# Making New Spaces from Old Ones - Part 2

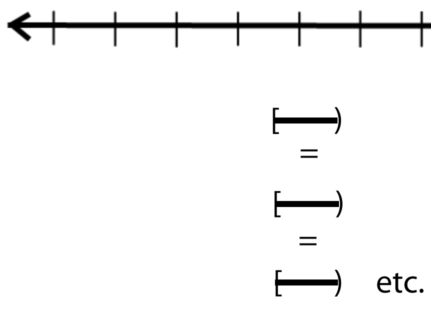
Renzo's math 570

## 1 The torus

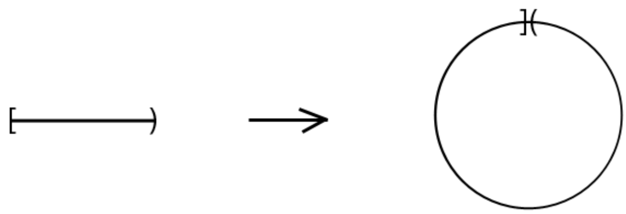
A **torus** is (informally) the topological space corresponding to the surface of a bagel, with topology induced by the euclidean topology.

**Problem 1.** *Realize the torus as a quotient space of the euclidean plane by an appropriate action of the group* $\mathbb{Z} \oplus \mathbb{Z}$

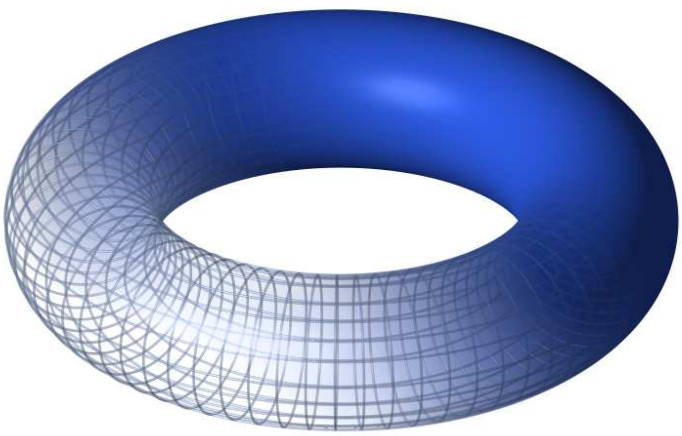
Let the group action $\mathbb{Z} \oplus \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $(m, n) \cdot (x, y) = (m + x, n + y)$, and consider the quotient space $\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$. Componentwise, each element $x \in \mathbb{R}$ is equal to $x+m$, for all $m \in \mathbb{Z}$ by the quotient operation, so in other words, our space is equivalent to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. Really, all we have done is slice up the real line, making cuts at every integer, and then equated the pieces:




Since this is just the two dimensional product space of $\mathbb{R}$ mod 1 (a.k.a. the quotient topology $Y/$ where $Y = [0, 1]$ and $= 0\ 1$), we could equivalently call it $S^1 \times S^1$, the unit circle cross the unit circle. Really, all we are doing is taking the unit interval $[0, 1)$ and connecting the ends to form a circle.

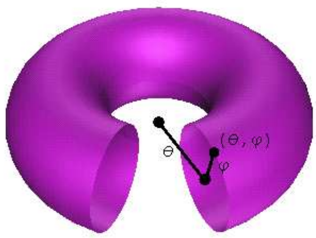


Now consider the torus.



Any point on the "shell" of the torus can be identified by its position with respect to the center of the torus (the donut hole), and its location on the circular outer rim - that is, the circle you get when you slice a thin piece out of a section of the torus.

Since both of these identifying factors are really just positions on two separate circles, the torus is equivalent to $S^1 \times S^1$, which is equal to $\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$ as explained above. To more formally see the equivalence between the torus ($S^1 \times S^1$) and $\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$, fix a coordinate system around the torus and take any point $(\theta, \varphi)$, where $\theta$ is the position of the point around the donut hole, and $\varphi$ is the position of the point on the cutaway circle. Now map this coordinate to $\mathbb{R}^2$ by taking $(\theta/2\pi, \varphi/2\pi)$. Since $0 \leq \theta, \varphi < 2\pi$, $0 \leq \theta/2\pi < 1$ and $0 \leq \varphi/2\pi < 1$, so $(\theta/2\pi, \varphi/2\pi) \in \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$. This mapping is 1-1 and onto by the basic properties of multiplication in $\mathbb{R}$, and similarly, we can take any element in $\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$ to a point on the torus by inverting the above map: $(2\pi x, 2\pi y) = (\theta, \varphi)$. Since these maps are continuous (they are just multiplication by a fixed number), $\mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$ is homeomorphic to the torus.




### 1.1 Cones

For any topological space $X$ we define a new topological space $CX$ called the **cone over** $X$. $CX$ is defined as an identification space of the product $X \times [0,1]$, where you identify all points of the form $(x,1)$ together.

**Problem 2.** *Draw some pictures and familiarize yourselves with this construction. Why is it called "cone"?*

For any topological space $X$ we define a new topological space $CX$ called the cone over $X$:

$$CX := X \times [0,1]/\sim, \tag{1}$$

where the equivalence relation $\sim$ defined by identifying all points of the form $(x,1)$ together. ( We can define this relation by making points $(x,1)$ and $(y,1)$ equivalent if $x - y \in X \times \{0\}$, then reflexivity, symmetry and transitivity hold.)

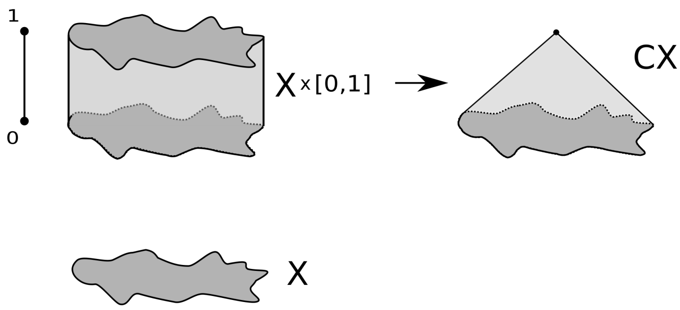



Figure 1: This construction is called "cone" as it has a "base" $X \times \{0\}$ and "vertex" $\{1\}$.

Some more simple examples of the cone are presented on Figure 3 and Figure 4.

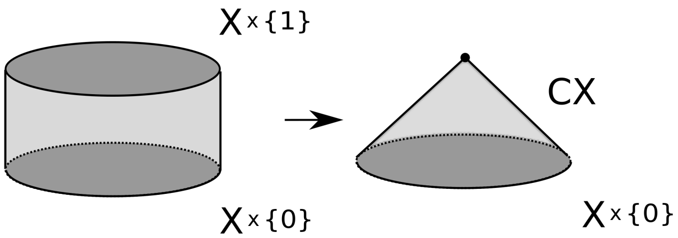



Figure 2: The "classical" example of $CX$ is the cone over a disk.

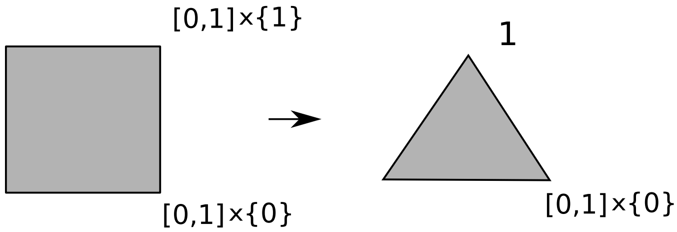



Figure 3: The cone over the interval $[0, 1]$ is a triangle.

**Problem 3.** *Show that the cone over a closed disc is homeomorphic to a closed three dimensional ball, but the cone over an open disc is not homeomorphic to an open ball.*

We can map the cone over a closed disc into $\mathbb{R}^3$ using scaled cylindrical coordinates. We begin with the function $r_{max}(h) = 1 - h$. We next parameterize the closed unit disc with polar coordinates $(r, \theta)$. We now map elements of the cone over the disc $(r, \theta, x)$ to $(r * r_{max}(x), \theta, x)$. Since $r_{max}(\theta) = 0$, all points of the form $(r, \theta, 1)$ map to the point $(0, 0, 1)$ (in Cartesian coordinates.)

The cone is of course homeomorphic to any rigid motion or scaling of this subset. So we now consider a cone with its base being the circle

$$\left\{x^2 + y^2 < \frac{3}{4}, z = -\frac{1}{2}\right\}$$

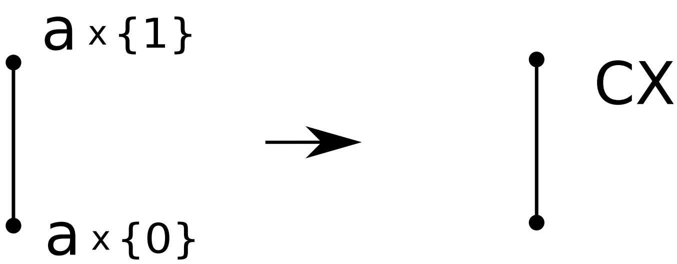



Figure 4: The cone over a point $\{a\}$ is the interval $[0, 1] \times \{a\}$.

If we describe this cone in terms of spherical coordinates, we find that its maximum value of $\rho$ is given by

$$\rho_{max} = \begin{cases} \frac{1}{\cos(\phi)+\sqrt{3}\sin(\phi)}, & \phi \in \left[0, \frac{2\pi}{3}\right] \\ -\frac{1}{2\cos(\phi)}, & \phi \in \left[\frac{2\pi}{3}, \pi\right] \end{cases}$$

We get these results by noting that for $\theta = 0$, the boundaries of the cone are given by $z = 1 - \sqrt{3}x$ and $z = -\frac{1}{2}$ and using the conversions $x = \rho\cos\theta\sin\phi, z = \rho\cos\phi$. See the attached figure.

This cone can now be mapped to the closed unit sphere by the map (in spherical coordinates)

$$(\rho, \theta, \phi) \to \left(\frac{\rho}{\rho_{max}(\phi)}, \theta, \phi\right)$$

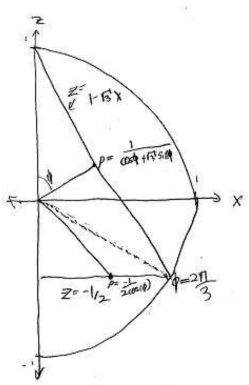



Figure 5:

This is a continuous and continously invertible function to the closed unit sphere, so the two sets are homeomorphic.

To see that the cone over an open disc is not homeomorphic to an open ball, we need the following lemma.

**Lemma 1.** *If $U \in \mathbb{R}^n$, $V \in \mathbb{R}^m$ for any $n, m$, $f : U \to V$ is a homeomorphism and $V$ is open in $\mathbb{R}^m$, then $U$ is open in $\mathbb{R}^n$.*

*Proof.* For every $v \in V$, there is an open ball $B_v$ centered at $v$ that is contained in $V$. $f^{-1}(B_v)$ is therefore open in $U$ and contained in $U$, and hence also open in $\mathbb{R}^n$ using the definition of the subspace topology. $U$ is equal to the union of all of these sets over $V$ since every point in $U$ must map to some point in $V$ and therefore be contained in $f^{-1}(B_v)$ for that $v$. Since $U$ is the union of infinitely many open sents in $R^n$, it is open in $\mathbb{R}^n$. □

Using the map from before, we can see that the cone over the open unit disc is homeomorhic to a set with a boundary point at the apex of the cone $(0, 0, 1)$. Since this set has a boundary point, it is not open and cannot be homemorphic to an open ball.

### 1.2 The Projective Plane

The projective plane is a space parameterizing all lines through the origin in $\mathbb{R}^3$. This means that there is a natural bijection between the set of points of the projective plane and the set of lines through the origin in three dimensional euclidean space. The following are some useful ways to visualize the set of points in this space.

$$\mathbb{P}^2 = \frac{\mathbb{R}^3 \smallsetminus \{0\}}{\{(X, Y, Z) = (\lambda X, \lambda Y, \lambda Z)\}} = \frac{Sphere}{\{P = -P\}}$$

There are three natural functions from the plane to $\mathbb{P}^2$:

$$\begin{array}{cccc} \varphi_z : & \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (x, y) & \mapsto & (x : y : 1) \end{array}$$

$$\begin{array}{cccc} \varphi_y : & \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (x, z) & \mapsto & (x : 1 : z) \end{array}$$

$$\begin{array}{cccc} \varphi_x : & \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (y, z) & \mapsto & (1 : y : z) \end{array}$$

**Problem 4.** *Induce a topology on $\mathbb{P}^2$ using our philosophy with respect to the three natural inclusion functions: the finest topology that makes all three inclusion functions continuous.*

*Describe this topology, and show that the images $\varphi_x(\mathbb{R}^2)$, $\varphi_y(\mathbb{R}^2)$, $\varphi_z(\mathbb{R}^2)$ become open dense sets of* $\mathbb{P}^2$

To start, we have three functions $\varphi_x$, $\varphi_y$ and $\varphi_z$, all mapping from $\mathbb{R}^2$ under the Euclidean topology to $\mathbb{P}^2$. We wish to use these to induce a topology on $\mathbb{P}^2$. Our philosophy dictates that we make a set in $\mathbb{P}^2$ open $\Leftrightarrow$ its preimages under $\varphi_x$, $\varphi_y$ and $\varphi_z$ are all open in $\mathbb{R}^2$. Note that this does, in fact, define a topology on $\mathbb{P}^2$ (i.e. we do not have to generate a topology from these open sets) since taking inverse images commutes with taking unions and intersections.

In order to describe this topology, it helps to view $\mathbb{P}^2$ as the set of all lines through the origin in $\mathbb{R}^3$. In other words, if a point $\alpha \in \mathbb{P}^2$ has projective coordinates $\alpha = (x : y : z)$ (with not all $x, y, z = 0$), we can view $\alpha$ as the point $(x, y, z)$ in $\mathbb{R}^3$ that happens to be identified with all the other points on the line through $(x, y, z)$ and the origin.

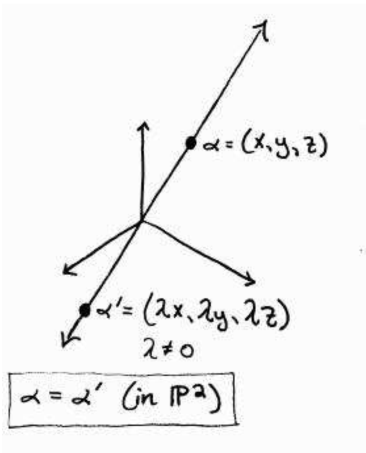



Figure 6: A point in $\mathbb{P}^2$

With this view, we see that $\varphi_x$, $\varphi_y$ and $\varphi_z$ each take $\mathbb{R}^2$ and "put it" on the planes $x = 1$, $y = 1$ and $z = 1$, respectively, in $\mathbb{R}^3$.

Note that if $\alpha = (x : y : z) \in \mathbb{P}^2$ is such that $z \neq 0$, then $\alpha = (\frac{1}{z}x : \frac{1}{z}y : \frac{1}{z}z) = (\frac{x}{z} : \frac{y}{z} : 1) \in \varphi_z(\mathbb{R}^2)$. To visualize this fact using our view above, first imagine a point $\alpha \in \mathbb{P}^2$ with $z = 0$. In $\mathbb{R}^3$, $\alpha$ is then sitting in the xy-plane, as is the line through the origin and $\alpha$. Now, if we bump $z$ even a little bit to something non-zero, then, in $\mathbb{R}^3$, $\alpha$ gets bumped out of the xy-plane and so does the line through the origin and $\alpha$. Since the line now has some slope in the z-direction, at some point the line will intersect with the plane z=1. But, since $\alpha$ is identified with all points on this line, this is the same as saying that $\alpha$ is in the image of $\varphi_z$.

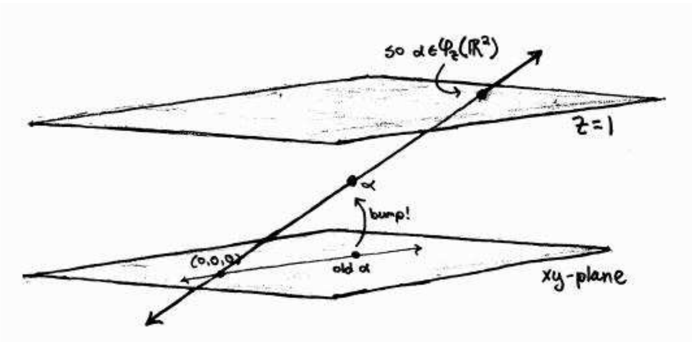



Figure 7: Any $\alpha$ with $z \neq 0$ intersects $\varphi_z(\mathbb{R}^2)$

Notice that a symmetric statement and arguement holds if either $x = 0$ or $y = 0$. For fun, note that if all of $x, y$ and $z$ are nonzero, then this means that the line in $\mathbb{R}^3$ through the origin and $\alpha$ has slope in each of the x, y and z-directions, and so it will intersect each of the planes $x = 1$, $y = 1$ and $z = 1$ somewhere (i.e. $\alpha$ is in the image of each of $\varphi_x$, $\varphi_y$ and $\varphi_z$).

So what do open sets in $\mathbb{P}^2$ look like? Well imagine we have one: some collection of lines through the origin in $\mathbb{R}^3$. Then by design, it's preimage under $\varphi_z$ is open in $\mathbb{R}^2$. But because $\varphi_z$ essentially plops $\mathbb{R}^2$ onto the plane $z = 1$, this is the same as saying that the slice of our open set that intersects the plane $z = 1$ is open in the Euclidean topology. So imagine that this slice looks just like an open ball in $\mathbb{R}^2$. So, we have an open ball sitting in the plane $z = 1$, right above the xy-plane ($z = 0$). In order to see what the open set in $\mathbb{P}^2$ that we started with looks like, we now just need to make lines connecting the origin to every point in the open ball sitting up in the plane $z = 1$. In doing this, we make a solid, infinite cone (excluding its surface) lying in $\mathbb{R}^3$. (See Figure 8)

Since slicing such an infinite cone using $z = 1$, $x = 1$ or $y = 1$ will give open sets of $\mathbb{R}^2$, we know that this cone is in fact open in $\mathbb{P}^2$. Now, since open sets in $\mathbb{R}^2$ don't just look like open balls, we see that open sets in $\mathbb{P}^2$ look like crazy, weird infinite cones in $\mathbb{R}^3$ (since these cones can have crazy-weird shapes and there can be a lot of cones shooting out from the origin).

We can use this view of open sets to show that:

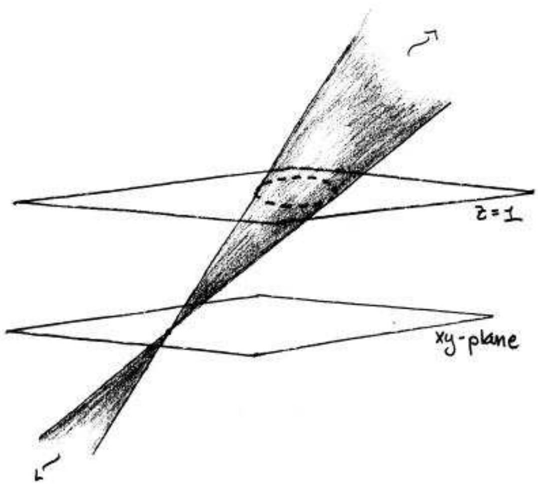



Figure 8: Filling out a cone

1. $\varphi_x(\mathbb{R}^2)$, $\varphi_y(\mathbb{R}^2)$ and $\varphi_z(\mathbb{R}^2)$ are open in $\mathbb{P}^2$.

   Let $\alpha \in \varphi_z(\mathbb{R}^2)$. Then we can view $\alpha$ as sitting in the plane $z = 1$. Now, take an open (Euclidean) ball around $\alpha$ in the plane and fill it out using lines through the origin as before. The cone we get is an open set of $\mathbb{P}^2$, and since every one of the lines we drew intersects the plane $z = 1$, the open set is lying inside $\varphi_z(\mathbb{R}^2)$. Thus $\varphi_z(\mathbb{R}^2)$ is open in $\mathbb{P}^2$, and similarly, so are $\varphi_x(\mathbb{R}^2)$ and $\varphi_y(\mathbb{R}^2)$.

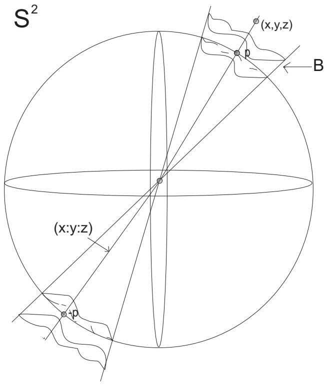
2. $\varphi_z(\mathbb{R}^2)$ (and thus $\varphi_x(\mathbb{R}^2)$ and $\varphi_y(\mathbb{R}^2)$) is dense in $\mathbb{P}^2$.

   If $\alpha \in \varphi_z(\mathbb{R}^2)$, we're happy. If $\alpha \notin \varphi_z(\mathbb{R}^2)$, then $z = 0$ and so $\alpha$ is sitting in the xy-plane. Now imagine some crazy cone encompassing the line through the origin and $\alpha$. Because the piece of our crazy cone that encompasses $\alpha$ is itself a *cone* (minus its surface), there will be lines inside of that piece that have some z-slope. Thus, these lines will intersect somewhere with the plane $z = 1$, and so the intersection of this cone (and thus the crazy cone we started with) and the plane $z = 1$ (i.e. $\varphi_z(\mathbb{R}^2)$), is nonempty.

**Problem 5.** *Define a natural map from the sphere to $\mathbb{P}^2$. Define a topology on $\mathbb{P}^2$ using our philosophy with respect to this map: the finest topology that makes this map continuous. Show that this topology is the same as the topology defined in the previous problem.*

A natural map from the sphere, $S^2$, to $\mathbb{P}^2$ would be to send a point on the sphere $(x, y, z) : x^2+y^2+z^2 = 1$ to the equivalence class $(x : y : z)$, which one might also view as the line through the origin which passes through$(x, y, z)$

on the surface of the sphere. We wish to define a topology on $\mathbb{P}^2$ which is the finest topology that makes this map continuous. Pre-images of sets in $\mathbb{P}^2$ under this map may be envisioned as follows: let $B$ be a set in $\mathbb{P}^2$. We may think of this set as a set of lines through the origin. A point $(x : y : z) \in B$ will correspond to the line through the origin that passes through $(x, y, z)$, and then intersects with the surface of the sphere at a point $p$ and also at $-p$. The collection of these points on the surface of the sphere will be the inverse images of the set $B$ in $S^2$. We define the open sets of our topology to be sets whose pull-back (projection) to the sphere are the intersection of infinite, open cones with the surface of the sphere. So, our topology is generated by a basis of analogous infinite open cones whose vertex is at the origin in $\mathbb{P}^2$.




We wish to show that this is the same topology as generated in problem 18, $\tau_{18}$. In Problem 18, the topology is defined via maps to charts. The topology defined is analogous to using open cones in $\mathbb{R}^3$ which pass through the three planes $x = 1$, $y = 1$, $z = 1$, interesecting as open sets, which pull back to open sets in the charts. (See the write up for 18 for a much better description). In Problem 19, the topology is defined via projection onto the surface of the sphere. The topology may be generated by a basis consisting of "open cones" in $\mathbb{P}^2$. These project to open sets on the surface of the sphere (think intersection with the sphere). In both topologies, any open set in $\mathbb{P}^2$ may be written as the union/intersection of open cones. Thus, as both use the same basis to generate, they are the same topology.

**Problem 6.** *Realize the projective plane as a quotient of some space via the action of the cyclic group $\mathbb{Z}/2\mathbb{Z}$.*

The space we will use to get $\mathbb{P}^2$ as a quotient space via the action of $\mathbb{Z}/2\mathbb{Z}$ is the sphere $S^2$.

Visual Approach Overlay a sphere centered at the origin and the projective plane $\mathbb{P}^2$ (thought of as the set of all lines through the origin). Next, notice that each line through the origin hits the sphere at two points, say, $(x, y, z)$ and $(-x, -y, -z)$. Thus, if we could quotient out by "minus" then we would have each point of the sphere (mod "minus") corresponding to a line of $\mathbb{P}^2$.

A reminder: if you would like to think of $\mathbb{P}^2$ as points and lines, you must realize for $\mathbb{P}^2$ in this setting, points are lines through the origin and lines are all lines on a plane intersecting the origin.

**Axiomatic Approach** Using the above notion of identifying antipodal points we can check our idea against the axioms of a projective plane.

1. A line contains at least two distinct points.
2. Any two distinct lines meet in a unique point.
3. There exists at least four points of which no three are collinear.

What are the points and lines in terms of the sphere? Points are a pair of antipodal points on the sphere. Lines are great circles of the sphere. Now check the axioms.

1. A great circle contains an infinite number of antipodal pairs.
2. Any two great circles meet in a unique antipodal pair.
3. There are at least four antipodal pairs of which no three are collinear.

Thus, we can see that using antipodal points will get us to where we want to go on this problem.

**$\mathbb{Z}/2\mathbb{Z}$ Action** Here we will think of $\mathbb{Z}/2\mathbb{Z}$ as a multiplicative group.

Define our group action

$$\varphi : \mathbb{Z}/2\mathbb{Z} \times S^2 \to S^2$$

by $\varphi(1, (x, y, z)) \mapsto (x, y, z)$ and $\varphi(-1, (x, y, z)) \mapsto (-x, -y, -z)$. These clearly satisfy the necessary conditions of continuity, identity element acting trivially, composition.

We can now define an equivalence relation on $S^2$ to be

$$P = (a, b, c) \sim (d, e, f) = Q$$

if $a = -d$, $b = -e$, and $c = -f$ or, for ease of notation, $P = -Q$. (In general terms, there exists a group element taking one point to the other.) Antipodal points are therefore orbits and the topological space $\mathbb{P}^2$ obtained by inducing a topology on the quotient set, $S^2/\sim$, is called the orbit space.