# Some Cool Theorems

Renzo's math 570

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We now look at some really cool applications of homotopy theory and the fundamental group. Each group should work on all four problems, then each group will be assigned one to write up. Have fun!

## 1 The Fundamental Theorem of Algebra

**Theorem 1** (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial $p(x) \in \mathbb{C}[x]$ has at least one root.*

**Hints:**

- assume that $p(x)$ has no roots. Then $p(x)$ defines a function from $\mathbb{C}$ to $\mathbb{C}^* \sim S^1$.
- consider a circle of very large radius in $\mathbb{C}$, and consider the image via $p$ of such circle as an element in the fundamental group of $\mathbb{C}^*$.
- observe that on the one hand such element is trivial, on the other it is equal to $deg(p)$, hence deducing that $p$ must be constant.

## 2 The Brouwer Fixed Point Theorem

**Theorem 2** (Brouwer)**.** *Any continuous map from the two dimensional unit disc to itself must have (at least) one fixed point.*

**Hints:**

- assume $f : D^2 \to D^2$ has no fixed points.
- use $f$ to construct a function $D^2 \to S^1$ that is the identity when restricted to the boundary of the disc.
- use functoriality of the fundamental group to deduce a contradiction.

## 3 The Fundamental Group of The Circle

**Theorem 3.**

$$\pi_1(S^1) = \mathbb{Z}$$

**Hints:**

- use a natural map $\mathbb{R} \to S^1$ to define a group homomorphism between $\mathbb{Z}$ and $\pi_1(S^1)$.
- show that such homomorphism is surjective by showing that any loop to the circle lifts to a path in $\mathbb{R}$.
- show that such homomorphism is injective by showing that an homotopy of loops to the circle lifts to a homotopy of paths to $\mathbb{R}$.

## 4 Borusk-Ulam Theorem

**Theorem 4** (Borusk-Ulam)**.** *Let*

$$f : S^2 \to \mathbb{R}^2$$

*Show that $f$ must identify two antipodal points.*

**Hints:**

- assume that $f(x) \neq f(-x)$ for every $x \in S^2$ and construct a map $g$ from $S^2 \to \mathbb{R}^2 \setminus \{0\} \sim S^1$.
- restrict $g$ to equator, and observe that this gives a trivial element in the fundamental group of $\mathbb{R}^2 \setminus \{0\}$.
- on the other hand observe that $g(x) = -g(-x)$. Show that a map $S^1 \to S^1$ with such a property is not null-homotopic, and deduce a contradiction. In order to do so observe that $g$ descends to the a $\mathbb{Z}/2\mathbb{Z}$ quotient of both the domain and range circles.