Homework 1

Renzo's math 570

1 Life with a metric

This homework set is supposed to present the basic concept of topology in the context of metric spaces. Even though this is a luxury that we can not always afford, I believe that here is where our intuition lies. Understanding these ideas should be the best motivation for the abstract (and seemingly kooky) general definitions we will make in class.

Write up at least 5 exercises, and number 8 is a must!

Definition 1. *A metric space is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a function (called* ***distance function*** *or* ***metric****) that satisfies the following three axioms:*

positivity *the distance between any two points is always non-negative. Further,*

$$d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2.$$

symmetry

$$d(x_1, x_2) = d(x_2, x_1).$$

triange inequality

$$d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3).$$

Exercise 1. *Which of the following is a metric function on $\mathbb{R}^n$?*

For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$

$$d(x, y) =$$

1. *(euclidean)* $\sqrt{\sum (x_i - y_i)^2}$
2. *(taxicab)* $\sum |x_i - y_i|$
3. $max_{i=1,\ldots,n}|x_i - y_i|$
4. $min_{i=1,\ldots,n}|x_i - y_i|$

5. *(discrete)*

$$\begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Definition 2. *An **open ball** centered at x_0 of radius r is the set of points:*

$$B_r(x_0) = \{x \in X s.t. d(x, x_0) < r\}.$$

*An **closed ball** centered at x_0 of radius r is the set of points:*

$$\overline{B_r(x_0)} = \{x \in X s.t. d(x, x_0) \leq r\}.$$

Definition 3. *Given a subset $U \subseteq X$, $x_0 \in U$ is an **interior point** of U if there exists some r such that the open ball of radius r centered at x_0 is completely contained in U. The **interior** of U is the set of interior points of U.*

*Given a subset $U \subseteq X$, $x_0 \in X$ (note, not necessarily in U) is an **limit point** for U if any open ball centered at x_0 has nonempty intersection with U. The closure of U is the set of limit points of U.*

Definition 4. *An **open set** is a subset $U \subseteq X$ that satisfies one of the following equivalent statements:*

1. *every point of U is an interior point.*

2. *U is equal to its interior.*

3. *U can be realized as a union of open balls.*

Exercise 2. *Show the equivalence of the above three statements.*

Definition 5. *A **closed set** is a subset $F \subseteq X$ that satisfies one of the following equivalent statements:*

1. *F contains all of his limit points.*

2. *U is equal to its closure.*

Exercise 3. *Show that the complement of an open set is closed and viceversa.*

Exercise 4. *What are limit points for subsets in a space with the discrete metric?*

Exercise 5. *Prove that a set is open/closed in the euclidean metric iff it is open/closed in the taxi-cab metric.*

Exercise 6. *What are open and closed sets in a space with the discrete metric?*

Exercise 7. *Assume X is a metric space. Prove the following:*

- ϕ, X *are open/closed.*
- *the arbitrary union of open sets is open.*
- *the finite intersection of open sets is open.*
- *the arbitrary intersection of closed sets is closed.*
- *the finite union of closed sets is closed.*

Definition 6. *A function between metric spaces*

$$f : (X, d_X) \to (Y, d_Y)$$

is **continuous** *if for any $x_0 \in X$, for any ϵ, there is a radius r (depending on both x_0 and ϵ) such that;*

$$f(B_r(x_0)) \subseteq B_\epsilon(f(x_0)).$$

Exercise 8. *Prove the following theorem:*

f is continuous if and only if, for any open set $U \subseteq Y$ the preimage $f^{-1}(U)$ is an open set in X.

Prove that the same staement holds if you replace open by closed.

Exercise 9. *Understand continuous functions when $X = Y = \mathbb{R}$ and d_X, d_Y are all 4 choices of two metrics between euclidean and discrete.*