Homework 2

Renzo's math 570

Here we go with the second homework set. Do at least 7 problems, and you must do the problem on the Zariski topology, and exercise 12!

1 Another weird topology

Let $X = \mathbb{R}$ and define a topology on it:

$$\tau_{hl} := \{(a, +\infty)\}_{a \in \mathbb{R}} \cup \{\phi, X\}$$

Exercise 1. *Verify that this indeed defines a topology on the real line.*

We call this the **half line** topology. This is yet another slightly kooky topology that we will use to train ourselves with this abstract nonsense.

Exercise 2. *What are the interior points, the limit points, the interior and the closure for the following sets:*

1. $V_1 = \{0\}$.
2. $V_2 = (0, 1)$.
3. $V_3 = (0, +\infty)$.
4. $V_4 = (-\infty, 0)$.

Exercise 3. *For what subsets of* $\mathbb{R}$ *is* 0 *a limit point?*

Exercise 4. *How does the half line topology compare with the euclidean topology? (is it coarser, finer, neither?)*

Now for some yoga about continuity of functions.

Exercise 5. *Decide if the following statements are true or false:*

1. *If* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function when both domain and codomain have the euclidean topology, then* f *is continuous when the domain has the half line topology and the codomain has the euclidean topology.*
2. *If* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function when both domain and codomain have the euclidean topology, then* f *is continuous when the domain has the euclidean topology and the codomain has the half line topology.*

Exercise 6. *Describe continuous function from $\mathbb{R} \to \mathbb{R}$ when:*

1. *the domain and the codomain have the half line topology.*
2. *the domain has the half line topology and the codomain has the euclidean topology.*

2 Dense Sets

Let X be a topological space. A set A is called **dense** if its closure is X.

Exercise 7. *Give some example of some dense sets in some familiar topologies. What sets are dense in the discrete and stupid topology?*

Remember the **finite complement topology** on a set X, where the open sets are ϕ, X and those sets such that their complement is finite.

Exercise 8. *What sets are dense in the finite complement topology? Make sure to discuss both the case where X is a finite or an infinite set.*

Exercise 9. *What sets are dense in the half line topology?*

Exercise 10 (The Zariski Topology)**.** *Consider the following topology on the plane: a set is defined to be closed if it the zero set of a system of polynomials in two variables, i.e. $C = \{(x, y) \text{ such that } f_1(x, y) = f_2(x, y) = \cdots = 0\}$.*

1. *Verify that this is indeed a topology on the plane.*
2. *Check that all open sets that are not ϕ are dense.*
3. *Define in the appropriate way the Zariski topology on the line. What topology is it? (i.e. have you seen it before?)*

3 Separating Stuff

A topological space X is called **Hausdorff** if, for any pair of points $x, y \in X$, there are two open sets O_x, O_y such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \phi$.

Exercise 11. *Show that if a space X has a **metrizable** topology, then it is Hausdorff.*

A topological space is called **T1** if for any pair of points $x, y \in X$, there is an open set O_x such that $x \in O_x$, $y \notin O_x$.

Exercise 12. *Show that being $T1$ is equivalent to the fact that points are closed sets.*

Exercise 13. *Is $\mathbb{R}$ with the half line topology a Hausdorff topological space? Is it T1?*

Exercise 14. *Can you think of a topological space that is T1 but not Hausdorff?*