Project 3: The Snake Lemma and the Long Exact Sequence in Homology

Renzo’s math 571

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This project develops a fundamental homological algebra tool for us: short exact sequences of complexes give rise to long exact sequences in homology. When this tool is applied in a geometric context, it allows to relate various homology groups.

The main statement we want to prove is the following.

**Theorem 1.** Let

\[ 0 \to A_\bullet \overset{f}{\to} B_\bullet \overset{g}{\to} C_\bullet \to 0 \quad (1) \]

be a short exact sequence of complexes of abelian groups (or \( R \)-modules, if you prefer, with \( R \) a commutative group). This induces a long exact sequence in homology:

\[ \ldots \to H_k(A_\bullet) \overset{f_*}{\to} H_k(B_\bullet) \overset{g_*}{\to} H_k(C_\bullet) \overset{\delta}{\to} H_{k-1}(A_\bullet) \to \ldots \quad (2) \]

The key ingredient for the proof is to define the connecting homomorphism \( \delta \).

**Problem 1** (Snake Lemma). Given (1), there is a canonical way to define a group homomorphism:

\[ \delta_k : H_k(C_\bullet) \to H_{k-1}(A_\bullet) \]

Given an element \( c \in \ker \partial \subseteq C_n \), obtain in a natural way an element \( a \in A_{n-1} \). This is not a function, but rather a correspondence, since you can obtain different \( a \)'s starting from the same \( c \). However show that this correspondence induces a well defined function at the level of homology. There are a few things to show here:

1. \( a \in \ker \partial \)
2. any two \( a \)'s that you may associate to the same \( c \) differ by a boundary, and therefore represent the same homology class.
3. if $c$ is a boundary, then the associated homology class $a$ is the zero class.

**Problem 2.** Having defined the connecting homomorphism, now you need to show that the sequence (2) is exact.

There are six things to check:

1. $\text{Im}(f_*) \subseteq \text{Ker}(g_*)(\text{aka } g_*f_* = 0)$.
2. $\text{Im}(f_*) \supseteq \text{Ker}(g_*)$.
3. $\text{Im}(g_*) \subseteq \text{Ker}(\delta_*)(\text{aka } \delta g_* = 0)$.
4. $\text{Im}(g_*) \supseteq \text{Ker}(\delta_*)$.
5. $\text{Im}(\delta) \subseteq \text{Ker}(f_*)(\text{aka } f_*\delta = 0)$.
6. $\text{Im}(\delta) \supseteq \text{Ker}(f_*)$. 