The Torus as the Square (via identification)

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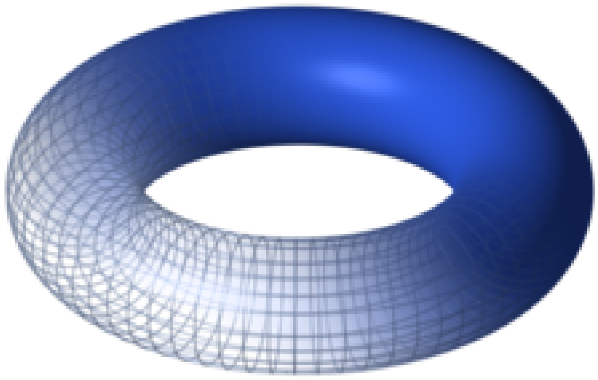
1 The Torus

The torus is the solid of revolution that one gets as a result of revolving a circle around an axis that is co-planar with and not touching the circle (Wikipedia!). What do we mean?

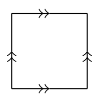


Ta da! Easy, right? Well, we want to be thinking of what happens if we were to be an ant that is walking on the surface of this doughnut. What sort of world do we live in? If we keep traveling in one direction, what sort of paths can we make that come back to our starting points? Obviously, we can 'see' how such a surface works, and how living on it would be (assuming we are constrained only to that surface – it's our entire universe!) because we're in a higher dimension.

We can see that the surface of the torus can be thought of as $S_1 x\ S_1$. Why is that? Any two angles will be able to tell us specifically where we are located at on our surface!

Proof : Mental Exercise (or someone else's problem ...).

Now, we want to show that a square can be thought of as the torus where we glue some points together, namely opposite edges with the arrows matching when we glue. Like so ...



Simple stuff, right? So, let's say $S = \{(x, y) : 0 \leq x, y \leq 1\}$. Now, let $\mathbb{T} = S_1\ x\ S_1$. Let the equivalance relationship $\sim$ be defined as the points such that $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$. We claim that $S\ /\sim\ \cong \mathbb{T}$.

Proof : It will suffice to show that we can find homeomorphic functions f, g such that $f : S\ /\sim\ \mapsto \mathbb{T}$ and $g : \mathbb{T} \mapsto S\ /\sim$ with $f \circ g = id_{S/\sim}$ and $g \circ f = id_{\mathbb{T}}$.

So, S_1 is the unit circle, so we can identify a point on the unit circle by a single angle θ, with $0 \leq \theta < 2\pi$, such that our point in $\mathbb{C}$ is $e^{i\theta}$. Now, let $f(x, y) = (x/2\pi, y/2\pi)$ and $g(\theta, \phi) = (2\theta\pi, 2\phi\pi)$.

Let's make sure that these functions are well-defined.

Let $(x_1, y_1) = (x_2, y_2)$. Then, $f(x_1, y_1) = (2x_1\pi, 2y_1\pi)$ and $f(x_2, y_2) = (2x_2\pi, 2y_2\pi) = f(x_1, x_2)$.

So, f is well-defined. Let's make sure g is, too! Take $(\theta_1, \phi_1) = (\theta_2, \phi_2)$. Now, $g(\theta_1, \phi_1) = (\theta_1/2\pi, \phi_1/2\pi)$ and $g(\theta_2, \phi_2) = (\theta_2/2\pi, \phi_2/2\pi) = (\theta_1/2\pi, \phi_1/2\pi) = g(\theta_1, \phi_1)$. g is also well-defined.

It's easy to see that these are inverses of one another and that the functions are continuous, so we have a homemeomorphism between the $S\ /\sim$ and $\mathbb{T}$.