the free object in the category of sheaves over an algebraic variety, hence the terminology locally free for sheaves of sections of vector bundles.
Invertible Sheaves, Line Bundles and Cartier Divisors We have already explored the correspondence between line bundles, Cartier divisors, and invertible sheaves. Recall it again, and remember to point out that operation in Pic corresponds naturally to tensoring of line bundles. On a much more basic level, we are talking about multiplication of sections.
Being Generated by Global Sections and Base Point Freeness Define the notions. Note that being generated by global sections is equivalent to admitting a surjection:

$$\mathcal{O}^n \to \mathcal{F}$$

Base point freeness is in general a weaker notion. The two notions coincide for line bundles however: what these concepts are "sheafifying are the notions of linear independence and vanishing of vectors. In a one dimensional vector space the two things are equivalent.
Pull-back of Sheaves Sheaves pull-back, even though it is kind of an awkward notion to define purely algebraically. When sheaves are sheaves of sections of line bundles or vector bundles, then things are much nicer to understand because such geometric objects naturally pull-back. Note for $f : X \to Y$ that we get for free a natural map on global sections:

$$H^0(Y, \mathcal{F}) \to H^0(X, f^*\mathcal{F}).$$

Show this map in the example of the function $w = z^d$ on $\mathbb{P}^1$.