Homework 2
Renzo’s math 570

Here we go with the second homework set. Do at least 7 problems, and you must do the problem on the Zariski topology, and exercise 12!

1 Another weird topology

Let $X = \mathbb{R}$ and define a topology on it:

$$\tau_{hl} := \{(a, +\infty)\}_{a \in \mathbb{R}} \cup \{\emptyset, X\}$$

Exercise 1. Verify that this indeed defines a topology on the real line.

We call this the half line topology. This is yet another slightly kooky topology that we will use to train ourselves with this abstract nonsense.

Exercise 2. What are the interior points, the limit points, the interior and the closure for the following sets:

1. $V_1 = \{0\}$.
2. $V_2 = (0, 1)$.
3. $V_3 = (0, +\infty)$.
4. $V_4 = (-\infty, 0)$.

Exercise 3. For what subsets of $\mathbb{R}$ is 0 a limit point?

Exercise 4. How does the half line topology compare with the euclidean topology? (is it coarser, finer, neither?)

Now for some yoga about continuity of functions.

Exercise 5. Decide if the following statements are true or false:

1. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function when both domain and codomain have the euclidean topology, then $f$ is continuous when the domain has the half line topology and the codomain has the euclidean topology.

2. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function when both domain and codomain have the euclidean topology, then $f$ is continuous when the domain has the euclidean topology and the codomain has the half line topology.
Exercise 6. Describe continuous function from $\mathbb{R} \rightarrow \mathbb{R}$ when:
1. the domain and the codomain have the half line topology.
2. the domain has the half line topology and the codomain has the euclidean topology.

2 Dense Sets

Let $X$ be a topological space. A set $A$ is called dense if its closure is $X$.

Exercise 7. Give some example of some dense sets in some familiar topologies. What sets are dense in the discrete and stupid topology?

Remember the finite complement topology on a set $X$, where the open sets are $\emptyset$, $X$ and those sets such that their complement is finite.

Exercise 8. What sets are dense in the finite complement topology? Make sure to discuss both the case where $X$ is a finite or an infinite set.

Exercise 9. What sets are dense in the half line topology?

Exercise 10 (The Zariski Topology). Consider the following topology on the plane: a set is defined to be closed if it the zero set of a system of polynomials in two variables, i.e. $C = \{(x, y) \text{ such that } f_1(x, y) = f_2(x, y) = \cdots = 0\}$.

1. Verify that this is indeed a topology on the plane.
2. Check that all open sets that are not $\emptyset$ are dense.
3. Define in the appropriate way the Zariski topology on the line. What topology is it? (i.e. have you seen it before?)

3 Separating Stuff

A topological space $X$ is called Hausdorff if, for any pair of points $x, y \in X$, there are two open sets $O_x, O_y$ such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \emptyset$.

Exercise 11. Show that if a space $X$ has a metrizable topology, then it is Hausdorff.

A topological space is called T1 if for any pair of points $x, y \in X$, there is an open set $O_x$ such that $x \in O_x$, $y \not\in O_x$.

Exercise 12. Show that being T1 is equivalent to the fact that points are closed sets.

Exercise 13. Is $\mathbb{R}$ with the half line topology a Hausdorff topological space? Is it T1?

Exercise 14. Can you think of a topological space that is T1 but not Hausdorff?