

# Projects

Renzo's math 472

## 1 Compactifications

Compact spaces are very nice for many different reasons, most of which will only become evident much further along in your mathematical journey. Given a non compact space  $X$ , one would sometimes like to construct a homeomorphism from  $X$  to an open dense set of a compact space  $\tilde{X}$ . (Informally, this means, adding some points so that  $X$  "becomes compact".)  $\tilde{X}$  is called a **compactification** of  $X$ .

Assume throughout that  $X$  is:

- Hausdorff.
- **Locally compact:** this simply means that every point of  $x$  is contained in a compact neighborhood.

Define the **one-point compactification**  $\tilde{X}$  of  $X$ :

**as a set:**  $\tilde{X}$  has precisely one more point than the points of  $X$ . We suggestively call this point **infinity**.

$$\tilde{X} = X \cup \{\infty\}$$

**topology:** we discuss open sets according to whether they contain  $\infty$  or not:

- if  $\infty \notin U$ , then  $U$  is open in  $\tilde{X}$  if and only if it is open in  $X$ .
- if  $\infty \in U$ , then  $U$  is open in  $\tilde{X}$  if and only if its complement is compact.

**Problem 1.** Show that for any open set  $U$  in  $\tilde{X}$ ,  $U \cap X$  is open in  $X$ .

**Problem 2.** If  $X$  has a finite number of connected components each of which is not compact, then prove that:

- $X$  is homeomorphic to an open dense set in  $\tilde{X}$ .
- $\tilde{X}$  is compact.

- $\tilde{X}$  is connected.
- $\tilde{X}$  is Hausdorff.

What happens if you start with an  $X$  that is already compact?

**Problem 3.** Show that if  $X$  and  $Y$  are homeomorphic, so are their one point compactifications.

**Problem 4.** Show that the one point compactification of the plane (with euclidean topology) is (homeomorphic to) the sphere.

**Problem 5.** What is the one point compactification of:

- an open disc in the plane.
- a closed disc in the plane minus a point.
- an open interval of the real line.
- a bunch of open intervals of the real line.

**Problem 6.** Find an (or even better, some) example(s) of two spaces  $X$  and  $Y$  that are NOT homeomorphic, but such that their one point compactifications are homeomorphic.

## 2 The Projective Plane

The projective plane is a space parameterizing all lines through the origin in  $\mathbb{R}^3$ . This means that there is a natural bijection between the set of points of the projective plane and the set of lines through the origin in three dimensional euclidean space. The following are some useful ways to visualize the set of points in this space.

$$\mathbb{P}^2 = \frac{\mathbb{R}^3 \setminus \{0\}}{\{(X, Y, Z) = (\lambda X, \lambda Y, \lambda Z)\}} = \frac{Sphere}{\{P = -P\}}$$

There are three natural functions from the plane to  $\mathbb{P}^2$ :

$$\begin{aligned} \varphi_z : \mathbb{R}^2 &\longrightarrow \mathbb{P}^2 \\ (x, y) &\mapsto (x : y : 1) \end{aligned}$$

$$\begin{aligned} \varphi_y : \mathbb{R}^2 &\longrightarrow \mathbb{P}^2 \\ (x, z) &\mapsto (x : 1 : z) \end{aligned}$$

$$\begin{aligned} \varphi_x : \mathbb{R}^2 &\longrightarrow \mathbb{P}^2 \\ (y, z) &\mapsto (1 : y : z) \end{aligned}$$

**Problem 1.** Define a topology on  $\mathbb{P}^2$  as the finest topology that makes all three inclusion functions continuous.

Describe this topology, and show that the images  $\varphi_x(\mathbb{R}^2)$ ,  $\varphi_y(\mathbb{R}^2)$ ,  $\varphi_z(\mathbb{R}^2)$  become open dense sets of  $\mathbb{P}^2$

**Problem 2.** Define a natural map from the sphere to  $\mathbb{P}^2$ . Define a topology on  $\mathbb{P}^2$  as the finest topology that makes this map continuous. Show that this topology is the same as the topology defined in the previous problem.

**Problem 3.** Prove that  $\mathbb{P}^2$  is compact. Deduce that  $\mathbb{P}^2$  is a compactification of the plane, but NOT a one-point compactification.