

# QUIVER REPRESENTATIONS IN TORIC GEOMETRY

ALASTAIR CRAW

*To Miles Reid on his 60th birthday*

ABSTRACT. This article is based on my lecture notes from summer schools at the Universities of Utah (June 2007) and Warwick (September 2007). We provide an introduction to explicit methods in the study of moduli spaces of quiver representations and derived categories arising in toric geometry. The first main goal is to present the noncommutative geometric approach to semiprojective toric varieties via quivers. To achieve this, we use geometric invariant theory to construct both semiprojective toric varieties and moduli spaces of quiver representations. The second main goal builds on the first by presenting an introduction to explicit methods in derived categories of coherent sheaves in toric geometry. We recall the notion of tilting bundles with examples, and describe the McKay correspondence as a derived equivalence in some detail following Bridgeland, King and Reid. We also describe extensions of their result beyond the  $G$ -Hilbert scheme to other fine moduli spaces of bound quiver representations.

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## 1. INTRODUCTION

This article provides an introduction to explicit methods in the study of moduli spaces of quiver representations and derived categories arising in toric geometry. Our guiding principle is to understand when a fine moduli space description can be given for a class of toric varieties so that the resulting universal family gives insight into geometric properties of the toric variety itself. For instance, if the universal family defines a collection of line bundles on the variety, do the classes of these bundles freely generate the Picard group of line bundles, the Grothendieck group of vector bundles, or the bounded derived category of coherent sheaves? We provide the necessary background from toric geometry, representations of quivers and derived categories in order to study this question. We also go beyond the toric category in several places, notably in presenting a detailed description of the work of Bridgeland, King and Reid that establishes the McKay correspondence as an equivalence of derived categories in dimension three.

To begin, we provide a brief and novel introduction to toric varieties by describing the class of semiprojective toric varieties. A brief tour of geometric invariant theory is given in Section 2, where results for diagonalisable group actions on affine toric varieties are listed, including a full description of finite group quotients and the geometric interpretation of  $\text{Proj}$  of a  $\mathbb{Z}$ -graded ring. This paves the way for the description of semiprojective toric varieties in Section 3. We do not assume that toric varieties are normal, but we do restrict attention only to toric varieties that are projective over an affine toric variety. In the normal case, we describe the relation to polyhedra and fans, and describe the canonical GIT description of Cox, giving worked examples for the first Hirzebruch surface  $\mathbb{F}_1$  and the minimal resolution of the  $A_2$ -singularity.

The second part of the article introduces the construction of normal semiprojective toric varieties as fine moduli spaces of bound quiver representations by Craw–Smith [31]. Section 4 presents the abelian categories of quiver representations and bound quiver representations with examples, before recalling the construction of fine moduli spaces of quiver representations by King [50]. We restrict the dimension vector to ensure that the resulting moduli space is toric, or at worst (in the case of bound quivers) where each irreducible component is a semiprojective toric variety. Section 5 uses these moduli spaces to extend the classical notion of the linear series  $|L|$  of a single basepoint-free line bundle  $L$ , obtaining the multilinear series  $|\mathcal{L}|$  of a collection of basepoint-free line bundles  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$  on a normal semiprojective toric variety  $X$ . Under nice conditions, the image under the natural morphism  $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$  is shown to represent a functor, and hence carries a tautological bundle that encodes a description of  $X$  as a fine moduli space of bound quiver representations.

In the final part, comprising Sections 6–8, we use quiver representations to study derived categories in toric geometry. Fine moduli space constructions and derived category questions and have been linked closely since the pioneering work on abelian varieties by Mukai [56]. This relationship blossomed with the development of Fourier–Mukai transforms by Bondal, Orlov, Bridgeland and others, culminating in the striking moduli space construction of threefold flops by Bridgeland [14]. Huybrechts [43] provides an excellent introduction to Fourier–Mukai transforms but, as Huybrechts remarks in his preface: “*questions related to representation theory of, e.g., quivers, or to modules over (noncommutative) rings, have not been touched upon.*”. The final part of this article introduces some of these ideas, focusing largely (but not entirely) on the toric case. Section 6 introduces the link between derived categories and moduli space descriptions for semiprojective toric varieties that goes back to King’s work on tilting bundles in toric geometry [51]. We present several of King’s results, and describe briefly the extension to smooth toric DM stacks by Kawamata [49] and, more recently, Borisov–Hua [12]. Section 7 provides a detailed look at the proof of the celebrated theorem of Bridgeland–King–Reid [16] that establishes the derived McKay correspondence for the  $G$ -Hilbert scheme in dimension three. Here we gain nothing by restricting to the toric case, so we do not do so. In Section 8, we conclude by moving beyond the  $G$ -Hilbert scheme to other moduli spaces of quiver representations for which the universal family determines explicitly the bounded derived category of coherent sheaves, describing briefly the work of Craw–Ishii [26].

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## 2. GIT FOR DIAGONALISABLE GROUP ACTIONS

This section reviews some of the basic elements in the construction of orbit spaces in algebraic geometry. Our primary examples are provided by quotients of affine space by a finite group, and the projective spectrum of a graded ring. Throughout these notes we write  $\mathbb{k}$  for an algebraically closed field of characteristic zero.

**2.1. Affine quotients by finite groups.** Let  $R$  be a finitely generated integral  $\mathbb{k}$ -algebra. For any set of  $\mathbb{k}$ -algebra generators  $f_1, \dots, f_n \in R$ , write  $\psi: \mathbb{k}[x_1, \dots, x_n] \rightarrow R$  for the surjective  $\mathbb{k}$ -algebra homomorphism obtained by sending  $x_i$  to  $f_i$  for  $1 \leq i \leq n$ . The prime ideal  $\text{Ker}(\psi)$  that records the relations between the generators  $f_1, \dots, f_n$  cuts out the affine variety

$$\text{Spec}(R) = \{p \in \mathbb{A}_{\mathbb{k}}^n : f(p) = 0 \text{ for all } f \in \text{Ker}(\psi)\}$$

with coordinate ring  $R$ . An affine variety need not be presented as a subvariety of affine space, though one may always choose an embedding. A different choice of  $\mathbb{k}$ -algebra generators for  $R$  provides an alternative embedding of  $X$  into affine space.

**Example 2.1.** Let  $V$  be a  $\mathbb{k}$ -vector space of dimension  $n$  with dual space  $V^*$ . The symmetric algebra  $R = \bigoplus_{k \geq 0} \text{Sym}^k(V^*)$  is isomorphic to the polynomial ring in  $n$  variables, and we identify  $V$  with affine space  $\mathbb{A}_{\mathbb{k}}^n = \text{Spec}(R)$ . Similarly, we regard  $\text{GL}(V) = \text{GL}(n, \mathbb{k})$  as the affine variety whose coordinate ring is the localisation of  $\mathbb{k}[x_{i,j} : 1 \leq i, j \leq n]$  at the determinant function.

**Example 2.2.** Let  $G$  denote the cyclic group of order three acting on  $V = \mathbb{A}_{\mathbb{k}}^2$  with generator the diagonal matrix  $\text{diag}(\omega, \omega)$  for  $\omega$  a primitive third root of unity. The dual action of  $G$  on  $V^*$  extends to an action on the polynomial ring  $\mathbb{k}[x, y] = \bigoplus_{k \geq 0} \text{Sym}^k(V^*)$ , and a monomial is invariant under the action of  $G$  precisely when it has total degree divisible by three. It follows that the algebra of  $G$ -invariant functions is

$$\mathbb{k}[x, y]^G \cong \mathbb{k}[x^3, x^2y, xy^2, y^3].$$

The  $\mathbb{k}$ -algebra homomorphism  $\psi: \mathbb{k}[y_1, y_2, y_3, y_4] \rightarrow \mathbb{k}[x, y]^G$  determined by setting  $\psi(y_1) = x^3$ ,  $\psi(y_2) = x^2y$ ,  $\psi(y_3) = xy^2$  and  $\psi(y_4) = y^3$  has kernel

$$(2.1) \quad \text{Ker}(\psi) = (y_1y_3 - y_2^2, y_1y_4 - y_2y_3, y_2y_4 - y_3^2).$$

Thus,  $\text{Spec}(\mathbb{k}[x, y]^G)$  is the singular surface in  $\mathbb{A}_{\mathbb{k}}^4$  cut out by the ideal  $\text{Ker}(\psi)$ .

**Example 2.3.** Let  $G$  denote the cyclic group of order 7 that acts on  $\mathbb{A}_{\mathbb{k}}^2$  with generator the diagonal matrix  $\text{diag}(\epsilon, \epsilon^4)$ , for  $\epsilon$  a primitive 7th root of unity. The algebra of  $G$ -invariants  $R = \mathbb{k}[x, y]^G$  is generated minimally by the monomials  $x^7, x^3y, x^2y^3, xy^5, y^7$ , and the induced surjective map of  $\mathbb{k}$ -algebras  $\varphi: \mathbb{k}[y_1, \dots, y_5] \rightarrow R$  determines the affine subvariety of  $\mathbb{A}_{\mathbb{k}}^5$  cut out by the equations

$$(2.2) \quad \text{rank} \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1^2 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1.$$

This rank conditions means simply that the equations are the binomials obtained as the  $2 \times 2$  minors of this matrix.

This example can be generalised significantly. For now we observe only that for any finite subgroup  $G$  of  $\text{GL}(n, \mathbb{k})$ , the set of  $G$ -orbits in  $\mathbb{A}_{\mathbb{k}}^n$  admits the structure of an affine variety.

**Proposition 2.4.** *For  $R = \mathbb{k}[x_1, \dots, x_n]$  and  $G \subset \text{GL}(n, \mathbb{k})$  a finite subgroup, we have:*

- (i) *the invariant ring  $R^G$  is a finitely generated integral  $\mathbb{k}$ -algebra; and*

(ii) the morphism  $\pi: \mathbb{A}_{\mathbb{k}}^n \rightarrow \text{Spec}(R^G)$  induced by the inclusion  $\iota: R^G \hookrightarrow R$  is surjective and closed. Moreover, for all points  $p, p' \in \mathbb{A}_{\mathbb{k}}^n$ ,

$$(2.3) \quad \pi(p) = \pi(p') \iff G \cdot p = G \cdot p'.$$

In particular, the affine quotient  $\mathbb{A}_{\mathbb{k}}^n/G := \text{Spec}(R^G)$  parametrise all  $G$ -orbits in  $\mathbb{A}_{\mathbb{k}}^n$ .

*Proof.* The map  $\iota: R^G \hookrightarrow R$  is integral, so  $R$  is finitely-generated as an  $R^G$ -module. Since  $R$  is a finitely-generated  $\mathbb{k}$ -algebra, part (1) follows from the result of Artin-Tate (see [2, Prop 7.8]). The going-up theorem [2, Ex 5.10] implies that  $\pi$  is a surjective and closed map, and, moreover, that any pair of closed, disjoint and  $G$ -invariant subsets  $Z_1, Z_2 \subset \mathbb{A}_{\mathbb{k}}^n$  satisfy  $\pi(Z_1) \cap \pi(Z_2) = \emptyset$ . This gives the statement from (2.3).  $\square$

*Exercise 2.5.* For  $r \in \mathbb{N}$  and  $a_1, \dots, a_n \in \mathbb{N}$  satisfying  $a_i < r$  for  $1 \leq i \leq n$ , consider the action of the cyclic group  $G = \mathbb{Z}/r$  on  $\mathbb{A}_{\mathbb{k}}^n$  with generator the diagonal matrix  $\text{diag}(\epsilon^{a_1}, \dots, \epsilon^{a_n})$  for  $\epsilon$  a primitive  $r$ th root of unity. This is the *action of type*  $\frac{1}{r}(a_1, \dots, a_n)$ , and the variety  $\mathbb{A}_{\mathbb{k}}^n/G$  is the *cyclic quotient singularity of type*  $\frac{1}{r}(a_1, \dots, a_n)$ . Generalise Example 2.3 by writing down the defining equations for the quotient singularity of type  $\frac{1}{r}(1, 1, \dots, 1)$ .

**Remark 2.6.** For  $X = \text{Spec}(R)$ , choose an embedding in  $\mathbb{A}_{\mathbb{k}}^n$  and suppose that a finite subgroup  $G \subset \text{GL}(n, \mathbb{k})$  acts on  $X$  by restriction. The proof of Proposition 2.4 applies verbatim to show that the *affine quotient*  $X/G := \text{Spec}(R^G)$  parametrises all  $G$ -orbits in  $X$ .

**2.2. Diagonalisable algebraic groups.** In order to study further group actions on algebraic varieties we first establish the actions that are of interest to us.

An affine algebraic group is an affine variety  $G$  over  $\mathbb{k}$  equipped with three morphisms, namely *multiplication*  $\nu: G \times G \rightarrow G$ ; an *identity*  $e: \text{Spec}(\mathbb{k}) \rightarrow G$ ; and an *inverse*  $\iota: G \rightarrow G$ , where:

- (i)  $\nu \circ (\text{id}_G \times \nu) = \nu \circ (\nu \times \text{id}_G): G \times G \times G \rightarrow G$ ;
- (ii)  $\text{id}_G = \nu \circ (e \times \text{id}_G) = \nu \circ (\text{id}_G \times e): G \rightarrow G$ ;
- (iii)  $\nu \circ (\text{id}_G \times \iota) \circ \Delta = \nu \circ (\iota \times \text{id}_G) \circ \Delta = e \circ \pi: G \rightarrow G$ .

Here,  $\text{id}_G: G \rightarrow G$  is the identity morphism,  $\Delta: G \rightarrow G \times G$  is the diagonal embedding, and  $\pi: G \rightarrow \text{Spec}(\mathbb{k})$  is the natural map.

*Exercise 2.7.* Draw commutative diagrams expressing each condition and convince yourself that these conditions are simply the axioms for a group stated in the category of affine varieties.

Let  $\Lambda$  be a finitely generated abelian group with group algebra  $\mathbb{k}[\Lambda]$ . The group generators of  $\Lambda$  determine finitely many  $\mathbb{k}$ -algebra generators of  $\mathbb{k}[\Lambda]$ , and we write  $G_\Lambda := \text{Spec}(\mathbb{k}[\Lambda])$ . The  $\mathbb{k}$ -algebra homomorphisms  $\nu^*: \mathbb{k}[\Lambda] \rightarrow \mathbb{k}[\Lambda] \otimes_{\mathbb{k}} \mathbb{k}[\Lambda]$ ,  $e^*: \mathbb{k}[\Lambda] \rightarrow \mathbb{k}$  and  $\iota^*: \mathbb{k}[\Lambda] \rightarrow \mathbb{k}[\Lambda]$  defined by setting  $\nu^*(\lambda) = \lambda \otimes \lambda$ ,  $e^*(\lambda) = 1$  and  $\iota^*(\lambda) = \lambda^{-1}$  for  $\lambda \in \Lambda$  induce morphisms  $\nu: G_\Lambda \times G_\Lambda \rightarrow G_\Lambda$ ,  $e: \text{Spec}(\mathbb{k}) \rightarrow G_\Lambda$  and  $\iota: G_\Lambda \rightarrow G_\Lambda$  that make  $G_\Lambda$  an affine algebraic group. An affine algebraic group is *diagonalisable* if it is isomorphic to  $G_\Lambda$  for some finitely generated abelian group  $\Lambda$ .

**Proposition 2.8.** *The assignment  $\Lambda \mapsto G_\Lambda$  establishes a contravariant equivalence from the category of finitely generated abelian groups to the category of diagonalisable algebraic groups, with quasi-inverse given by sending each group  $G$  to its character group  $G^*$ .*

*Proof.* See, for example, Milne [55].  $\square$

To explain the terminology, we mention the following result [55].

**Lemma 2.9.** *Let  $V$  be a finite dimensional  $\mathbb{k}$ -vector space and  $G \subset \mathrm{GL}(V)$  an algebraic subgroup. There is a basis of  $V$  in which  $G$  acts diagonally on  $V$  if and only if  $G = G_\Lambda$  for some finitely generated abelian group  $\Lambda$ .*

**Example 2.10.** Let  $\Lambda$  be a cyclic group.

- If  $\Lambda$  has order  $r \in \mathbb{N}$  then  $\mathbb{k}[\Lambda] \cong \mathbb{k}[t]/\langle t^r - 1 \rangle$ , so  $G_\Lambda \cong \mu_r := \{p \in \mathbb{A}_{\mathbb{k}}^1 : t^r = 1\}$ .
- If  $\Lambda \cong \mathbb{Z}$  then  $G_\Lambda \cong \mathbb{k}^\times := \mathrm{Spec}(\mathbb{k}[t, t^{-1}])$  is the algebraic torus of dimension one.

Direct sums of cyclic groups induce tensor products of group algebras that are compatible with the algebraic groups structure. Thus, the structure theorem finitely generated abelian groups implies that every diagonalisable algebraic group is isomorphic to a product

$$G_\Lambda \cong \mathbb{k}^\times \times \cdots \times \mathbb{k}^\times \times \mu_{r_1}^{m_1} \times \cdots \times \mu_{r_t}^{m_t}$$

of finitely many copies of the algebraic torus  $\mathbb{k}^\times$  with a finite abelian group.

**2.3. On actions and grading.** An *algebraic action* of an affine algebraic group  $G$  on an affine variety  $X = \mathrm{Spec}(R)$  is a morphism  $\mu: G \times X \rightarrow X$  of affine varieties such that the following compatibility conditions hold:

- (1)  $\mathrm{id}_X = \mu \circ (e \times \mathrm{id}_X): X \rightarrow X$ ;
- (2)  $\mu \circ (\mathrm{id}_G \times \mu) = \mu \circ (\nu \times \mathrm{id}_X): G \times G \times X \rightarrow X$ .

These conditions simply the natural conditions for the action of a group on a set restated in the category of affine varieties.

**Example 2.11.** Let  $V$  be a finite-dimensional  $\mathbb{k}$ -vector space and  $G$  an affine algebraic group. Recall from Example 2.1 that  $\mathrm{GL}(V)$  is an affine variety. Algebraic actions of  $G$  on  $V$  correspond precisely to those representations  $\rho: G \rightarrow \mathrm{GL}(V)$  that are morphisms in the category of affine varieties.

We consider only algebraic actions of a diagonalisable algebraic group  $G$  on an affine variety  $X = \mathrm{Spec}(R)$ . In the presence of such an action,  $G$  acts dually on  $R$  via

$$(2.4) \quad (g \cdot f)(p) = f(g^{-1} \cdot p) \quad \text{for all } g \in G, f \in R, p \in X.$$

Let  $\iota: R^G \hookrightarrow R$  denote the inclusion of the subalgebra of  $G$ -invariant functions, i.e., functions  $f \in R$  such that  $g \cdot f = f$ . The following algebraic reformulation is extremely useful.

**Theorem 2.12.** *Let  $\Lambda$  be a finitely generated abelian group. The following are equivalent:*

- an algebraic action of the diagonalisable group  $G_\Lambda$  on an affine variety  $X = \mathrm{Spec}(R)$ ;*
- a  $\Lambda$ -grading of  $R = \bigoplus_{\chi \in \Lambda} R_\chi$ .*

*Proof.* Applying the  $\mathrm{Spec}$  functor to the morphism  $\mu: G_\Lambda \times X \rightarrow X$  and to the morphisms associated to  $G_\Lambda$  gives a  $\mathbb{k}$ -algebra homomorphism  $\mu^*: R \rightarrow \mathbb{k}[\Lambda] \otimes_{\mathbb{k}} R$  satisfying conditions

- (1\*)  $\mathrm{id}_R = (e^* \otimes \mathrm{id}_R) \circ \mu^*: R \rightarrow R$ ;
- (2\*)  $(\mathrm{id}_{\mathbb{k}[\Lambda]} \otimes \mu^*) \circ \mu^* = (\nu^* \otimes \mathrm{id}_R) \circ \mu^*: R \rightarrow \mathbb{k}[\Lambda] \otimes \mathbb{k}[\Lambda] \otimes R$ .

For each character  $\chi \in \Lambda$  of the group  $G_\Lambda$ , define

$$R_\chi := \{f \in R : \mu^*(f) = \chi \otimes f\}.$$

Note that  $R_\chi \cap R_{\chi'} = 0$  when  $\chi \neq \chi'$ , and that  $R_\chi \otimes_{\mathbb{k}} R_{\chi'} \subseteq R_{\chi+\chi'}$  since  $\mu^*$  is a  $\mathbb{k}$ -algebra homomorphism. For  $f \in R$  decompose  $\mu^*(f) = \sum_{\chi \in \Lambda} \chi \otimes f_\chi$ . Applying condition (2\*) to  $f \in R$  gives  $\mu^*(f_\chi) = \chi \otimes f_\chi$ , so  $f_\chi \in R_\chi$ . On the other hand, condition (1\*) gives  $f = \sum_{\chi \in \Lambda} f_\chi$ , from which we obtain  $R = \bigoplus_{\chi \in \Lambda} R_\chi$ . Conversely, given a  $\Lambda$ -grading of  $R$  we construct a  $\mathbb{k}$ -algebra

homomorphism  $\mu^*: R \rightarrow \mathbb{k}[\Lambda] \otimes_{\mathbb{k}} R$  by setting  $\mu^*(f) = \chi \otimes f$  for each  $f \in R_\chi$ . Reversing the argument above shows that the induced morphism  $\mu: G_\Lambda \times X \rightarrow X$  is an algebraic action.  $\square$

For a character  $\chi \in \Lambda$  of  $G = G_\Lambda$ , the elements  $f \in R_\chi$  are *homogeneous of degree  $\chi$  with respect to  $G$* , or simply  *$G$ -semi-invariants of weight  $\chi$* . Explicitly, an element  $f \in R$  is semi-invariant of weight  $\chi$  if

$$g \cdot f = \chi(g^{-1})f \quad \text{for all } g \in G.$$

In light of (2.4), this means that  $f(g \cdot p) = \chi(g)f(p)$  for all  $g \in G$  and  $p \in X$ . The zero-graded piece  $R_0$  coincides with the  $G$ -invariant subalgebra of  $R$ .

**2.4. Proj of a  $\mathbb{Z}$ -graded ring.** The most common examples of diagonalisable group actions on affine varieties arise from finitely generated  $\mathbb{k}$ -algebras that are  $\mathbb{Z}$ -graded:

$$R = \bigoplus_{\chi \in \Lambda} R_\chi \quad \text{for } \Lambda = \mathbb{Z}.$$

Assume  $R_0 = \mathbb{k}$ . Choose homogeneous generators  $f_0, f_1, \dots, f_n \in R$  and suppose for now that each has degree one. If we grade the polynomial ring  $\mathbb{k}[x_0, x_1, \dots, x_n]$  by setting  $\deg(x_i) = 1$  for  $0 \leq i \leq n$  then the surjective map of  $\mathbb{k}$ -algebras  $\psi: \mathbb{k}[x_0, x_1, \dots, x_n] \rightarrow R$  where  $\psi(x_i) = f_i$  for  $0 \leq i \leq n$  preserves the  $\Lambda$ -grading. The ideal  $\text{Ker}(\psi)$  that records the relations between the generators  $f_0, \dots, f_n$  is  $\Lambda$ -homogeneous, and cuts out a projective subvariety

$$\text{Proj}(R) := \{p \in \mathbb{P}_{\mathbb{k}}^n : f(p) = 0 \text{ for all homogeneous } f \in \text{Ker}(\psi)\};$$

this is the *projective spectrum* of the graded ring  $R$ . A different choice of generators for  $R$  leads to an alternative embedding of  $\text{Proj}(R)$ .

A geometric interpretation of  $\text{Proj}(R)$  is obtained from Theorem 2.12 above. Indeed, for  $X = \text{Spec}(R)$  the map  $\psi: \mathbb{k}[x_0, x_1, \dots, x_n] \rightarrow R$  defines a closed immersion  $\varphi: X \rightarrow \mathbb{A}_{\mathbb{k}}^{n+1}$  sending  $p \in X$  to the point  $(f_0(p), f_1(p), \dots, f_n(p)) \in \mathbb{A}_{\mathbb{k}}^{n+1}$ . The  $\Lambda$ -grading determines actions of the algebraic torus  $\mathbb{k}^\times = \text{Spec}(\mathbb{k}[\Lambda])$  on both  $X$  and  $\mathbb{A}_{\mathbb{k}}^{n+1}$ , making  $\varphi$  into a  $\mathbb{k}^\times$ -equivariant map. Since each variable  $x_i$  has degree one, let  $t \in \mathbb{k}^\times$  act on  $\mathbb{A}_{\mathbb{k}}^{n+1}$  as

$$t \cdot (p_0, p_1, \dots, p_n) = (tp_0, \dots, tp_n).$$

If we remove from  $X$  the common zero-locus of the generators  $f_1, \dots, f_n \in R$ , then the restriction of  $\varphi$  descends to a morphism on  $\mathbb{k}^\times$ -orbits that fits in to the right-hand square of the commutative diagram

$$(2.5) \quad \begin{array}{ccccc} \text{Spec}(R[f_i^{-1}]) & \longrightarrow & X \setminus \mathbb{V}(f_0, f_1, \dots, f_n) & \xrightarrow{\varphi} & \mathbb{A}_{\mathbb{k}}^{n+1} \setminus \{0\} \\ \pi \downarrow & & \pi \downarrow & & \downarrow \pi \\ \text{Spec}(R[f_i^{-1}]^{\mathbb{k}^\times}) & \longrightarrow & \text{Proj}(R) & \xrightarrow{\bar{\varphi}} & \mathbb{P}_{\mathbb{k}}^n \end{array}$$

where the horizontal maps in the right-hand square are closed immersions, the horizontal maps in the left-hand square are open embeddings, and the vertical maps identify points in the same  $\mathbb{k}^\times$ -orbit. The locus  $X \setminus \mathbb{V}(f_0, f_1, \dots, f_n)$  on which the quotient map  $\pi|_X$  is well defined is covered by the principal affine open subsets  $X \setminus \mathbb{V}(f_i) = \text{Spec}(R[f_i^{-1}])$  for  $0 \leq i \leq n$ , and by examining the standard local coordinate charts on  $\mathbb{P}_{\mathbb{k}}^n$ , we see that  $\text{Proj}(R)$  is covered by local charts of the form  $\text{Spec}(R[f_i^{-1}]^{\mathbb{k}^\times})$  for  $0 \leq i \leq n$ , where

$$R[f_i^{-1}]^{\mathbb{k}^\times} := \left\{ \frac{f}{f_i^j} : f \in R_j \text{ for } j \geq 0 \right\}$$

is the  $\mathbb{k}^\times$ -invariant subalgebra of the localisation of  $R$  at  $f_i$ . Note that  $\text{Spec}(R[f_i^{-1}]^{\mathbb{k}^\times})$  is well defined since  $R[f_i^{-1}]^{\mathbb{k}^\times}$  is a finitely generated  $\mathbb{k}^\times$ -algebra (see Theorem 2.18 to follow).

**Example 2.13.** Associate to the line bundle  $\mathcal{O}_{\mathbb{P}^1}(3)$  the  $\mathbb{Z}$ -graded algebra

$$R := \bigoplus_{\chi \in \mathbb{Z}} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3\chi)) \cong \mathbb{k}[tx_1^3, tx_1^2x_2, tx_1x_2^2, tx_2^3],$$

where the exponent of  $t$  in a monomial gives the degree. The obvious homomorphism of graded  $\mathbb{k}$ -algebras  $\psi: \mathbb{k}[y_1, y_2, y_3, y_4] \rightarrow \mathbb{k}[t, x_1, x_2]$  has kernel the  $\mathbb{k}^\times$ -homogeneous ideal  $I$  from equation (2.1), and  $\text{Proj}(R)$  is the twisted cubic curve cut out by  $I$  in the complete linear series  $|\mathcal{O}_{\mathbb{P}^1}(3)| = \mathbb{P}_{\mathbb{k}}^3$ . The affine cone  $\text{Spec}(R)$  over  $\text{Proj}(R)$  featured in Example 2.2.

*Exercise 2.14.* Generalise the previous example to the  $d$ -uple Veronese embedding on  $\mathbb{P}^n$ , and present  $\text{Spec}(R)$  as the quotient of affine space by a finite group (compare Exercise 2.5).

To extend the construction to algebras  $R$  whose generators do not all have degree one we recall weight projective space. Fix weights  $a_0, \dots, a_n \in \mathbb{Z}_{>0}$ , and let  $t \in \mathbb{k}^\times$  act on  $\mathbb{A}_{\mathbb{k}}^{n+1}$  as

$$t \cdot (p_0, p_1, \dots, p_n) = (t^{a_0}p_0, t^{a_1}p_1, \dots, t^{a_n}p_n).$$

Equivalently, grade the polynomial ring  $\mathbb{k}[x_0, x_1, \dots, x_n]$  by setting  $\deg(x_i) = a_i$  for  $0 \leq i \leq n$ . For  $0 \leq i \leq n$ , the coordinate ring of  $U_i := \mathbb{A}_{\mathbb{k}}^{n+1} \setminus (x_i = 0)$  is isomorphic to the quotient singularity of type  $\frac{1}{a_i}(a_0, \dots, \hat{a}_i, \dots, a_n)$  from Exercise 2.5. These varieties glue nicely along intersections and provide the standard open cover of *weighted projective space*  $\mathbb{P}_{\mathbb{k}}(a_0, \dots, a_n)$ .

Suppose now that a  $\mathbb{k}$ -algebra  $R$  has generators  $f_0, f_1, \dots, f_n$  where  $f_i$  has degree  $a_i > 0$  for  $0 \leq i \leq n$ . For  $X = \text{Spec}(R)$ , the induced closed immersion  $\varphi: X \rightarrow \mathbb{A}_{\mathbb{k}}^{n+1}$  is  $\mathbb{k}^\times$ -homogeneous and descends to a map  $\bar{\varphi}: \text{Proj}(R) \rightarrow \mathbb{P}_{\mathbb{k}}(a_0, \dots, a_n)$  on  $\mathbb{k}^\times$ -orbits, where  $\text{Proj}(R)$  is the variety obtained by gluing the affine open sets

$$R[f_i^{-1}]^{\mathbb{k}^\times} := \left\{ \frac{f}{f_i^j} : f \in R_{ja_i} \text{ for } j \geq 0 \right\}$$

for  $0 \leq i \leq n$ .

**Remark 2.15.** We need not always assume that  $R_0 = \mathbb{k}$ , in which case  $\text{Proj}(R)$  is a subvariety of  $\mathbb{P}_{R_0}^n$ . Thus, while  $\text{Proj}(R)$  need not be projective, it is always projective over  $\text{Spec}(R_0)$ .

**2.5. Affine GIT.** The left-hand vertical map in diagram (2.5) and the map analysed in Proposition 2.4 both arise from the inclusion of a  $G$ -invariant subalgebra  $R^G \hookrightarrow R$  for some diagonalisable group action on a finitely generated  $\mathbb{k}$ -algebra. The relation between morphisms of this kind and  $G$ -orbits in  $\text{Spec}(R)$  can be straightforward as in Proposition 2.4, but it need not be as the following example shows.

**Example 2.16.** Consider the standard diagonal action of  $\mathbb{k}^\times$  on  $\mathbb{A}_{\mathbb{k}}^{n+1} = \text{Spec}(\mathbb{k}[x_0, \dots, x_n])$  that defines  $\mathbb{P}_{\mathbb{k}}^n$ . Note that  $\mathbb{k}^\times$  acts on the open subset  $U_i := \mathbb{A}_{\mathbb{k}}^{n+1} \setminus (x_i = 0)$  for  $0 \leq i \leq n$ .

- (i) For the action of  $\mathbb{k}^\times$  on the whole of  $\mathbb{A}_{\mathbb{k}}^{n+1}$  we have  $\text{Spec}(\mathbb{k}[x_0, \dots, x_n]^{\mathbb{k}^\times}) \cong \text{Spec}(\mathbb{k})$ , so the map  $\mathbb{A}_{\mathbb{k}}^{n+1} \rightarrow \text{Spec}(\mathbb{k})$  induced by the inclusion  $\mathbb{k} \hookrightarrow \mathbb{k}[x_0, \dots, x_n]$  identifies every point. In this case, the closure of each orbit in  $\mathbb{A}_{\mathbb{k}}^{n+1}$  contains the origin, so the variety  $\text{Spec}(\mathbb{k}[x_0, \dots, x_n]^{\mathbb{k}^\times})$  may be thought of as parametrising  $\mathbb{k}^\times$ -orbit closures.
- (ii) For the  $\mathbb{k}^\times$ -action on the principal open set  $U_i$  we have  $\text{Spec}((\mathbb{k}[x_0, \dots, x_n][x_i^{-1}])^{\mathbb{k}^\times}) \cong \text{Spec}(\mathbb{k}[\frac{x_1}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i}]) \cong \mathbb{A}_{\mathbb{k}}^n$ . The  $\mathbb{k}^\times$ -orbits in  $U_i$  are all fixed ratios  $[p_0 : \dots : p_n]$  with  $p_i = 1$ . Such orbits are not closed in  $\mathbb{A}_{\mathbb{k}}^{n+1}$ , but they are closed in  $U_i$ . Thus, the variety  $\text{Spec}(\mathbb{k}[x_0, \dots, x_n][x_i^{-1}]^{\mathbb{k}^\times})$  parametrises genuine  $\mathbb{k}^\times$ -orbits in  $U_i$ .

**Remark 2.17.** Note that Proposition 2.4 (compare Remark 2.6) provides a special case.

To state the general result, consider the action of a diagonalisable group  $G$  on an affine variety  $X = \text{Spec}(R)$ . A point  $p \in X$  is *stable* with respect to  $G$  if the orbit  $G \cdot p$  is closed in  $X$ , and if the stabiliser subgroup  $\{g \in G : gp = p\}$  is finite. Let  $X^s \subseteq X$  denote the locus of stable points in  $X$ . The main result of Geometric Invariant Theory for the action of  $G$  on an affine variety now applies because  $G$  is isomorphic to the product of an algebraic torus  $(\mathbb{k}^\times)^r$  with a finite group and hence is linearly reductive (see Dolgachev [32, §3.3, §6], Mukai [56, §4.3, §5] or Mumford et. al. [57, §1.2]).

**Theorem 2.18.** *Let a diagonalisable group  $G$  act on an affine variety  $X = \text{Spec}(R)$ :*

- (i) *the subalgebra  $R^G$  is a finitely generated integral  $\mathbb{k}$ -algebra;*
- (ii) *the stable locus  $X^s \subseteq X$  is open, and there is a commutative diagram*

$$\begin{array}{ccc} X^s & \longrightarrow & X = \text{Spec}(R) \\ \downarrow \pi^s & & \downarrow \pi \\ X^s/G & \longrightarrow & \text{Spec}(R^G) \end{array}$$

*where the horizontal maps are open embeddings, and  $\pi$  is surjective and closed. Moreover, for all  $p, p' \in X$  we have*

$$\pi(p) = \pi(p') \iff \overline{G \cdot p} \cap \overline{G \cdot p'} \neq \emptyset,$$

*and each fibre of  $\pi$  contains a unique closed  $G$ -orbit in  $X$ ; and*

- (iii) *each the fibre of  $\pi$  over a point of  $X^s/G$  is equal to a unique closed orbit.*

*The variety  $X/G := \text{Spec}(R^G)$  is the affine quotient of  $X$  by  $G$ .*

We prefer whenever possible to work only with stable points, when the affine quotient  $X^s/G$  really does parametrise  $G$ -orbits in  $X^s$ , at the expense of having to throw away part (possibly all) of  $X$ . For example, every point of  $X$  is stable for the action of a finite group, in which case Theorem 2.18 reduces to Proposition 2.4. On the other hand, for the  $\mathbb{k}^\times$ -action given in Example 2.16, no point of  $\mathbb{A}_{\mathbb{k}}^{n+1}$  is  $\mathbb{k}^\times$ -stable whereas every point of  $U_i \subset \mathbb{A}_{\mathbb{k}}^{n+1}$  is  $\mathbb{k}^\times$ -stable. This illustrates clearly that stability depends on the choice of  $G$  and  $X$ .

**Remark 2.19.** The quotient map  $\pi: X \rightarrow X/G$  satisfies a universal property, namely, that any morphism  $\tau: X \rightarrow Z$  that is constant on  $G$ -orbits factors through  $\pi: X \rightarrow X/G$ . Given the categorical nature of this property,  $X/G$  is often called the *categorical quotient* of  $X$  by  $G$ . The restriction to the stable locus  $\pi^s: X^s \rightarrow X^s/G$  is a *geometric quotient*.

**2.6. Projective GIT.** Let a diagonalisable group  $G$  act algebraically on  $X = \text{Spec}(R)$ , so  $R$  is graded by a finitely generated abelian group  $\Lambda$  by Theorem 2.12. To produce a  $\mathbb{Z}$ -graded ring, choose a character  $\chi \in \Lambda$  of  $G$  and let  $R_{j\chi}$  denote the  $\mathbb{k}$ -vector subspace of  $R$  spanned by the  $G$ -semi-invariants of weight  $\chi$ . The *algebra of  $\chi$ -semi-invariants*

$$(2.6) \quad \bigoplus_{j \in \mathbb{Z}} R_{j\chi}$$

is a  $\mathbb{Z}$ -graded ring. We assume that  $R_{j\chi} = 0$  for  $j < 0$ .

**Lemma 2.20.** *The graded ring  $\bigoplus_{k \geq 0} R_{k\chi}$  is a finitely-generated  $\mathbb{k}$ -algebra.*

*Proof.* Lift the  $G$ -action on  $X = \text{Spec}(R)$  to the action on  $X \times \mathbb{A}_{\mathbb{k}}^1 = \text{Spec}(R[y])$  for which  $y$  is semi-invariant of weight  $\chi^{-1}$ . For  $j \geq 0$ , the function  $fy^j \in R[y]$  is  $G$ -invariant if and only if  $f$  is semi-invariant of weight  $\chi^j$ . The graded ring  $\bigoplus_{j \geq 0} R_{j\chi}$  is therefore the  $G$ -invariant subring of  $R[y]$ . The result follows from Theorem 2.18.  $\square$

The *GIT quotient* of  $X = \text{Spec}(R)$  by the action of the diagonalisable group  $G$  linearised by the character  $\chi \in \Lambda$  is defined to be the projective spectrum

$$X//_{\chi}G := \text{Proj} \left( \bigoplus_{j \geq 0} R_{j\chi} \right).$$

If  $X//_{\chi}G$  is nonempty then it is projective over the affine variety  $\text{Spec}(R_0)$ . In fact, most of the statements that one might make about GIT quotients work well only under the assumption that  $X//_{\chi}G \neq \emptyset$ , in which case we say that  $\chi$  is *effective*. While this is not always the case, we restrict attention only to effective characters for the sake of simplicity.

**Example 2.21.** Consider the action of  $\mathbb{k}^{\times}$  on  $\mathbb{A}_{\mathbb{k}}^3 = \text{Spec}(\mathbb{k}[x, y, z])$  where the variables  $x, y, z$  have degree one, two and three respectively. For  $\chi = 6$ , the subalgebra of  $\mathbb{k}[x, y, z]$  spanned by all monomials  $x^a y^b z^c$  for which  $6|a + 2b + 3c$  is generated by  $\{x^6, x^4 y, x^2 y^2, y^3, x^3 z, xyz, z^2\}$ . The GIT quotient  $\mathbb{A}_{\mathbb{k}}^3//_{\chi}\mathbb{k}^{\times}$  is the weighted projective plane  $\mathbb{P}_{\mathbb{k}}(1, 2, 3)$  presented as a subvariety of  $\mathbb{P}_{\mathbb{k}}^6$ .

To describe the local structure of the GIT quotient, we say that a point  $p \in X$  is  $\chi$ -*semistable* with respect to  $G$  if there exists  $j > 0$  and  $f \in R_{j\chi}$  satisfying  $f(p) \neq 0$ . The affine open subset  $X_{\chi}^{\text{ss}} := X \setminus \mathbb{V}(f_0, f_1, \dots, f_n)$  of  $X$  consisting of all such points is the  $\chi$ -*semistable locus* in  $X$ . The restriction of  $\pi$  to  $X_{\chi}^{\text{ss}}$  and to  $X \setminus \mathbb{V}(f_i)$  gives a commutative diagram

$$(2.7) \quad \begin{array}{ccc} \text{Spec}(R[f_i^{-1}]) & \longrightarrow & X_{\chi}^{\text{ss}} \\ \downarrow & & \downarrow \pi \\ \text{Spec}(R[f_i^{-1}]^G) & \longrightarrow & X//_{\chi}G \end{array}$$

where the horizontal maps are open embeddings, and the left-hand vertical map is the affine quotient map arising from the  $G$ -action on  $X \setminus \mathbb{V}(f_i)$ . Theorem 2.18 implies the following.

**Theorem 2.22.** *The categorical quotient  $X//_{\chi}G$  parametrises equivalence classes of  $G$ -orbit-closures for all  $\chi$ -semistable points in  $X_{\chi}^{\text{ss}}$ . The geometric quotient  $X//_{\chi}G$  parametrises  $G$ -orbits of  $\chi$ -stable points in  $X_{\chi}^{\text{ss}}$ .*

For the action of a diagonalisable group  $G$  on an affine variety  $X$ , any fractional character  $\chi \in G^* \otimes_{\mathbb{Z}} \mathbb{Q}$  is called *generic* if every  $\chi$ -semistable of  $X$  is  $\chi$ -stable. Following Thaddeus [64] and Dolgachev–Hu [33], define an equivalence relation on the set of generic fractional characters by setting  $\chi \sim \chi'$  if and only if every  $\chi$ -stable point of  $X$  is  $\chi'$ -stable, and vice versa. This gives a polyhedral decomposition of the set of effective fractional characters called the (*polarised*) *GIT chamber decomposition* for the given action of  $G$  on  $X$ . As  $\chi$  varies in a chamber the GIT quotient  $X//_{\chi}G$  remains unchanged as a variety, though the graded ring that defines it varies.

**Remark 2.23.** We present several examples of this phenomenon later in these notes, including:

- (i) an investigation of the GIT chamber decomposition for an action of  $(\mathbb{k}^{\times})^2$  on  $\mathbb{A}_{\mathbb{k}}^4$  that defines the first Hirzebruch surface  $\mathbb{F}_1$  (see Exercise 3.24); and
- (ii) an explicit construction of the GIT chamber decomposition arising from an action of  $(\mathbb{k}^{\times})^2$  on an affine fourfold associated to the quotient singularity of type  $\frac{1}{3}(1, 2)$ ; this is better known as the Weyl chamber decomposition of type  $A_2$  (see Exercise 8.4).

### 3. SEMIPROJECTIVE TORIC VARIETIES

This section introduces semiprojective toric varieties using the projective GIT construction for diagonalisable group actions. This simple class of toric varieties includes the overwhelming majority of examples that one meets in the course of everyday life. We do not assume that such toric varieties are normal, but in that case we obtain naturally a convex geometric interpretation via the fan. We also present a simple proof of Cox's quotient description of a normal toric variety in the semiprojective case.

**3.1. Affine toric varieties.** For  $d \in \mathbb{N}$ , let  $S \subseteq \mathbb{Z}^d$  be a finitely generated subsemigroup containing the zero element. Without loss of generality, we may assume that  $S$  generates  $\mathbb{Z}^d$  over  $\mathbb{Z}$ , in which case  $S$  is the image of the subsemigroup  $\mathbb{N}^n \subset \mathbb{Z}^n$  under a lattice map  $\pi: \mathbb{Z}^n \rightarrow \mathbb{Z}^d$ . Identify the semigroup algebras of  $\mathbb{N}^n$  and  $\mathbb{Z}^d$  with the polynomial ring  $\mathbb{k}[x_1, \dots, x_n]$  and the ring of Laurent polynomials  $\mathbb{k}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$  respectively. Write  $\mathbb{k}[S] = \{t^u \in \mathbb{k}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] : u \in S\}$  for the semigroup algebra of  $S$ , where the element  $u = (u_1, \dots, u_d) \in S$  defines the monomial  $t^u := \prod_{1 \leq i \leq d} t_i^{u_i}$ . The map  $\pi: \mathbb{N}^n \rightarrow \mathbb{Z}^d$  factors through  $S$  and induces diagrams of semigroup algebras and affine varieties as shown:

$$(3.1) \quad \begin{array}{ccc} \mathbb{k}[x_1, \dots, x_n] & & \mathbb{A}_{\mathbb{k}}^n \\ \downarrow \psi & \searrow \pi & \uparrow \psi^* \\ \mathbb{k}[S] & \xrightarrow{\tau} & \mathbb{k}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] & \xleftarrow{\pi^*} & (\mathbb{k}^\times)^d \\ & & \text{Spec}(\mathbb{k}[S]) & \xleftarrow{\tau^*} & \end{array}$$

Since  $S$  generates  $\mathbb{Z}^d$  over  $\mathbb{Z}$ , the algebra  $\mathbb{k}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$  can be obtained from  $\mathbb{k}[S]$  by localising suitably, so the righthand horizontal map is an open embedding that presents the algebraic torus  $(\mathbb{k}^\times)^d$  as a Zariski dense open subset of  $\text{Spec}(\mathbb{k}[S])$ . Moreover, the lefthand vertical map surjects, so the righthand vertical map is a closed immersion with image cut out by the *toric ideal* of  $\pi$ , namely the prime ideal

$$\text{Ker}(\psi) = (x^u - x^{u'} \in \mathbb{k}[x_1, \dots, x_n] : u - u' \in \text{Ker}(\pi)).$$

An affine variety is a *toric variety* if it is of the form  $\text{Spec}(\mathbb{k}[S])$  for some subsemigroup  $S \subset \mathbb{Z}^d$  as above. With this definition, toric varieties need not be normal.

**Example 3.1.** Define  $\pi: \mathbb{N}^4 \rightarrow \mathbb{Z}^2$  by the matrix  $\begin{bmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{bmatrix}$ , so  $S \subset \mathbb{Z}^2$  is generated by the columns of the matrix. The toric variety  $\text{Spec}(\mathbb{k}[S])$  is not normal.

**Remark 3.2.** For further evidence that nonnormal toric varieties arise naturally in a geometric context, see Example 3.6 and Remark 8.3.

To state the normality criterion for a toric variety, associate to a subsemigroup  $S \subseteq \mathbb{Z}^d$  with generators  $s_1, \dots, s_n \in S$  the polyhedral cone

$$\text{cone}(S) := \left\{ \sum_{1 \leq i \leq n} \lambda_i s_i \in \mathbb{Q}^d : \lambda_i \in \mathbb{Q}_{\geq 0} \right\}.$$

Clearly,  $S$  is contained in the set  $\text{cone}(S) \cap \mathbb{Z}^d$  of integral points of the cone, but this inclusion can be strict, e.g., consider the vector  $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$  from Example 3.1.

**Lemma 3.3.** *The affine toric variety  $\text{Spec}(\mathbb{k}[S])$  is normal if and only if the semigroup  $S$  consists of the integral points of a polyhedral cone. When this holds,  $S = \text{cone}(S) \cap \mathbb{Z}^d$ .*

*Exercise 3.4.* Prove Lemma 3.3.

For a simple class of examples, recall from Exercise 2.5 that the quotient singularity of type  $\frac{1}{r}(1, a)$  is the affine quotient of  $\mathbb{A}_{\mathbb{k}}^2$  by the action of the cyclic group  $\mathbb{Z}/r$  with generator the matrix  $\text{diag}(\epsilon, \epsilon^a)$  for  $\epsilon$  a primitive  $r$ th root of unity. Proposition 2.4 shows that the coordinate ring of  $\mathbb{A}_{\mathbb{k}}^2/(\mathbb{Z}/r)$  is the  $\mathbb{Z}/r$ -invariant subring of  $\mathbb{k}[x, y]$ . The monomial  $x^{u_1}y^{u_2}$  is  $\mathbb{Z}/r$ -invariant if and only if  $u_1 + au_2 \equiv 0 \pmod{r}$ , and we write  $S = \{(u_1, u_2) \in \mathbb{N}^2 : u_1 + au_2 \equiv 0 \pmod{r}\}$  for the subsemigroup of (exponents of)  $\mathbb{Z}/r$ -invariant monomials. To describe explicitly the generators of  $S$ , define the Jung–Hirzebruch continued fraction expansion of  $r/(r-a)$  to be

$$\frac{r}{r-a} = b_1 - \frac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_t}}}.$$

The subsemigroup  $S \subset \mathbb{Z}^2$  is generated by  $\{u_0, u_1, \dots, u_{t+1}\}$ , where  $u_0 = (r, 0)$ ,  $u_1 = (r-a, 1)$  and  $u_{i+1} := b_i u_i - u_{i-1}$  for  $1 \leq i \leq t$ .

More generally, the cyclic quotient singularity of type  $\frac{1}{r}(a_1, \dots, a_n)$  from Exercise 2.5 is toric. For the action of  $\mathbb{Z}/r$  on  $\mathbb{A}_{\mathbb{k}}^n$  with generator matrix  $\text{diag}(\epsilon^{a_1}, \dots, \epsilon^{a_n})$ , a monomial  $x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$  is  $\mathbb{Z}/r$ -invariant if and only if the exponent vector lies in the subsemigroup

$$S = \left\{ (u_1, \dots, u_n) \in \mathbb{N}^n : \sum_{i=1, \dots, n} a_i u_i \equiv 0 \pmod{r} \right\}$$

of (exponents of)  $\mathbb{Z}/r$ -invariant monomials. In fact, the lattice point  $(u_1, \dots, u_n) \in \mathbb{N}^n$  lies in  $S$  if and only if its inner product with the rational vector  $\frac{1}{r}(a_1, \dots, a_n)$  is zero; this explains the terminology. In any event, the cyclic quotient  $\mathbb{A}^n/(\mathbb{Z}/r) \cong \text{Spec}(\mathbb{k}[S])$  is a toric variety.

**Example 3.5.** The singularity of type  $\frac{1}{7}(1, 4)$  from Exercise 2.3 is  $\text{Spec}(\mathbb{k}[S])$  where the semigroup  $S$  is obtained as the image of the map  $\pi: \mathbb{N}^5 \rightarrow \mathbb{Z}^2$  with matrix  $\begin{bmatrix} 7 & 3 & 2 & 1 & 0 \\ 0 & 1 & 3 & 5 & 7 \end{bmatrix}$ .

**3.2. Semiprojective toric varieties.** Let  $S \subseteq \mathbb{Z}^d$  be a finitely generated subsemigroup that generates  $\mathbb{Z}^d$  over  $\mathbb{Z}$  as above, and suppose in addition that  $\nu: \mathbb{Z}^d \rightarrow \mathbb{Z}$  is a  $\mathbb{Z}$ -linear map satisfying  $\nu(S) \subseteq \mathbb{N}$ . Set  $M := \text{Ker}(\nu)$ . The semigroup algebra  $\mathbb{k}[S]$  admits a  $\mathbb{Z}$ -grading  $\bigoplus_{j \geq 0} \mathbb{k}[S]_j$ , where  $\mathbb{k}[S]_j$  has a  $\mathbb{k}$ -vector space basis consisting of monomials  $t^u \in \mathbb{k}[S]$  with  $\nu(u) = j$ . The graded pieces satisfy  $\mathbb{k}[S]_j = 0$  for  $j < 0$  by assumption, and  $\mathbb{k}[S]_0$  is the semigroup algebra of  $S \cap M$ , so the variety

$$\text{Proj}(\mathbb{k}[S]) := \text{Proj}\left(\bigoplus_{j \geq 0} \mathbb{k}[S]_j\right)$$

is projective over the affine toric variety  $\text{Spec}(\mathbb{k}[S]_0)$ . A variety is a *semiprojective toric variety* if it is of the form  $\text{Proj}(\mathbb{k}[S])$  for some  $\mathbb{Z}$ -graded semigroup algebra  $\mathbb{k}[S]$  as above, and it is a *projective toric variety* if  $\mathbb{k}[S]_0 = \mathbb{k}$ .

**Example 3.6.** Let  $X$  be a normal projective toric variety and let  $L \in \text{Pic}(X)$  be very ample with complete linear series  $|L| = \text{Proj}\left(\bigoplus_{j \geq 0} H^0(X, L^j)\right) \cong \mathbb{P}_{\mathbb{k}}^m$  for  $m = \dim_{\mathbb{k}} H^0(X, L) - 1$ . The closed immersion  $\varphi_{|L|}: X \rightarrow |L|$  obtained by evaluating a  $\mathbb{k}$ -vector space basis of  $H^0(X, L)$  at points of  $X$  presents  $X$  as a toric subvariety of  $|L|$ . The affine cone  $\text{Spec}\left(\bigoplus_{j \geq 0} H^0(X, L^j)\right)$  over  $\varphi_{|L|}(X)$  need not be normal in general. See [38, Example 4.2] for details.

Put geometrically (compare Theorem 2.12), every algebraic variety that is obtained as the GIT quotient of an affine toric variety  $\text{Spec}(\mathbb{k}[S])$  by an action of  $\mathbb{k}^\times$  is a semiprojective toric variety. In fact, one can say the following.

**Proposition 3.7.** *Let  $G$  be a diagonalisable group acting on an affine toric variety  $\text{Spec}(\mathbb{k}[S])$ . Then  $\text{Spec}(\mathbb{k}[S])//_{\chi} G$  is a semiprojective toric variety for any character  $\chi \in G^*$ .*

*Proof.* Let  $\Lambda = G^*$  denote the character group of  $G$ . Theorem 2.12 shows that the  $G$ -action on  $\text{Spec}(\mathbb{k}[S])$  gives a  $\Lambda$ -grading of  $\mathbb{k}[S]$  and hence a semigroup homomorphism  $\pi: S \rightarrow \Lambda$ . For  $\chi \in \Lambda$ , the  $\mathbb{k}$ -algebra  $\bigoplus_{j \geq 0} \mathbb{k}[S]_{j\chi}$  of  $\chi$ -semi-invariant functions is the semigroup algebra of

$$S_\chi = \bigoplus_{j \geq 0} \left( S \cap \pi^{-1}(j\chi) \right).$$

The GIT quotient  $\text{Spec}(\mathbb{k}[S]) //_\chi G$  is the semiprojective toric variety  $\text{Proj}(\mathbb{k}[S_\chi])$ .  $\square$

The class of semiprojective toric varieties is an especially nice class: while it includes all affine and projective toric varieties, it also includes many varieties that arise as toric resolutions of affine toric singularities (see Example 3.25).

**Example 3.8.** For the graded ring  $\mathbb{k}[x, y, z]$  from Example 2.21 with  $\deg(x) = 1$ ,  $\deg(y) = 2$  and  $\deg(z) = 3$ , the corresponding toric variety  $\text{Proj}(\mathbb{k}[x, y, z])$  is weighted projective space  $\mathbb{P}_{\mathbb{k}}(1, 2, 3)$ . As  $t^{u_i}$  varies over the generators  $x, y$  and  $z$ , the ring  $\mathbb{k}[x, y, z][t^{-u_i}]^{\mathbb{k}^\times}$  is  $\mathbb{k}[\frac{y}{x^2}, \frac{z}{x^3}]$ ,  $\mathbb{k}[\frac{x^2}{y}, \frac{xz}{y^2}, \frac{z^2}{y^3}]$  and  $\mathbb{k}[\frac{x^3}{z}, \frac{xy}{z}, \frac{y^3}{z^2}]$  respectively. The corresponding charts on  $\mathbb{P}_{\mathbb{k}}(1, 2, 3)$  are isomorphic to the affine toric varieties  $\mathbb{A}_{\mathbb{k}}^2$ , the singularity of type  $\frac{1}{2}(1, 1)$  and the singularity of type  $\frac{1}{3}(1, 2)$ .

*Exercise 3.9.* For the graded ring  $R = \mathbb{k}[x_0, \dots, x_n]$  with  $\deg(x_i) = a_i > 0$  for  $0 \leq i \leq n$ , prove that the charts of  $\text{Proj}(R)$  corresponding to the generators  $x_0, \dots, x_n$  of  $R$  are the quotient singularities of type  $\frac{1}{a_i}(a_0, \dots, \hat{a}_i, \dots, a_n)$  for  $0 \leq i \leq n$ . In particular, weighted projective space admits a cover by affine toric varieties.

The semiprojective toric varieties in Example 3.8 and Exercise 3.9 each admit a cover by affine toric varieties. This statement holds true in general.

**Proposition 3.10.** *Let  $X$  be a semiprojective toric variety and set  $T_X := \text{Spec}(\mathbb{k}[M])$ . Then:*

- (i)  $X$  admits a cover by affine toric varieties, each containing the algebraic torus  $T_X$  as a Zariski dense open subset;
- (ii) the action of  $T_X$  on itself extends to an action on  $X$ .

*Proof.* Consider  $X = \text{Proj}(\mathbb{k}[S])$  where  $\mathbb{k}[S]$  is a  $\mathbb{Z}$ -graded semigroup algebra. For (i), choose  $u_1, \dots, u_m \in S$  so that the monomials  $t^{u_1}, \dots, t^{u_m}$  form a minimal  $\mathbb{k}[S]_0$ -algebra generating set for  $\mathbb{k}[S]$ . The Proj construction shows that  $X$  is covered by charts  $\text{Spec}(\mathbb{k}[S][t^{-u_i}]^{\mathbb{k}^\times})$  for  $1 \leq i \leq m$ . The  $\mathbb{k}$ -vector space underlying  $\mathbb{k}[S][t^{-u_i}]^{\mathbb{k}^\times}$  has basis consisting of elements  $t^u / (t^{u_i})^j$  for  $j \geq 0$  and for  $t^u \in \mathbb{k}[S]$  of degree  $j \deg(u_i)$ . It follows that  $\mathbb{k}[S][t^{-u_i}]^{\mathbb{k}^\times}$  is isomorphic to the semigroup algebra  $\mathbb{k}[S_i]$  of the subsemigroup  $S_i := \{u - ju_i \in M : u \in S, j \in \mathbb{N}\}$  of  $M$ . Since  $S$  generates  $\mathbb{Z}^d$  over  $\mathbb{Z}$ , it follows that  $S_i$  generates  $M$  over  $\mathbb{Z}$ . Therefore  $X$  admits a cover by the affine toric varieties  $\text{Spec}(\mathbb{k}[S_i])$  for  $1 \leq i \leq m$  and, since  $\mathbb{k}[S_i]$  can be localised to obtain  $\mathbb{k}[M]$ , the torus  $T_X$  is dense in  $\text{Spec}(\mathbb{k}[S_i])$  for  $1 \leq i \leq m$ . This completes the proof of (i). For (ii), since  $S_i \subset M$  we obtain by restriction an  $M$ -grading of  $\mathbb{k}[S_i]$  and hence a  $T_X$ -action on  $\text{Spec}(\mathbb{k}[S_i])$  that extends the  $T_X$ -action on itself. These actions agree where charts overlap, so the action extends to the whole of  $X$ .  $\square$

**Remark 3.11.** The existence of a Zariski dense algebraic torus in  $X$  whose action on itself extends to an action on  $X$  can be used to characterise semiprojective toric varieties.

**3.3. A dual approach via convex geometry.** In the special case when the toric variety  $X = \text{Proj}(\mathbb{k}[S])$  is normal, the affine cover by toric charts admits a simple description in terms of polyhedral cones. This approach relies on some elementary facts about duality in convex geometry that we now recall.

Let  $S \subseteq \mathbb{Z}^d$  be a finitely generated subsemigroup that generates  $\mathbb{Z}^d$  over  $\mathbb{Z}$ , and let

$$(3.2) \quad 0 \longrightarrow M \longrightarrow \mathbb{Z}^d \xrightarrow{\nu} \mathbb{Z} \longrightarrow 0$$

be a short exact sequence of lattice maps satisfying  $\nu(S) \subseteq \mathbb{N}$ . Write  $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$  for the lattice dual to  $M$  and  $\langle \cdot, \cdot \rangle: M \times N \rightarrow \mathbb{Z}$  for the dual pairing. Let  $\sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{Q}$  be a *polyhedral cone*, that is, the  $\mathbb{Q}_{\geq 0}$ -span of a finite set of vectors in  $N \otimes_{\mathbb{Z}} \mathbb{Q}$ . The *dual cone*

$$\sigma^{\vee} := \{u \in M \otimes_{\mathbb{Z}} \mathbb{Q} : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}$$

is also a polyhedral cone. A *face* of  $\sigma$  is the intersection of  $\sigma$  with a supporting hyperplane, and  $\sigma$  is *strongly convex* if  $\sigma^{\vee}$  spans  $M \otimes_{\mathbb{Z}} \mathbb{Q}$ . Gordan's Lemma asserts that the integral points  $\sigma^{\vee} \cap M$  form a finitely generated subsemigroup of  $M$ , giving rise to a normal affine toric variety  $\text{Spec}(\mathbb{k}[\sigma^{\vee} \cap M])$ . This dual approach to constructing a semigroup algebra may appear rather convoluted at first, but the cones  $\sigma \subseteq N \otimes_{\mathbb{Z}} \mathbb{Q}$  that encode the toric charts of a normal semiprojective toric variety satisfy the following nice convex-geometry property.

**Proposition 3.12.** *Every normal semiprojective toric variety  $X = \text{Proj}(\mathbb{k}[S])$  is covered by toric charts of the form  $\text{Spec}(\mathbb{k}[\sigma_i^{\vee} \cap M])$  for  $1 \leq i \leq m$ , where  $\sigma_1, \dots, \sigma_m \subseteq N \otimes_{\mathbb{Z}} \mathbb{Q}$  are strongly convex polyhedral cones such that  $\sigma_j \cap \sigma_k$  is a face of both  $\sigma_j$  and  $\sigma_k$  for  $1 \leq j, k \leq m$ .*

*Proof.* Choosing  $\ell \geq 0$  sufficiently large ensures that the  $\mathbb{Z}$ -graded  $\mathbb{k}[S]_0$ -algebra  $\bigoplus_{j \geq 0} \mathbb{k}[S]_{j\ell}$  is generated in degree  $j = 1$ , so  $X$  is covered by the charts  $\text{Spec}(\mathbb{k}[S][t^{-u}]^{\mathbb{k}^{\times}})$  as  $u$  ranges over some subset of  $S \cap \nu^{-1}(\ell)$ . To remove redundancy in this cover, let  $\text{conv}(S \cap \nu^{-1}(\ell))$  denote the polyhedron obtained as the convex hull of the points  $S \cap \nu^{-1}(\ell)$  in the vector space  $\nu^{-1}(\ell) \otimes_{\mathbb{Z}} \mathbb{Q}$ , and let  $u_1, \dots, u_m \in S$  be the vertices of  $\text{conv}(S \cap \nu^{-1}(\ell))$ . As in the proof of Proposition 3.10,  $\mathbb{k}[S][t^{-u_i}]^{\mathbb{k}^{\times}}$  is isomorphic to the semigroup algebra of  $S_i := \{u - u_i \in M : u \in S \cap \nu^{-1}(\ell)\}$ . Set  $N := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ , and for each  $1 \leq i \leq m$  define the cone  $\sigma_i := \text{cone}(S_i)^{\vee}$  in  $N \otimes_{\mathbb{Z}} \mathbb{Q}$ . Since  $X$  is normal, the semigroup  $\sigma_i^{\vee} \cap M = \text{cone}(S_i) \cap M$  is equal to  $S_i$  by Lemma 3.3 and hence  $X$  is covered by the charts  $\text{Spec}(\mathbb{k}[\sigma_i^{\vee} \cap M])$  for  $1 \leq i \leq m$ . Each cone  $\sigma_i$  is strongly convex since  $\sigma_i^{\vee} = \text{cone}(S_i)$  spans  $M \otimes_{\mathbb{Z}} \mathbb{Q}$ . We leave the proof of the final statement as a simple exercise in convex geometry.  $\square$

*Exercise 3.13.* Complete the proof of Proposition 3.12.

**Remark 3.14.** Normality of  $\text{Spec}(\mathbb{k}[S])$  implies normality of  $\text{Spec}(\mathbb{k}[S]//_{\chi}G)$  by Lemma 3.3, so Proposition 3.12 applies in particular to those varieties obtained as GIT quotients of normal affine toric varieties by the action of a diagonalisable group (see Proposition 3.7).

**Example 3.15.** Consider the action of  $T = (\mathbb{k}^{\times})^2$  on  $\mathbb{A}_{\mathbb{k}}^4 = \text{Spec}(\mathbb{k}[x_1, x_2, x_3, x_4])$  given by

$$(t_1, t_2) \cdot (p_1, p_2, p_3, p_4) = (t_1 p_1, t_1^{-1} t_2 p_2, t_1 p_3, t_2 p_4),$$

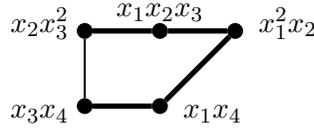
and let  $\pi: \mathbb{Z}^4 \rightarrow \mathbb{Z}^2$  be the map defined by the matrix

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

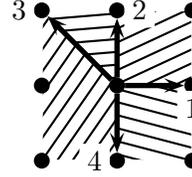
The character  $\chi = (1, 1)$  of  $(\mathbb{k}^{\times})^2$  defines the  $\mathbb{Z}$ -graded semigroup  $S_{\chi} = \bigoplus_{j \geq 0} (\mathbb{N}^4 \cap \pi^{-1}(j, j))$  whose semigroup algebra  $\mathbb{k}[S_{\chi}] = \bigoplus_{j \geq 0} \mathbb{k}[x_1, x_2, x_3, x_4]_{j\chi}$  determines the projective toric variety  $\mathbb{A}_{\mathbb{k}}^4//_{\chi}T = \text{Proj}(\mathbb{k}[S_{\chi}])$ . To compute the toric charts that cover  $\mathbb{A}_{\mathbb{k}}^4//_{\chi}T$ , note that the  $\mathbb{k}$ -algebra  $\mathbb{k}[S_{\chi}]$  is generated in degree  $j = 1$  by the five monomials shown in Figure 1(a). The algebra of  $T$ -invariant functions on  $U_1 := \mathbb{A}_{\mathbb{k}}^4 \setminus (x_3 x_4 = 0)$  is  $\mathbb{k}[U_1]^T = \mathbb{k}\left[\frac{x_1 x_4}{x_3 x_4}, \frac{x_2^2 x_2}{x_3 x_4}, \frac{x_1 x_2 x_3}{x_3 x_4}, \frac{x_2 x_3^2}{x_3 x_4}\right]$  and hence

the affine quotient is  $U_1/T \cong \text{Spec}(\mathbb{k}[\frac{x_1}{x_3}, \frac{x_2x_3}{x_4}]) \cong \mathbb{A}_{\mathbb{k}}^2$ . Computing the other charts similarly shows that the chart corresponding to the monomial  $x_1x_2x_3$  is redundant.

Alternatively, by examining the proof of Proposition 3.12 we see that the charts can be read off directly from the polytope  $\text{conv}(S_\chi \cap \nu^{-1}(1))$  shown in Figure 1(a): each vertex determines a chart, and the coordinates have exponent vectors given by the edge-vectors emanating from the corresponding vertex. For each chart  $U_i$ , the cone generated by the pair of edges vectors



(a) Polytope  $\text{conv}(S_\chi \cap \nu^{-1}(1))$



(b) Cones  $\sigma_i$  in  $N \otimes_{\mathbb{Z}} \mathbb{Q}$

FIGURE 1. The Hirzebruch surface  $\mathbb{F}_1$  as a toric variety

emanating from the vertex is the cone  $\sigma_i^\vee$  whose integral points  $\sigma_i^\vee \cap M$  define the semigroup algebra  $\mathbb{k}[\sigma_i^\vee \cap M] \cong \mathbb{k}[U_i]^T$ . The dual cone  $\sigma_i$  is generated by the inward-pointing normal vectors of  $\sigma_i^\vee$ , and forms one of the two-dimensional cones in Figure 1(b). The patching data for the local toric charts shows that  $\mathbb{A}_{\mathbb{k}}^4//_\chi T$  is the Hirzebruch surface  $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ .

*Exercise 3.16.* Repeat the analysis of Example 3.15 for each of the characters  $\chi = (1, 0)$ ,  $(0, 1)$ ,  $(-1, 2)$  and  $(1, 2)$ . Explain your findings in terms of  $\pi$  and the birational geometry of  $\mathbb{F}_1$ .

**Example 3.17.** Let  $G \subset \text{GL}(n, \mathbb{k})$  be a finite abelian subgroup of order  $r$ . Choose coordinates  $x_1, \dots, x_n$  on  $\mathbb{A}_{\mathbb{k}}^n$  to diagonalise the action and write  $g = \text{diag}(\epsilon^{\alpha_1(g)}, \dots, \epsilon^{\alpha_n(g)})$  where  $\epsilon$  is a primitive  $r^{\text{th}}$  root of unity and  $0 \leq \alpha_j(g) < r$  for  $1 \leq j \leq n$ . Define the lattice

$$N := \mathbb{Z}^n + \sum_{g \in G} \mathbb{Z} \cdot \frac{1}{r}(\alpha_1(g), \dots, \alpha_n(g))$$

and set  $M = \text{Hom}(N, \mathbb{Z})$ . A Laurent monomial  $x_1^{m_1} x_2^{m_2} \dots x_n^{m_n} \in \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  is  $G$ -invariant if and only if the exponent vector lies in  $M$ . Thus, if  $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q}$  denotes the positive orthant, then  $\mathbb{k}[\sigma^\vee \cap M] = \mathbb{k}[x_1, \dots, x_n]^G$  and hence the abelian quotient singularity  $\mathbb{A}_{\mathbb{k}}^n/G$  is the normal toric variety  $\text{Spec}(\mathbb{k}[\sigma^\vee \cap M])$  for  $N$  and  $\sigma$  as above.

**3.4. Fans and lattice polyhedra.** For a lattice  $N$  with dual lattice  $M$ , a *fan*  $\Sigma$  in  $N \otimes_{\mathbb{Z}} \mathbb{Q}$  is a finite collection of strongly convex polyhedral cones  $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q}$  such that:

- (i) for  $\sigma, \sigma' \in \Sigma$ , the cone  $\sigma \cap \sigma'$  is a face of both  $\sigma$  and  $\sigma'$ ;
- (ii) for  $\sigma \in \Sigma$ , each face of  $\sigma$  also lies in  $\Sigma$ .

We may assume that the cones in any given fan do not all lie in a hyperplane in  $N \otimes_{\mathbb{Z}} \mathbb{Q}$ .

Each cone  $\sigma \in \Sigma$  determines the normal toric variety  $U_\sigma = \text{Spec}(\mathbb{k}[\sigma^\vee \cap M])$  with dense algebraic torus  $\text{Spec}(\mathbb{k}[M])$ . Moreover, each face of  $\sigma$  is of the form  $\tau = \sigma \cap u^\perp$  for some  $u \in M$ , and one can show that  $\mathbb{k}[\tau^\vee \cap M]$  is the localisation of  $\mathbb{k}[\sigma^\vee \cap M]$  at  $t^u$ . It follows that  $U_\tau$  is a principal open affine subset of  $U_\sigma$ , and these normal affine varieties glue to give

$$X_\Sigma := \bigsqcup_{\sigma \in \Sigma} U_\sigma / \sim_{\text{glue}} .$$

We work only with fans arising from lattice polyhedra as follows. For an  $n$ -dimensional lattice polyhedron  $P \subseteq \mathbb{Q}^n$  and for  $w \in (\mathbb{Q}^n)^*$  we denote by  $\text{face}_w(P)$  the face of  $P$  minimising  $w$ . Given a face  $F$  of  $P$ , the *inner normal cone*  $\mathcal{N}_P(F)$  is the set of  $w \in (\mathbb{Q}^n)^*$  such that  $\text{face}_w(P) = F$ ,

and the *inner normal fan*  $\mathcal{N}(P)$  of  $P$  is the fan consisting of the inner normal cones  $\{\mathcal{N}_P(F)\}$  as  $F$  varies over the faces of  $P$ .

**Proposition 3.18.** *Let  $X_\Sigma$  be a normal toric variety. Then  $X_\Sigma$  is semiprojective if and only if  $\Sigma$  is the inner normal fan of a lattice polyhedron.*

To construct the semiprojective toric variety directly from the polyhedron  $P \subseteq \mathbb{Q}^n$  let  $\tilde{P}$  be the polyhedral cone over  $P$  obtained as the closure in  $\mathbb{Q}^n \oplus \mathbb{Q}$  of the set  $\{(\lambda p, \lambda) : p \in P, \lambda \in \mathbb{Q}_{\geq 0}\}$ . The semigroup  $S_P := \tilde{P} \cap (\mathbb{Z}^n \oplus \mathbb{Z})$  determines the normal affine toric variety  $\text{Spec}(\mathbb{k}[S_P])$  of dimension  $n+1$ . The second projection  $\nu: \mathbb{Z}^n \oplus \mathbb{Z} \rightarrow \mathbb{Z}$  gives  $S_P$  a  $\mathbb{Z}$ -grading and hence defines the normal semiprojective toric variety  $X_P := \text{Proj}(\mathbb{k}[S_P])$  of dimension  $n$ . The polyhedron  $P$  is equal to  $\text{conv}(S_P \cap \nu^{-1}(1))$ , the convex hull of the lattice points  $S_P \cap \nu^{-1}(1)$  in the vector space  $\nu^{-1}(1) \otimes_{\mathbb{Z}} \mathbb{Q}$ . The  $n$ -dimensional cones  $\mathcal{N}_P(F)$  as  $F$  ranges over the vertices of  $P$  coincide with the cones  $\sigma_1, \dots, \sigma_m$  from Proposition 3.12 that determine an affine toric cover of  $X_P$ . More generally, as  $F$  ranges over the faces of  $P$  we produce the fan  $\Sigma = \mathcal{N}(P)$  that encodes the data to reconstruct toric variety  $X_P$ . It is worth recording two special cases:

**Example 3.19.** If  $P \subseteq \mathbb{Q}^n$  is a polyhedral cone then its normal fan consists of the faces of the dual cone  $\sigma := P^\vee \subset N_{\mathbb{Q}}$ . The toric variety  $X_P$  is  $\text{Spec}(\mathbb{k}[\sigma^\vee \cap M])$  for  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ .

**Example 3.20.** If  $P \subseteq \mathbb{Q}^n$  is a polytope (= bounded polyhedron) then  $S_P \cap \nu^{-1}(0)$  is zero. Thus, the zero-graded piece of the algebra  $\mathbb{k}[S_P]$  is  $\mathbb{k}$ , so  $X_P$  is projective.

**Remark 3.21.** Geometric properties of a normal toric variety  $X$  are encoded by its fan  $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q}$ . An  $n$ -dimensional toric variety  $X$  is smooth if and only if the generators of each  $n$ -dimensional cone  $\sigma \in \Sigma$  form a  $\mathbb{Z}$ -basis of the lattice  $N$ . Moreover,  $X$  is an orbifold if each  $n$ -dimensional cone has precisely  $n$  cone generators, in which case the fan  $\Sigma$  is *simplicial*. For more on the relation between a normal toric variety and its fan, see Fulton [35] or Oda [59].

*Exercise 3.22.* A very simple class of toric varieties is the class of smooth toric del Pezzo surfaces of which there are only five examples:  $\mathbb{P}^1 \times \mathbb{P}^1$ ,  $\mathbb{P}^2$  and  $\mathbb{P}^2$  blown-up in one, two, or three points; the fans are shown in Figures 1 and 2. Show that each fan can be realised as the inner normal fan of a *reflexive* lattice polygon, that is, a lattice polygon  $P$  such that the origin is the only lattice point in the interior of  $P$ .

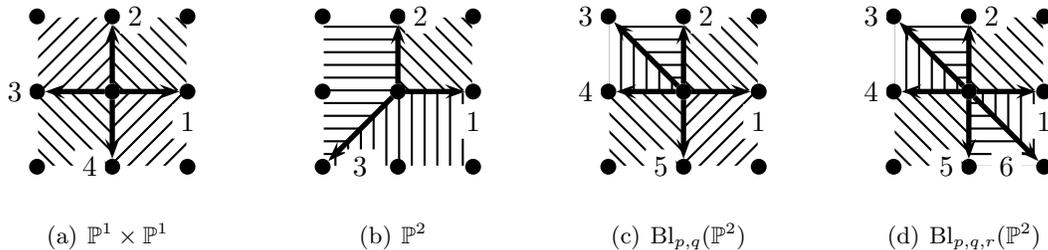


FIGURE 2. Fans for the remaining four smooth toric del Pezzo surfaces

**3.5. Cox's construction.** Every semiprojective toric variety  $X$  is by construction the GIT quotient of an affine toric variety by an action of  $\mathbb{k}^\times$ . A result of Cox [23] asserts that if  $X$  is normal then it can be constructed as a GIT quotient of affine space  $\mathbb{A}_{\mathbb{k}}^n$ , though one must replace the  $\mathbb{k}^\times$ -action by that of a diagonalisable group  $G_\Lambda$ . This construction is ubiquitous in the study of toric varieties, and it is simple to describe in the semiprojective case.

For a normal semiprojective toric variety  $X$  of dimension  $n$  with fan  $\Sigma$ , write  $\Sigma(1)$  for the set of *rays* (one-dimensional cones) in  $\Sigma$ . We associate to each ray  $\rho \in \Sigma(1)$ , a  $T_X$ -invariant Weil divisor  $D_\rho$  in  $X$  as follows. For each cone  $\sigma$  containing  $\rho$ , the hyperplane  $\rho^\perp$  dual to  $\rho$  cuts out a facet of  $\sigma^\vee$ . This facet determines a divisor in  $U_\sigma$  (given by the vanishing of the function on  $U_\sigma$  defined by the inward-pointing normal vector), and  $D_\rho$  is the divisor in  $X_\Sigma$  obtained by gluing these divisors together. These divisors generate the free abelian group  $\mathbb{Z}^{\Sigma(1)}$  of  $T_X$ -invariant (Weil) divisors, and there is an exact sequence

$$(3.3) \quad 0 \longrightarrow M \xrightarrow{\text{div}} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\pi} A_{n-1}(X) \longrightarrow 0$$

where  $M := \text{Hom}(N, \mathbb{Z})$  is the dual lattice. Here,  $\text{div}(u) = \sum_{\rho \in \Sigma(1)} \langle u, v_\rho \rangle D_\rho$ , where  $v_\rho$  is the primitive lattice point on the cone  $\rho$ .

The *total coordinate ring* of  $X$  is the polynomial ring  $R_X := \mathbb{k}[x_\rho : \rho \in \Sigma(1)]$  obtained as the semigroup algebra of the semigroup  $\mathbb{N}^{\Sigma(1)}$  of  $T_X$ -invariant effective divisors. For each cone  $\sigma \in \Sigma$ , write  $\hat{\sigma}$  for the set of one-dimensional cones in  $\Sigma$  that are not contained in  $\sigma$ , and consider the *irrelevant ideal*

$$B_X := \left( \prod_{\rho \in \hat{\sigma}} x_\rho \in R_X : \sigma \in \Sigma(n) \right)$$

whose generators are indexed by the top dimensional cones in  $\Sigma$ . The map  $\pi$  from (3.3) induces a grading of  $R_X$  by the finitely generated  $\mathbb{Z}$ -module  $A_{n-1}(X)$ , where  $\deg(x^u) := \pi(u)$ . The diagonalisable group  $G := \text{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{k}^\times)$  acts on the affine space  $\mathbb{A}_{\mathbb{k}}^{\Sigma(1)} = \text{Spec}(R_X)$  by Theorem 2.12, and the ideal  $B_X$  is homogeneous with respect to this action.

**Theorem 3.23** (Cox [23]). *Let  $X$  be a normal semiprojective toric variety with simplicial fan  $\Sigma$ . Then  $X$  is isomorphic to:*

- (i) *the GIT quotient  $\mathbb{A}_{\mathbb{k}}^{\Sigma(1)} //_L G$  for any relatively ample line bundle  $L$  on  $X$ ;*
- (ii) *the geometric quotient of  $\mathbb{A}_{\mathbb{k}}^{\Sigma(1)} \setminus \mathbb{V}(B_X)$  by the action of  $G$ .*

*Proof.* Since the character group of  $G$  is  $A_{n-1}(X)$ , we may regard the line bundle  $L$  as a character of  $G$  and hence  $\mathbb{A}_{\mathbb{k}}^{\Sigma(1)} //_L G$  is well defined. For any  $j \in \mathbb{N}$ , the set  $\mathbb{N}^{\Sigma(1)} \cap \pi^{-1}(L^{\otimes j})$  coincides with the set of effective  $T_X$ -invariant divisors  $D$  in  $X$  satisfying  $\mathcal{O}_X(D) = L^{\otimes j}$ . It follows that the  $L^{\otimes j}$ -graded piece of ring  $R_X$  is  $H^0(X, L^{\otimes j})$ , and hence

$$\bigoplus_{j \geq 0} H^0(X, L^{\otimes j}) \cong \bigoplus_{j \geq 0} (R_X)_{L^{\otimes j}}.$$

Since  $L$  is relatively ample,  $\text{Proj}$  of the left hand side is equal to  $X$ . Part (i) follows since  $\text{Proj}$  of the right hand side is the GIT quotient  $\mathbb{A}_{\mathbb{k}}^{\Sigma(1)} //_L G$ . To establish the isomorphism between (i) and (ii), we assume without loss of generality that  $L$  is (relatively) very ample. Then for each toric chart  $U_\sigma$  in the cover of  $X$ , the corresponding face of the polyhedron  $\text{conv}(S \cap \pi^{-1}(L))$  defines a section  $s_\sigma \in H^0(X, L)$  obtained as a product  $s_\sigma = \prod_{\rho \in \hat{\sigma}} x_\rho^{m_\rho} \in R_X$  for  $m_\rho > 0$ . The GIT quotient  $\mathbb{A}_{\mathbb{k}}^{\Sigma(1)} //_L G$  is the geometric quotient of the  $L$ -stable locus  $\mathbb{A}_{\mathbb{k}}^{\Sigma(1)} \setminus \mathbb{V}((s_\sigma : \sigma \in \Sigma))$  by the action of  $G$ . The result follows since the radical of the ideal  $(s_\sigma : \sigma \in \Sigma)$  is  $B_X$ .  $\square$

**Example 3.24.** The toric fan of  $X = \mathbb{F}_1$  is shown in Figure 1. For  $i = 1, \dots, 4$ , write  $D_i$  for the toric divisor corresponding to the ray indexed by  $i$ . For the standard basis  $e_1, e_2 \in \mathbb{Z}^2 = M$ , we have  $\text{div}(e_1) = D_1 - D_3$  and  $\text{div}(e_2) = D_2 + D_3 - D_4$ . If we choose  $\mathcal{O}_X(D_1)$  and  $\mathcal{O}_X(D_4)$  as

the basis for  $\text{Pic}(X) \cong A_1(X)$ , then the sequence (3.3) becomes

$$(3.4) \quad 0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow 0.$$

The line bundle  $L = \mathcal{O}_X(D_1 + D_4)$  is very ample. The monomials shown in Figure 1(a) form both a  $\mathbb{k}$ -vector space basis for the  $L$ -graded piece of  $R_X = \mathbb{k}[x_1, x_2, x_3, x_4]$  and a set of  $\mathbb{k}$ -algebra generators of  $\bigoplus_{j \geq 0} (R_X)_L^{\otimes j}$ . The GIT diagram (2.5) shows that these  $\mathbb{k}$ -algebra generators cut out the  $L$ -unstable locus, so  $\mathbb{A}_{\mathbb{k}}^4 //_L G$  is the categorical quotient of  $\mathbb{A}_{\mathbb{k}}^4 \setminus \mathbb{V}(J)$  by the action of  $G$  for the ideal  $J = (x_1x_3, x_1x_4, x_1^2x_2, x_1x_2x_3, x_2x_3^2)$ . To compute  $\mathbb{V}(J)$  we may of course replace  $J$  by its radical, namely  $\text{rad}(J) = (x_1x_3, x_1x_4, x_1x_2, x_2x_3)$ . In this case, the result boils down to the observation that  $\text{rad}(J) = B_X$ .

**Example 3.25.** Consider the toric variety  $X$  whose fan  $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q}$  is shown in Figure 3(a), where  $N = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{3}(1, 2)$ . It follows from Remark 3.21 that  $X$  is smooth. If we list the rays  $\rho_0, \dots, \rho_3$  as shown then  $\mathcal{O}_X(D_0)$  and  $\mathcal{O}_X(D_3)$  provide a basis for  $\text{Pic}(X) \cong A_1(X)$ . The lattice

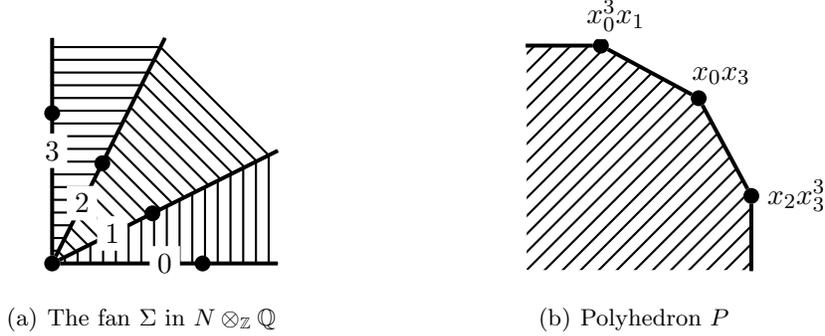


FIGURE 3. The minimal resolution of the singularity of type  $\frac{1}{3}(1, 2)$

$M$  dual to  $N$  is the sublattice of  $(\mathbb{Z}^2)^\vee$  consisting of the exponents of monomials in  $\mathbb{k}[x, y]$  that are invariant under the group action of type  $\frac{1}{3}(1, 2)$  described in Section 3.1. If we choose the exponents of the monomials  $x^3, xy$  as a  $\mathbb{Z}$ -basis for  $M$  then sequence (3.3) becomes

$$(3.5) \quad 0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}} \mathbb{Z}^2 \longrightarrow 0.$$

In this case,  $L = \mathcal{O}_X(D_0 + D_3)$  is very ample, and the monomials in the total coordinate ring corresponding to the vertices of the polyhedron  $P = \text{conv}(\mathbb{N}^4 \cap \pi^{-1}(L))$  appear in Figure 3(b). The GIT quotient  $\mathbb{A}_{\mathbb{k}}^4 //_L G$  is the categorical quotient of  $\mathbb{A}_{\mathbb{k}}^4 \setminus \mathbb{V}(x_0^3x_1, x_0x_3, x_2x_3^3)$  by the action of  $G$ . Again, the radical of the ideal is the irrelevant ideal  $B_X = (x_0x_1, x_0x_3, x_2x_3)$ .

*Exercise 3.26.* Show that the affine quotient  $\mathbb{A}_{\mathbb{k}}^4 //_0 G \cong \text{Spec}(\mathbb{k}[\mathbb{N}^4 \cap M])$  is isomorphic to the cyclic quotient singularity  $\mathbb{A}_{\mathbb{k}}^2 / (\mathbb{Z}/3)$ . Thus, the natural morphism  $\mathbb{A}_{\mathbb{k}}^4 //_L G \rightarrow \mathbb{A}_{\mathbb{k}}^4 //_0 G$  obtained by variation of GIT quotient is the minimal resolution  $X \rightarrow \mathbb{A}_{\mathbb{k}}^2 / (\mathbb{Z}/3)$ .

#### 4. MODULI SPACES OF QUIVER REPRESENTATIONS

We now turn our attention to quivers and to the abelian category of quiver representations. Certain toric varieties, known as toric quiver varieties, arise naturally as fine moduli spaces for

representations of a quiver with a fixed dimension vector; the key to this construction is King's notion of  $\theta$ -stability. We extend the moduli construction to the case of bound quivers (= quivers with relations), and indicate some important examples which illustrate that the introduction of relations makes the study of the resulting moduli spaces much more delicate.

**4.1. On quivers, path algebras and the incidence map.** A *quiver*  $Q$  is a directed graph given by a set  $Q_0$  of vertices, a set  $Q_1$  of arrows, and maps  $t, h: Q_1 \rightarrow Q_0$  that specify the tail and head of each arrow. We assume that both  $Q_0, Q_1$  are finite sets, and that  $Q$  is connected, i.e., the graph obtained by forgetting the orientation of arrows in  $Q$  is connected. A nontrivial *path*  $p$  in  $Q$  of length  $\ell \in \mathbb{N}$  from vertex  $i \in Q_0$  to vertex  $j \in Q_0$  is a sequence of arrows  $a_1 \cdots a_\ell$  with  $h(a_k) = t(a_{k+1})$  for  $1 \leq k < \ell$ ; we set  $t(p) := t(a_1)$  and  $h(p) := h(a_\ell)$ . In addition, each vertex  $i \in Q_0$  gives a trivial path  $e_i$  of length zero with  $t(e_i) = h(e_i) = i$ . A cycle is a nontrivial path with the same head and tail, and  $Q$  is *acyclic* if it contains no cycles. A *tree* is a connected acyclic quiver.

Let  $\mathbb{k}$  be a field and  $Q$  a finite, connected quiver. The *path algebra*  $\mathbb{k}Q$  of  $Q$  is the associative  $\mathbb{k}$ -algebra whose underlying  $\mathbb{k}$ -vector space has as basis the set of paths in  $Q$ , where the product of basis elements equals the basis element defined by concatenation of the paths if possible, or zero otherwise. The identity element in  $\mathbb{k}Q$  is  $\sum_{i \in Q_0} e_i$ , where  $e_i$  is the trivial path of length zero at  $i \in Q_0$ . Note that  $\mathbb{k}Q$  has finite dimension over  $\mathbb{k}$  if and only if  $Q$  contains no cycles. The algebra  $\mathbb{k}Q$  is graded by path length, and the zero-graded subring  $(\mathbb{k}Q)_0 \subset \mathbb{k}Q$  spanned by the trivial paths  $e_i$  for  $i \in Q_0$  is a semisimple ring in which the elements  $e_i$  are orthogonal idempotents.

**Example 4.1.** The *Beilinson quiver* for  $\mathbb{P}^n$  is the quiver with  $n + 1$  vertices  $Q_0 = \{0, 1, \dots, n\}$  that has  $n + 1$  arrows from the  $i$ th vertex to the  $(i + 1)$ -st vertex for each  $i = 0, 1, \dots, n - 1$ . If we augment  $Q$  by adding  $n + 1$  arrows from the  $n$ th vertex to the 0th vertex then we obtain a cyclic quiver, namely the *McKay quiver* for the group action of type  $\frac{1}{n+1}(1, 1, \dots, 1)$  on  $\mathbb{A}_{\mathbb{k}}^{n+1}$ . For more on these quivers, see Examples 4.5 and 4.7.

The characteristic functions  $\chi_i: Q_0 \rightarrow \mathbb{Z}$  for  $i \in Q_0$  and  $\chi_a: Q_1 \rightarrow \mathbb{Z}$  for  $a \in Q_1$  form the standard integral bases of the vertex space  $\mathbb{Z}^{Q_0}$  and the arrows space  $\mathbb{Z}^{Q_1}$  respectively. The *weight lattice* of  $Q$  is the sublattice of  $\mathbb{Z}^{Q_0}$  given by

$$\text{Wt}(Q) := \left\{ \sum_{i \in Q_0} \theta_i \chi_i \in \mathbb{Z}^{Q_0} : \sum_{i \in Q_0} \theta_i = 0 \right\}.$$

Since  $Q$  is connected, the incidence map  $\text{inc}: \mathbb{Z}^{Q_1} \rightarrow \mathbb{Z}^{Q_0}$  defined by  $\text{inc}(\chi_a) = \chi_{h(a)} - \chi_{t(a)}$  determines a short exact sequence

$$(4.1) \quad 0 \longrightarrow \text{Cir}(Q) \longrightarrow \mathbb{Z}^{Q_1} \xrightarrow{\text{inc}} \text{Wt}(Q) \longrightarrow 0$$

where  $\text{Cir}(Q) \subseteq \mathbb{Z}^{Q_1}$  denotes that sublattice of elements  $f = \sum_{a \in Q_1} f_a \chi_a \in \mathbb{Z}^{Q_1}$  such that

$$\sum_{\{a \in Q_1: t(a)=i\}} f_a = \sum_{\{a \in Q_1: h(a)=i\}} f_a$$

holds for each vertex  $i \in Q_0$ . The incidence map plays a central role in what follows.

**4.2. The category of quiver representations.** As is typical in representation theory, our interest lies not just with  $\mathbb{k}Q$ , but with modules over  $\mathbb{k}Q$ . To help us visualise these modules we use the terminology of quiver representations.

A *representation of the quiver*  $Q$  consists of a  $\mathbb{k}$ -vector space  $W_i$  for each  $i \in Q_0$  and a  $\mathbb{k}$ -linear map  $w_a: W_{t(a)} \rightarrow W_{h(a)}$  for each  $a \in Q_1$ . More compactly, we write  $W = ((W_i)_{i \in Q_0}, (w_a)_{a \in Q_1})$ .

A representation is finite dimensional if each vector space  $W_i$  has finite dimension over  $\mathbb{k}$ , and the dimension vector of  $W$  is the tuple of nonnegative integers  $(\dim_{\mathbb{k}} W_i)_{i \in Q_0}$ . The *support* of a representation  $W$  is the quiver with vertex set  $\{i \in Q_0 : W_i \neq 0\}$  and arrow set  $\{a \in Q_1 : w_a \neq 0\}$ . A map between representations  $W$  and  $W'$  is a family  $\psi_i: W_i \rightarrow W'_i$  for  $i \in Q_0$  of  $\mathbb{k}$ -linear maps that are compatible with the structure maps, that is, such that the diagrams

$$\begin{array}{ccc} W_{t(a)} & \xrightarrow{w_a} & W_{h(a)} \\ \downarrow \psi_{t(a)} & & \downarrow \psi_{h(a)} \\ W'_{t(a)} & \xrightarrow{w'_a} & W'_{h(a)} \end{array}$$

commute for all  $a \in Q_1$ . With composition defined componentwise, we obtain the category of finite-dimensional representations of  $Q$ , denoted  $\text{rep}_{\mathbb{k}}(Q)$ .

**Proposition 4.2.** *The category  $\text{rep}_{\mathbb{k}}(Q)$  is equivalent to the category of finitely-generated left  $\mathbb{k}Q$ -modules. In particular,  $\text{rep}_{\mathbb{k}}(Q)$  is an abelian category.*

*Proof.* Let  $W = ((W_i)_{i \in Q_0}, (w_a)_{a \in Q_1})$  be a finite-dimensional representation of  $Q$ . Define a  $\mathbb{k}Q$ -module structure on the  $\mathbb{k}$ -vector space  $M := \bigoplus_{i \in Q_0} W_i$  by extending linearly from

$$e_i m = \begin{cases} m & \text{for } m \in W_i, \\ 0 & \text{for } m \in W_j \text{ with } j \neq i, \end{cases} \quad \text{and} \quad a \cdot m = \begin{cases} w_a(m_{t(a)}) & \text{for } m \in W_{t(a)}, \\ 0 & \text{for } m \in W_j \text{ with } j \neq t(a), \end{cases}$$

for  $i \in Q_0$  and  $a \in Q_1$ . In the opposite direction, associate to each left  $\mathbb{k}Q$ -module  $M$  the quiver representation with  $W_i := e_i M$  for  $i \in Q_0$  and maps  $w_a: W_{t(a)} \rightarrow W_{h(a)}$  for  $a \in Q_1$  satisfying  $w_a(m) = a \cdot m$ . These operations are inverse to each other, and maps of representations of  $Q$  correspond to  $\mathbb{k}Q$ -module homomorphisms.  $\square$

**Remark 4.3.** This equivalence gives the notion of a subrepresentation of a quiver representation  $W$ , and the proof shows that quiver representations of dimension vector  $(\dim_{\mathbb{k}} W_i)_{i \in Q_0}$  determine left  $\mathbb{k}Q$ -modules that are isomorphic as  $\bigoplus_{i \in Q_0} \mathbb{k}e_i$ -modules to  $\bigoplus_{i \in Q_0} (\mathbb{k}^{\dim_{\mathbb{k}}(W_i)})e_i$ .

For an algebra  $A$ , let  $\text{mod}(A)$  denote the category of finitely generated left  $A$ -modules. If  $A^{\text{op}}$  denotes the opposite algebra where the product satisfies  $a \cdot b := ba$ , then  $\text{mod}(A^{\text{op}})$  is the category of finitely generated right  $A$ -modules. If  $Q^{\text{op}}$  denotes the quiver obtained from  $Q$  by reversing the orientation of the arrows then  $(\mathbb{k}Q)^{\text{op}} \cong \mathbb{k}(Q^{\text{op}})$ .

**4.3. Bound quiver representations.** In most geometric contexts, the algebra of interest is not isomorphic to the path algebra of a quiver  $Q$ , but is isomorphic to the quotient of a path algebra by an ideal of relations (see Section 5.1 for the geometric construction).

Formally, a *relation* in a quiver  $Q$  (with coefficients in  $\mathbb{k}$ ) is a  $\mathbb{k}$ -linear combination of paths of length at least two, each with the same head and the same tail. Any finite set of relations  $\varrho$  in  $Q$  determines a two-sided ideal  $\langle \varrho \rangle$  in the algebra  $\mathbb{k}Q$ . A *bound quiver*  $(Q, \varrho)$ , or equivalently a *quiver with relations*, is a quiver  $Q$  together with a finite set of relations  $\varrho$ . A *representation of the bound quiver*  $(Q, \varrho)$  is a representation of  $Q$  where each relation is satisfied in the sense that the corresponding  $\mathbb{k}$ -linear combination of homomorphisms between the vector spaces  $(W_i)_{i \in Q_0}$  should give the zero map. As before, finite-dimensional representations of  $(Q, \varrho)$  form a category denoted  $\text{rep}_{\mathbb{k}}(Q, \varrho)$ . The presence of relations  $\varrho$  in a quiver  $Q$  leads to a refinement of the equivalence of abelian categories constructed in Proposition 4.2.

**Proposition 4.4.** *The category  $\text{rep}_{\mathbb{k}}(Q, \varrho)$  is equivalent to the category of left  $\mathbb{k}Q/\langle \varrho \rangle$ -modules.*

*Proof.* See [3, Theorem III.1.6] for a proof.  $\square$

**Example 4.5** (The bound Beilinson quiver). For  $n \geq 2$ , the Beilinson quiver for  $\mathbb{P}_{\mathbb{k}}^n$  described in Example 4.1 admits a set of relations that has geometric significance. List the  $n + 1$  arrows from the  $i$ th vertex to the  $(i + 1)$ st vertex as  $a_{i,0}, \dots, a_{i,n}$  and define

$$\varrho = \{a_{i,j}a_{(i+1),\ell} - a_{i,\ell}a_{(i+1),j} : 0 \leq j < k \leq n, 0 \leq i \leq n - 1\}.$$

Since the length two paths  $a_{i,j}a_{(i+1),\ell}$  and  $a_{i,\ell}a_{(i+1),j}$  share tail at vertex  $i$  and head at vertex  $i + 1$ , the difference defines a relation for  $0 \leq j < k \leq n$  and  $0 \leq i \leq n - 1$ . The quotient  $\mathbb{k}Q/\langle\varrho\rangle$  is isomorphic as a  $\mathbb{k}$ -algebra to the endomorphism algebra of the vector bundle  $\bigoplus_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}_{\mathbb{k}}^n}(i)$  on  $\mathbb{P}_{\mathbb{k}}^n$ .

*Exercise 4.6.* Prove the assertion from Example 4.5, namely that the quotient algebra  $\mathbb{k}Q/\langle\varrho\rangle$  is isomorphic to the endomorphism algebra of  $\bigoplus_{0 \leq i \leq n} \mathcal{O}_{\mathbb{P}_{\mathbb{k}}^n}(i)$ .

**Example 4.7** (The bound McKay quiver). Let  $G \subset \mathrm{GL}(n, \mathbb{k})$  be a finite abelian subgroup. Irreducible representations of  $G$  are one-dimensional and hence define elements of the character group  $G^* := \mathrm{Hom}(G, \mathbb{k}^\times)$ . The given representation then decomposes as  $\mathbb{A}_{\mathbb{k}}^n = \rho_1 \oplus \dots \oplus \rho_n$  for some  $\rho_i \in G^*$ . The *McKay quiver* of  $G \subset \mathrm{GL}(n, \mathbb{k})$  is the quiver with a vertex for each character  $\rho \in G^*$ , and an arrow  $a_i^\rho$  from  $\rho\rho_i := \rho \otimes \rho_i$  to  $\rho$  for each  $\rho \in G^*$  and  $1 \leq i \leq n$ . If we label the arrow  $a_i^\rho$  by the monomial  $x_i$ , then the set

$$\varrho = \{a_i^{\rho \otimes \rho_j} a_j^\rho - a_j^{\rho \otimes \rho_i} a_i^\rho : \rho \in G^*, 1 \leq i, j \leq n\}$$

of relations in  $\mathbb{k}Q$  corresponds to the condition that the labelling monomials commute. We call  $(Q, \varrho)$  the *bound McKay quiver* of the abelian subgroup  $G \subset \mathrm{GL}(n, \mathbb{k})$ . It is well known (see Craw–Maclagan–Thomas [29]) that the quotient algebra  $\mathbb{k}Q/\langle\varrho\rangle$  is isomorphic to the *skew group algebra*  $\mathbb{k}[x_1, \dots, x_n] * G$ , i.e. to the free  $\mathbb{k}[x_1, \dots, x_n]$ -module with basis  $G$ , where the ring structure is given by

$$(sg) \cdot (s'g') := s(g \cdot s')gg'$$

for  $s, s' \in \mathbb{k}[x_1, \dots, x_n]$  and  $g, g' \in G$ . Under this isomorphism, the semisimple subalgebra  $\bigoplus_{i \in Q_0} \mathbb{k}e_i$  that plays a role in Remark 4.3 is isomorphic to the group algebra  $\mathbb{k}[G]$ .

**Remark 4.8.** Since these lectures were presented, Bocklandt–Schedler–Wemyss [8] provided an explicit description of an ideal of relations  $\varrho$  in the path algebra of the McKay quiver  $Q$  arising from an arbitrary finite subgroup  $G \subset \mathrm{SL}(n, \mathbb{k})$  for which  $\mathbb{k}Q/\langle\varrho\rangle$  is Morita equivalent to the skew group algebra  $\mathbb{k}[x_1, \dots, x_n] * G$ .

**4.4. Toric quiver varieties.** We now restrict attention to those representations  $W$  of a finite, connected quiver  $Q$  with dimension vector  $(1, 1, \dots, 1) \in \mathbb{Z}^{Q_0}$ . Such representations correspond via the equivalence of categories from Proposition 4.2 to finitely generated left  $\mathbb{k}Q$ -modules that are isomorphic as  $\bigoplus_{i \in Q_0} \mathbb{k}e_i$ -modules to  $\bigoplus_{i \in Q_0} \mathbb{k}e_i$ . For simplicity, we sometimes refer to such modules as *Q-constellations*.

Let  $W = ((W_i)_{i \in Q_0}, (w_a)_{a \in Q_1})$  be a representation of  $Q$ . If we pick a basis for each one-dimensional vector space  $W_i$  ( $0 \leq i \leq r$ ) then the representation space is

$$\mathbb{A}_{\mathbb{k}}^{Q_1} = \mathrm{Spec}(\mathbb{k}[y_a : a \in Q_1]) \cong \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{k}}(W_{t(a)}, W_{h(a)}).$$

The isomorphism classes of such representations are orbits by the action of the change of basis group  $(\mathbb{k}^\times)^{Q_0} \cong \prod_{i \in Q_0} \mathrm{GL}(W_i)$ , where  $t = (t_i)_{i \in Q_0}$  acts on  $w = (w_a)_{a \in Q_1}$  as

$$(t \cdot w)_a = t_{h(a)} w_a t_{t(a)}^{-1}.$$

The weights of this action are encoded by the incidence map (4.1), and hence the corresponding  $\mathbb{Z}^{Q_0}$ -grading of  $\mathbb{k}[y_a : a \in Q_1]$  is obtained from the semigroup homomorphism  $\text{inc}: \mathbb{N}^{Q_1} \rightarrow \mathbb{Z}^{Q_0}$ . Since the image of the incidence map is the weight lattice  $\text{Wt}(Q)$ , the algebra  $\mathbb{k}[y_a : a \in Q_1]$  admits only a grading by  $\text{Wt}(Q)$ . Geometrically, the diagonal one-dimensional subtorus of  $(\mathbb{k}^\times)^{Q_0}$  acts trivially, leading to a faithful action of the quotient torus  $T := \text{Hom}_{\mathbb{Z}}(\text{Wt}(Q), \mathbb{k}^\times) \cong (\mathbb{k}^\times)^{Q_0} / \mathbb{k}^\times$  on  $\mathbb{A}_{\mathbb{k}}^{Q_1}$ . The effective characters  $\theta \in T^* = \text{Wt}(Q)$  are those in the subsemigroup  $\text{inc}(\mathbb{N}^{Q_1})$ , and for any such character the GIT quotient

$$\mathbb{A}_{\mathbb{k}}^{Q_1} //_{\theta} T = \text{Proj} \left( \bigoplus_{j \geq 0} \mathbb{k}[y_a : a \in Q_1]_{j\theta} \right)$$

is a normal, semiprojective toric variety by Proposition 3.7 and Remark 3.14.

*Exercise 4.9.* Use sequence (4.1) to prove that  $\mathbb{A}_{\mathbb{k}}^{Q_1} //_{\theta} T$  is projective if and only if  $Q$  is acyclic.

Recall from Section 2.6 that a weight  $\theta \in \text{Wt}(Q)$  is *generic* if every  $\theta$ -semistable point of  $\mathbb{A}^{Q_1}$  is  $\theta$ -stable. Generalising only slightly the work of Hille [39], we call

$$\mathcal{M}_{\theta}(Q) := \mathbb{A}_{\mathbb{k}}^{Q_1} //_{\theta} T$$

a *toric quiver variety* whenever  $\theta \in \text{Wt}(Q)$  is generic. As Hille remarks, the normal toric variety  $\mathcal{M}_{\theta}(Q)$  is smooth, and in our case it is projective over  $\text{Spec}(\mathbb{k}[\mathbb{N}^{Q_1} \cap \text{Cir}(Q)])$ . Moreover, the exact sequence (4.1) arising from the incidence map of  $Q$  coincides with the sequence (3.3) from the Cox construction of  $\mathcal{M}_{\theta}(Q)$  as a toric variety, giving  $\text{Wt}(Q) = \text{Pic}(\mathcal{M}_{\theta}(Q))$ .

**Example 4.10.** For the quiver  $Q$  from Figure 4, consider the weight  $\vartheta := (-2, 1, 1) \in \text{Wt}(Q)$ . The algebra  $\bigoplus_{j \geq 0} \mathbb{k}[y_1, y_2, y_3, y_4]_{j\vartheta}$  is the semigroup algebra of  $S_{\vartheta} = \bigoplus_{j \geq 0} (\mathbb{N}^4 \cap \text{inc}^{-1}(j\vartheta))$

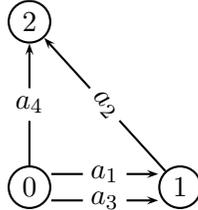


FIGURE 4. A quiver defining the Hirzebruch surface  $\mathbb{F}_1$

which is generated in degree  $j = 1$ . The computation of the set  $\mathbb{N}^4 \cap \text{inc}^{-1}(\vartheta)$  should be familiar from Example 3.15. Indeed, the incidence matrix of the quiver  $Q$  is obtained by adding a row at the top of the matrix from that example, where the new entry in each column is added to ensure that the column sum is zero. The parameter  $\chi = (1, 1)$  from Example 3.15 corresponds to  $\vartheta = (-2, 1, 1)$  here, and hence  $\mathcal{M}_{\vartheta}(Q) \cong \mathbb{F}_1$ .

*Exercise 4.11.* For the weight  $\vartheta = (-2, 1, 1)$ , identify the toric quiver variety defined by the quiver  $Q$  with incidence matrix

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

[The solution to this exercise does appear later in these notes.]

**4.5. Moduli spaces of  $\theta$ -stable representations.** The importance of toric quiver varieties  $\mathcal{M}_\theta(Q)$  becomes apparent only after establishing that each represents a functor. To describe this construction we first recall King's notion of  $\theta$ -stability. Let  $\theta \in T^* \otimes_{\mathbb{Z}} \mathbb{Q}$  be a fractional character that is not necessarily generic. A representation  $W$  of  $Q$  with dimension vector  $(1, \dots, 1) \in \mathbb{Z}^{Q_0}$  is said to be  $\theta$ -semistable if  $\theta(W) = 0$  and if for every proper, nonzero subrepresentation  $W' \subset W$  we have  $\theta(W') \geq 0$ . The notion of  $\theta$ -stability is obtained by replacing  $\geq$  with  $>$ . This translates immediately via the equivalence of categories from Proposition 4.2 into a notion of  $\theta$ -(semi)stability for  $Q$ -constellations.

*Exercise 4.12.* Let  $Q$  be a quiver with a distinguished source, denoted  $0 \in Q_0$ , that admits a path to every other vertex in  $Q$ . Consider any weight  $\vartheta \in \text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  with  $\vartheta_i > 0$  for  $i \neq 0$  (and hence  $\vartheta_0 < 0$ ). Show that:

- (i) every  $\vartheta$ -semistable representation is  $\vartheta$ -stable.
- (ii) a  $Q$ -constellation is  $\vartheta$ -stable if and only if it is a cyclic  $\mathbb{k}Q$ -module with generator  $e_0$ .

An abelian category has finite length if every object  $W$  has a Jordan-Hölder filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_{n-1} \subset E_n = W$$

such that each factor object  $F_i = E_i/E_{i-1}$  is simple. The set of  $\theta$ -semistable representations form the objects in a finite-length abelian subcategory of  $\text{rep}_{\mathbb{k}}(Q)$ , where the simple objects are precisely the  $\theta$ -stable representations. Moreover, two  $\theta$ -semistable representations of  $Q$  are said to be  $S$ -equivalent (with respect to  $\theta$ ) if their Jordan-Hölder filtrations have isomorphic composition factors.

**Proposition 4.13.** *Let  $W$  be a representation of  $Q$  defining the point  $w \in \mathbb{A}_{\mathbb{k}}^{Q_1}$  in the space of representations and let  $\theta \in \text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Then:*

- (i) *the point  $w \in \mathbb{A}_{\mathbb{k}}^{Q_1}$  is  $\theta$ -semistable in the sense of GIT if and only if the representation  $W$  is  $\theta$ -semistable in the sense defined above; and*
- (ii) *two  $\theta$ -semistable points of  $\mathbb{A}_{\mathbb{k}}^{Q_1}$  are identified under the map  $\pi: (\mathbb{A}^{Q_1})_{\theta}^{ss} \rightarrow \mathbb{A}^{Q_1} //_{\theta} T$  from Diagram 2.7 if and only if the corresponding representations are  $S$ -equivalent (w.r.t.  $\theta$ ).*

Combining this result with Theorem 2.22 gives the following.

**Corollary 4.14.** *The toric variety  $\mathcal{M}_\theta(Q)$  parametrises  $S$ -equivalence classes of  $\theta$ -semistable representations of  $Q$ . If  $\theta$  is generic then  $\mathcal{M}_\theta(Q)$  parametrises isomorphism classes of  $\theta$ -stable representations of  $Q$ .*

For the moduli description of  $\mathcal{M}_\theta(Q)$ , we choose our favourite vertex of the quiver  $Q$  and denote it  $0 \in Q_0$ ; at present this looks arbitrary, but we shall see in Section 5.1 that a natural candidate often presents itself. Consider the functor  $\text{rep}_\theta(Q)(-)$  that assigns to each connected scheme  $B$  the set of locally-free sheaves on  $B$  of the form

$$\left\{ \bigoplus_{i \in Q_0} \mathcal{W}_i : \begin{array}{l} \text{rank}(\mathcal{W}_i) = 1 \quad \forall i \in Q_0, \text{ with } \mathcal{W}_0 \cong \mathcal{O}_B \\ \exists \text{ morphism } \mathcal{W}_{t(a)} \rightarrow \mathcal{W}_{h(a)} \quad \forall a \in Q_1 \\ \text{each fibre } \bigoplus_{i \in Q_0} W_i \text{ is } \theta\text{-semistable (up to } S\text{-equiv)} \end{array} \right\} / \sim_{\text{isom}};$$

the requirement on the existence of the morphisms  $\mathcal{W}_{t(a)} \rightarrow \mathcal{W}_{h(a)}$  for all  $a \in Q_1$  is equivalent to the existence of a  $\mathbb{k}$ -algebra homomorphism  $\mathbb{k}Q \rightarrow \text{End}(\bigoplus_{i \in Q_0} \mathcal{W}_i)$ .

**Theorem 4.15** (King [50]). *If  $\theta$  is generic then  $\mathcal{M}_\theta(Q)$  represents the functor  $\text{rep}_\theta(Q)$ , i.e.,  $\mathcal{M}_\theta(Q)$  is the fine moduli space of  $\theta$ -stable representations of  $Q$ . In particular,  $\mathcal{M}_\theta(Q)$  carries*

- (i) a tautological locally free sheaf  $\bigoplus_{i \in Q_0} \mathcal{W}_i$  with  $\text{rank}(\mathcal{W}_i) = 1$  and  $\mathcal{W}_0 \cong \mathcal{O}_{\mathcal{M}_\theta(Q)}$ ; and
- (ii) a  $\mathbb{k}$ -algebra homomorphism  $\mathbb{k}Q \rightarrow \text{End}(\bigoplus_{i \in Q_0} \mathcal{W}_i)$ .

*Sketch proof.* Since  $\theta$  is generic, the  $S$ -equivalence classes of  $\theta$ -semistable representations coincide with the isomorphism classes of  $\theta$ -stable representations. Write  $\bigoplus_{i \in Q_0} \mathcal{O}_{\mathbb{A}^{Q_1}}$  for the universal locally free sheaf on the representation space  $\mathbb{A}_{\mathbb{k}}^{Q_1}$  whose fibre at a point is the corresponding representation of  $Q$ . One expects that the restriction of  $\bigoplus_{i \in Q_0} \mathcal{O}_{\mathbb{A}^{Q_1}}$  to the  $\theta$ -stable locus in  $\mathbb{A}_{\mathbb{k}}^{Q_1}$  will descend under the geometric quotient map  $(\mathbb{A}_{\mathbb{k}}^{Q_1})_\theta^s \rightarrow (\mathbb{A}_{\mathbb{k}}^{Q_1})_\theta^s/T$  to give the required tautological sheaf. This is not immediate since the change of basis group  $(\mathbb{k}^\times)^{Q_0}$  does not act faithfully on  $\mathbb{A}_{\mathbb{k}}^{Q_1}$ . However, our choice of the distinguished vertex  $0 \in Q_0$  and our condition  $\mathcal{W}_0 \cong \mathcal{O}_B$  on the functor  $\text{rep}_\theta(Q)(-)$  removes this ambiguity, at the expense of imposing the restriction  $\mathcal{W}_0 \cong \mathcal{O}_{\mathcal{M}_\theta(Q)}$  on the resulting tautological bundle.

**4.6. Moduli for bound quiver representations.** The GIT and moduli constructions for representations of  $Q$  lead naturally to similar constructions for representations of a bound quiver  $(Q, \varrho)$  as follows.

Let  $(Q, \varrho)$  be a bound quiver (see Section 4.3). The map sending a path  $p = a_1 \cdots a_j$  to the monomial  $y_{a_1} \cdots y_{a_j} \in \mathbb{k}[y_a : a \in Q_1]$  extends to a  $\mathbb{k}$ -linear map from  $\mathbb{k}Q$  to  $\mathbb{k}[y_a : a \in Q_1]$ , and let  $I_\varrho$  denote the ideal in  $\mathbb{k}[y_a : a \in Q_1]$  generated by the image of  $\langle \varrho \rangle$  under this map. A point in  $\mathbb{A}_{\mathbb{k}}^{Q_1}$  corresponds to a representation of the bound quiver  $(Q, \varrho)$  if and only if it lies in the subscheme  $\mathbb{V}(I_\varrho)$  cut out by  $I_\varrho$ . Since  $\varrho$  consists of  $\mathbb{k}$ -linear combinations of paths with the same head and tail, the ideal  $I_\varrho$  is homogeneous with respect to the  $\text{Wt}(Q)$ -grading of  $\mathbb{k}[y_a : a \in Q_1]$  and hence the subscheme  $\mathbb{V}(I_\varrho)$  is invariant under the action of  $T$ . For  $\theta \in \text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ , let  $(\mathbb{k}[y_a : a \in Q_1]/I_\varrho)_\theta$  denote the  $\theta$ -graded piece. Then

$$(4.2) \quad \mathbb{V}(I_\varrho) //_\theta T = \text{Proj} \left( \bigoplus_{j \in \mathbb{N}} (\mathbb{k}[y_a : a \in Q_1]/I_\varrho)_{j\theta} \right)$$

is the categorical quotient of the open subscheme  $\mathbb{V}(I_\varrho)^{\text{ss}} \subseteq \mathbb{V}(I_\varrho)$  parametrising  $\theta$ -semistable representations of  $(Q, \varrho)$ .

**Example 4.16.** The ideal of relations  $I_\varrho \subset \mathbb{k}[y_a : a \in Q_1]$  need not be prime, so the scheme  $\mathbb{V}(I_\varrho)$  may be reducible. For example, let  $(Q, \varrho)$  be the bound McKay quiver (see Example 4.7) for the finite group action of type  $\frac{1}{7}(1, 2)$ . Denote the arrows by  $a_i^{\rho_j}$  for  $1 \leq i \leq 2$  and  $0 \leq j \leq 6$ , so the representation space  $\mathbb{A}_{\mathbb{k}}^{14}$  has coordinate ring  $\mathbb{k}[y_i^{\rho_j} : 1 \leq i \leq 2, 0 \leq j \leq 6]$ . The set of relations  $\varrho$  from Example 4.7 defines the ideal

$$I_\varrho = \langle z_2^{\rho_j} z_1^{\rho_{j-1}} - z_1^{\rho_{j+1}} z_2^{\rho_{j-1}} : 0 \leq j \leq 6 \rangle,$$

where addition on the indices of exponents is taken modulo 7. This ideal has 8 associated primes, so  $\mathbb{V}(I_\varrho)$  has 8 irreducible components.

**Remark 4.17.** The brief introduction to GIT in Section 2 constructed quotient spaces arising from a diagonalisable group action on an affine variety. However, if the scheme  $\mathbb{V}(I_\varrho)$  is reducible then the GIT quotient  $\mathbb{V}(I_\varrho) //_\theta T$  may also be reducible. This highlights a crucial difference between toric quiver varieties  $\mathbb{A}_{\mathbb{k}}^{Q_1} //_\theta T$  and the subschemes  $\mathbb{V}(I_\varrho) //_\theta T$ . The former are normal semiprojective toric varieties, but the latter need not a priori be toric as Example 4.16 shows. In fact, since the ideal  $I_\varrho \subset \mathbb{k}[y_a : a \in Q_1]$  is generated by binomials, Eisenbud–Sturmfels [34] shows that each irreducible component of the scheme  $\mathbb{V}(I_\varrho)$  is an affine toric variety, so each irreducible component of the GIT quotient  $\mathbb{V}(I_\varrho) //_\theta T$  is a semiprojective toric variety. These components need not be normal as Example 4.19 demonstrates.

If  $\theta$  is generic then we write

$$\mathcal{M}_\theta(Q, \varrho) := \mathbb{V}(I_\varrho) //_\theta T$$

for the geometric quotient of the open subscheme  $\mathbb{V}(I_\varrho)_\theta^s \subseteq \mathbb{V}(I_\varrho)$  parametrising  $\theta$ -stable representations of  $(Q, \varrho)$  by the action of the torus  $T = \text{Hom}_{\mathbb{Z}}(\text{Wt}(Q), \mathbb{k}^\times)$ . The analogue of Theorem 4.15 holds for bound quiver representations, making  $\mathcal{M}_\theta(Q, \varrho)$  the *fine moduli space of  $\theta$ -stable representations of  $(Q, \varrho)$* .

*Exercise 4.18.* Establish the bound quiver analogue of Theorem 4.15 by introducing a functor  $\text{rep}_{\mathbb{k}}(Q, \varrho)$ , and hence give the fine moduli space interpretation of  $\mathcal{M}_\theta(Q, \varrho)$ .

**Example 4.19.** Let  $(Q, \varrho)$  denote the bound McKay quiver associated to a finite abelian subgroup  $G \subset \text{GL}(n, \mathbb{k})$  as constructed in Example 4.7, so  $\mathbb{k}Q/\langle \varrho \rangle$  is isomorphic to the skew group algebra  $\mathbb{k}[x_1, \dots, x_n] * G$  and  $\bigoplus_{i \in Q_0} \mathbb{k}e_i$  is isomorphic to the group algebra  $\mathbb{k}[G]$ . Write  $0 \in Q_0$  for the vertex arising from the trivial representation of  $G$ , and let  $\vartheta \in \text{Wt}(Q)$  be any weight satisfying  $\vartheta_i > 0$  for  $i \neq 0$ . Exercises 4.12(i) and 4.18 show that  $\mathcal{M}_\vartheta(Q, \varrho)$  is the fine moduli space of  $\vartheta$ -stable  $\mathbb{k}[x_1, \dots, x_n] * G$ -modules that are isomorphic as  $\mathbb{k}[G]$ -modules to  $\mathbb{k}[G]$ ; in this context, such modules are called  *$\vartheta$ -stable  $G$ -constellations* (see Craw [24]). Craw–Maclagan–Thomas [29, Examples 4.12 and 5.7] show that the scheme  $\mathcal{M}_\vartheta(Q, \varrho)$  is:

- (i) reducible for the subgroup of  $\text{GL}(3, \mathbb{k})$  that defines the action of type  $\frac{1}{14}(1, 9, 11)$ ; and
- (ii) not normal for a subgroup of  $\text{GL}(6, \mathbb{k})$  that is isomorphic to  $(\mathbb{Z}/5)^4$ .

**Remark 4.20** (The  $G$ -Hilbert scheme). For the bound McKay quiver  $(Q, \varrho)$  associated to a finite abelian subgroup  $G \subset \text{GL}(n, \mathbb{k})$ , and for any weight  $\vartheta \in \text{Wt}(Q)$  satisfying  $\vartheta_i > 0$  for  $i \neq 0$ , the scheme  $\mathcal{M}_\vartheta(Q, \varrho)$  represents the same functor as the  $G$ -Hilbert scheme  $G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^n)$ . Indeed, a  $\mathbb{k}[x_1, \dots, x_n]$ -module  $M$  is  $G$ -equivariant if and only if it carries a  $G$ -action such that

$$g \cdot (sm) = (g \cdot s)(g \cdot m)$$

for  $g \in G, s \in \mathbb{k}[x_1, \dots, x_n]$  and  $m \in M$ , from which it follows that left  $\mathbb{k}[x_1, \dots, x_n] * G$ -modules are precisely  $G$ -equivariant  $\mathbb{k}[x_1, \dots, x_n]$ -modules (compare also Lemma 7.5). If we consider only  $G$ -constellations, then Exercise 4.12(ii) asserts that a  $G$ -constellation is  $\vartheta$ -stable if and only if it is a cyclic  $\mathbb{k}[x_1, \dots, x_n] * G$ -module with generator the trivial representation. Combining these observations, we see that  $\mathcal{M}_\vartheta(Q, \varrho)$  is isomorphic to the  *$G$ -Hilbert scheme*

$$G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^n) := \left\{ G\text{-equivariant quotient modules } M = \mathbb{k}[x_1, \dots, x_n]/I \text{ with } M \cong_{\mathbb{k}[G]} \mathbb{k}[G] \right\}$$

that parametrises (coordinate rings of)  $G$ -homogeneous ideals  $I \subset \mathbb{k}[x_1, \dots, x_n]$  for which the quotient module  $\mathbb{k}[x_1, \dots, x_n]/I$  is isomorphic as a  $\mathbb{k}[G]$ -module to  $\mathbb{k}[G]$ . Moreover, the universal bundles for each moduli space are isomorphic as  $G$ -equivariant locally free sheaves. This moduli space (and its nonabelian analogue) plays the key role in Section 7.

## 5. NONCOMMUTATIVE CONSTRUCTION OF TORIC VARIETIES

This section demonstrates how quivers arise naturally in the study of normal toric varieties, and provides a quiver-theoretic (and hence noncommutative) construction of normal semiprojective toric varieties. Following Craw–Smith [31], we extend the classical notion of the linear series  $|L|$  of a single line bundle  $L$ , obtaining the multilinear series  $|\mathcal{L}|$  of a collection of line bundles  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$  on a normal semiprojective toric variety  $X$ . In particular, we show that every such toric variety is a fine moduli space of bound quiver representations for many sequences of line bundles.

**5.1. Quivers of sections.** Let  $X$  be a normal semiprojective toric variety with fan  $\Sigma$  and dense torus  $T_X$ . For  $r \geq 1$ , let  $\mathcal{L} := (L_0, \dots, L_r)$  be a sequence of distinct, basepoint-free line bundles on  $X$  with  $L_0 := \mathcal{O}_X$ . A  $T_X$ -invariant section  $s$  of  $L_j \otimes L_i^{-1}$  is *indecomposable* if the divisor of zeros  $\text{div}(s)$  cannot be expressed as  $\text{div}(s') + \text{div}(s'')$  where  $s'$  and  $s''$  are nonzero  $T_X$ -invariant sections of  $L_\ell \otimes L_i^{-1}$  and  $L_j \otimes L_\ell^{-1}$  respectively, and  $0 \leq \ell \leq r$ . The complete *quiver of sections* of the sequence  $\mathcal{L}$  is the quiver  $Q$  whose vertices  $Q_0 = \{0, \dots, r\}$  correspond to the line bundles in  $\mathcal{L}$ , and the arrows from  $i$  to  $j$  correspond to the indecomposable  $T_X$ -invariant elements of  $H^0(X, L_j \otimes L_i^{-1}) = \text{Hom}(L_i, L_j)$ .

**Lemma 5.1.** *Let  $Q$  be a complete quiver of sections on  $X$ . For each  $i \in Q_0$ , there is a path in  $Q$  from vertex 0 to vertex  $i$ . Moreover,  $Q$  is acyclic if and only if  $X$  is projective.*

*Proof.* The first statement holds since each  $L_i$  is effective. For the second, write  $\nu: S \rightarrow \mathbb{Z}$  for the  $\mathbb{Z}$ -graded semigroup giving  $X = \text{Proj}(\bigoplus_{j \geq 0} \mathbb{k}[S]_j)$ , so  $X$  is projective over  $\text{Spec}(\mathbb{k}[S]_0)$ . It follows for each  $0 \leq i \leq r$  that  $H^0(L_i \otimes L_i^{-1}) \cong \mathbb{k}[S]_0$ , so directed cycles in  $Q$  based at vertex  $i \in Q_0$  correspond to elements of the  $\mathbb{k}$ -vector space basis  $S_0 := S \cap \text{Ker}(\nu)$  of the algebra  $\mathbb{k}[S]_0$ . In particular,  $Q$  is acyclic if and only if  $\mathbb{k}[S]_0 = \mathbb{k}$ .  $\square$

Each quiver of sections carries naturally a set of relations. For each arrow  $a \in Q_1$  we write  $\text{div}(a)$  for the divisor of zeros of the section  $s \in H^0(X, L_j \otimes L_i^{-1})$  that defines  $a$ . Extend this to paths  $p = a_1 \dots a_\ell$  in  $Q$  by setting  $\text{div}(p) := \sum_{j=1}^{\ell} \text{div}(a_j)$ , and hence define

$$(5.1) \quad \varrho = \{p - p' \in \mathbb{k}Q : t(p) = t(p'), h(p) = h(p'), \text{div}(p) = \text{div}(p')\}.$$

Only indecomposable sections determine arrows in  $Q$ , so the paths whose differences form the elements of  $\varrho$  each comprise at least two arrows. We call  $(Q, \varrho)$  is the *complete bound quiver of sections* for  $\mathcal{L}$ . The significance of the relations  $\varrho$  is evident from the following result.

**Proposition 5.2.** *The quotient algebra  $\mathbb{k}Q/\langle \varrho \rangle$  is isomorphic to  $\text{End}(\bigoplus_{i \in Q_0} L_i)$ .*

*Proof.* Each path in  $Q$  arises from an element of  $\text{Hom}(L_i, L_j)$  for some  $i, j \in Q_0$ . The assignment sending each path to the corresponding homomorphism of line bundles extends to a surjective homomorphism of  $\mathbb{k}$ -algebras  $\eta: \mathbb{k}Q \rightarrow \text{End}(\bigoplus_{i \in Q_0} L_i)$  with kernel  $\langle \varrho \rangle$ .  $\square$

*Exercise 5.3.* Prove that the bound Beilinson quiver from Example 4.5 is the complete bound quiver of sections of  $\mathcal{L} = (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$ .

In light of Exercise 5.3, Proposition 5.2 provides a solution to Exercise 4.6.

**Example 5.4.** Let  $X$  denote the minimal resolution of the singularity of type  $\frac{1}{3}(1, 2)$  from Example 3.25. The bound McKay quiver  $(Q, \varrho)$  for the action of type  $\frac{1}{3}(1, 2)$  is the complete quiver of sections of  $\mathcal{L} = (\mathcal{O}_X, L_1, L_2)$  on  $X$ , where  $L_1 = \mathcal{O}_X(D_0)$  and  $L_2 = \mathcal{O}_X(D_3)$  form the  $\mathbb{Z}$ -basis of  $\text{Pic}(X)$  dual to the classes of the irreducible exceptional curves. Figure 5 shows the quiver, where each arrow  $a \in Q_1$  is labelled with the monomial  $x^{\text{div}(a)} \in R_X$ . Using sequence (3.5) from Example 3.25, we see that the sections  $x_0, x_2x_3^2 \in H^0(X, L_1)$  correspond to the pair of vertices of the polyhedron  $\text{conv}(\mathbb{N}^4 \cap \pi^{-1}(L_1))$ , while  $x_3, x_0^2x_1 \in H^0(X, L_2)$  correspond to the pair of vertices of  $\text{conv}(\mathbb{N}^4 \cap \pi^{-1}(L_2))$ . Clearly  $x_0 \in H^0(X, L_1)$  is indecomposable, while  $x_2x_3^2 \in H^0(X, L_1)$  decomposes as the product of  $x_3 \in H^0(L_2)$  and  $x_2x_3 \in H^0(L_1 \otimes L_2^{-1})$ , and similarly for the section of  $L_2$ . If we list the arrows as shown in Figure 5(b), then the relations  $\varrho = \{a_1a_4 - a_2a_5, a_3a_6 - a_4a_1, a_5a_2 - a_6a_3\}$  each derive from decomposing the section  $x_0x_1x_2x_3 \in H^0(\mathcal{O}_X)$  in two different ways when viewed as a cycle from vertex  $i \in Q_0$ .

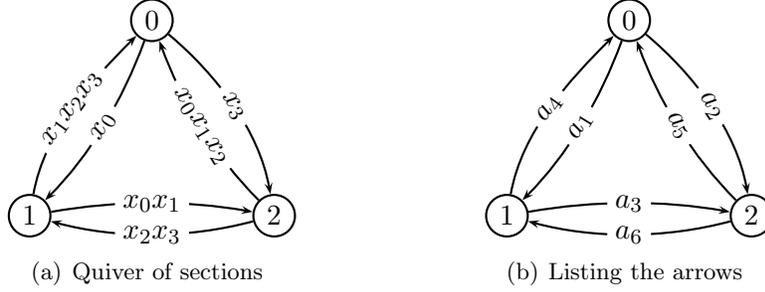


FIGURE 5. McKay quiver for the action of type  $\frac{1}{3}(1,2)$

*Exercise 5.5.* Let  $X$  denote the minimal resolution of the singularity of type  $\frac{1}{r}(1, r-1)$ . By first generalising Example 3.25, show that the bound McKay quiver for the action of type  $\frac{1}{r}(1, r-1)$  is the complete quiver of sections of  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_{r-1})$  on  $X$ , where  $L_1, \dots, L_{r-1}$  form the  $\mathbb{Z}$ -basis of  $\text{Pic}(X)$  dual to the classes of the irreducible exceptional curves.

**Remark 5.6.** Exercise 5.5 can be generalised to the case  $\frac{1}{r}(1, a)$  where  $\gcd(a, r) = 1$ . Let  $X$  denote the minimal resolution of the singularity of type  $\frac{1}{r}(1, a)$  and let  $L_1, \dots, L_m$  be the  $\mathbb{Z}$ -basis of  $\text{Pic}(X)$  dual to the irreducible exceptional curves in  $X$ . The complete bound quiver of sections  $(Q, \varrho)$  for  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_m)$  is the *bound Special McKay quiver* of type  $\frac{1}{r}(1, a)$  introduced in Craw [25] and the quotient algebra  $\mathbb{k}Q/\langle \varrho \rangle$  is the *reconstruction algebra* of Wemyss [66].

**5.2. Multilinear series.** Since the vertex  $0 \in Q_0$  in any complete quiver of sections  $Q$  provides a distinguished source admitting a path to every other vertex in  $Q$ , Exercise 4.12 shows that any weight  $\vartheta \in \text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  satisfying  $\vartheta_i > 0$  for  $i \neq 0$  (and hence  $\vartheta_0 < 0$ ) is generic. In this case, we write

$$|\mathcal{L}| := \mathcal{M}_{\vartheta}(Q) = \mathbb{A}^{Q_1} //_{\vartheta} T$$

for the fine moduli space of  $\vartheta$ -stable representations of  $Q$  of dimension vector  $(1, \dots, 1) \in \mathbb{Z}^{Q_0}$ , and call this toric variety the *multilinear series* of the sequence  $\mathcal{L}$ . Results from Section 4.4 show that  $|\mathcal{L}|$  is smooth and projective over  $\text{Spec}(\mathbb{k}[\mathbb{N}^{Q_1} \cap \text{Cir}(Q)])$ , and  $\text{Pic}(|\mathcal{L}|) \cong \text{Wt}(Q)$ . Moreover, our normalisation for the tautological bundle on the moduli space  $\mathcal{M}_{\vartheta}(Q)$  (see Section 4.5) implies that the isomorphism  $\text{Wt}(Q) \rightarrow \text{Pic}(|\mathcal{L}|)$  identifies  $\sum_{i \in Q_0} \theta_i \chi_i$  with  $\bigotimes_{i \in Q_0} \mathscr{W}_i^{\otimes \theta_i}$ . This leads to the following mild generalisation of Craw–Smith [31, Proposition 3.8] from the case when  $X$  is a normal projective toric variety:

**Proposition 5.7.** *Let  $Q$  be the complete quiver of sections for a sequence  $\mathcal{L}$  as above on a normal semiprojective toric variety  $X$ . The multilinear series  $|\mathcal{L}|$  is a smooth semiprojective toric variety obtained as the geometric quotient of  $\mathbb{A}^{Q_1} \setminus \mathbb{V}(B_Q)$  by  $T$  where*

$$B_Q := \left( \prod_{a \in Q'_1} y_a : Q' \text{ is a spanning tree of } Q \text{ rooted at } 0 \right).$$

Moreover, the tautological line bundles  $\mathscr{W}_i$  for  $i \in Q_0$  on the fine moduli space  $|\mathcal{L}| = \mathcal{M}_{\vartheta}(Q)$  correspond to the elements  $\chi_i - \chi_0$  for  $i \in Q_0$  in  $\text{Wt}(Q)$  under the isomorphism described above.

*Proof.* It remains to establish that the  $\vartheta$ -unstable locus is  $\mathbb{V}(B_Q)$ . The irrelevant ideal for the quotient  $\mathbb{A}^{Q_1} //_{\vartheta} T$  is generated by those monomials  $y^u \in \mathbb{k}[y_a : a \in Q_1]$  with  $u \in \mathbb{N}^{Q_1} \cap \text{inc}^{-1}(\vartheta)$ . If we set  $\vartheta := (-r, 1, 1, \dots, 1) \in \text{Wt}(Q)$ , then the set of arrows supporting every such exponent  $u$  also supports paths from the distinguished vertex 0 to every other vertex  $i \in Q_0 \setminus \{0\}$ . Forgetting

multiplicities and working only with the supporting arrows is equivalent to taking the radical of the ideal. Therefore, as in the proof of Theorem 3.23, the ideal  $B_Q$  is the radical of the ideal that cuts out the  $\vartheta$ -unstable locus in  $\mathbb{A}^{Q_1}$ .  $\square$

**Example 5.8.** Let  $L_1$  be a basepoint-free line bundle on  $X$  and set  $\mathcal{L} = (\mathcal{O}_X, L_1)$ . The complete quiver of sections of  $\mathcal{L}$  has arrow set  $Q_1 = \{a_0, \dots, a_m\}$  corresponding to a  $\mathbb{k}$ -vector space basis of  $H^0(X, L_1)$ . Since every arrow forms a spanning tree in  $Q$ , we have  $B_Q = (y_{a_0}, \dots, y_{a_m})$  and hence,  $|\mathcal{L}|$  is the geometric quotient of  $\mathbb{A}^{m+1} \setminus \{0\}$  by  $T := \text{Hom}_{\mathbb{Z}}(\text{Wt}(Q), \mathbb{k}^*)$ . Choosing  $\chi_1 - \chi_0$  as a basis for  $\text{Wt}(Q) \cong \mathbb{Z}$ , the action of  $T \cong \mathbb{k}^\times$  on  $\mathbb{A}^{m+1}$  is induced by

$$0 \longrightarrow \text{Cir}(Q) \longrightarrow \mathbb{Z}^{m+1} \xrightarrow{[1 \ \dots \ 1]} \mathbb{Z} \longrightarrow 0.$$

It follows that  $|\mathcal{L}|$  is isomorphic to the classical linear series  $|L_1| \cong \mathbb{P}_{\mathbb{k}}^m$ ; whence the name. In our case, the tautological bundle is  $L_1$  rather than  $L_1^{-1}$ ; we view projective space as parametrising hyperplanes rather than lines.

**Example 5.9.** For  $X = \mathbb{F}_1$ , consider the sequence  $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(D_1), \mathcal{O}_X(D_4))$  where we adopt the notation of Example 3.24. The complete quiver of sections for  $\mathcal{L}$  appears in Figure 4, and Example 4.10 shows that the  $|\mathcal{L}|$  is isomorphic to  $\mathbb{F}_1$ . In this case, the tautological line bundles coincide with the bundles of  $\mathcal{L}$ .

**Example 5.10.** Let  $X$  be the smooth toric threefold determined by the following fan  $\Sigma$  in  $\mathbb{Q}^3$ : the rays  $\Sigma(1)$  are generated by the vectors  $v_1 := (1, 0, 0)$ ,  $v_2 := (0, 1, 0)$ ,  $v_3 := (-1, -1, -1)$ ,  $v_4 := (0, 1, 1)$ ,  $v_5 := (1, 0, 1)$  and the two-dimensional cones are represented in Figure 6 (a). There is a flop  $X \dashrightarrow X'$  where the toric variety  $X'$  is the determined by the triangulation of  $\Sigma(1)$  where the cone generated by  $v_1, v_4$  is replaced by the cone with generators  $v_2, v_5$ . For  $(k, \ell) \in \mathbb{Z}^2$ , write  $\mathcal{O}_X(k, \ell) := \mathcal{O}_X(kD_3 + \ell D_2) \in \text{Pic}(X)$ . The complete quiver of sections for  $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(0, 1), \mathcal{O}_X(1, 0))$  satisfies  $X = |\mathcal{L}|$  in this case (compare Exercise 4.11).

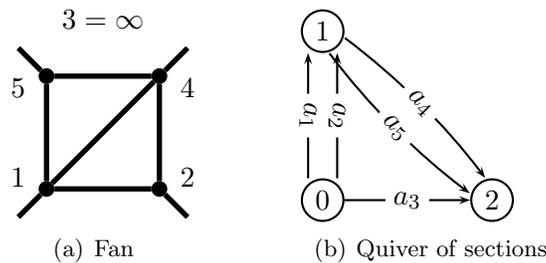


FIGURE 6. Projective threefold admitting a flop

**5.3. Morphism to the multilinear series.** We can provide the multigraded analogue of the morphism  $\varphi_{|L|}: X \rightarrow |L| \cong \mathbb{P}^m$  to the classical linear series for a basepoint-free line bundle  $L$  on a toric variety  $X$ . To motivate our generalisation we reconsider Example 2.13.

**Example 5.11.** For  $X = \mathbb{P}^1$ , consider  $\mathcal{L} = (\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(3))$  with complete quiver of sections  $Q$ . The image of  $\varphi_{|\mathcal{O}_{\mathbb{P}^1}(3)|}$  is obtained by taking Proj of the section ring

$$\bigoplus_{j \geq 0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3j)) \cong \mathbb{k}[y_1, y_2, y_3, y_4]/I,$$

where  $I = (y^u - y^v : u, v \in \mathbb{N}^4, u - v \in \text{Ker}(\pi))$  is the *toric ideal* defined by the matrix

$$\pi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

where the grading is recorded in the first variable.

We now associate a toric ideal to every complete quiver of sections. The map sending  $a \in Q_1$  to  $\text{div}(a) \in \mathbb{Z}^{\Sigma(1)}$  extends to give a  $\mathbb{Z}$ -linear map  $\text{div}: \mathbb{Z}^{Q_1} \rightarrow \mathbb{Z}^{\Sigma(1)}$ . The *section lattice*  $\mathbb{Z}(Q)$  is defined to be the image of the lattice map  $\pi := (\text{inc}, \text{div}): \mathbb{Z}^{Q_1} \rightarrow \text{Wt}(Q) \oplus \mathbb{Z}^{\Sigma(1)}$  sending  $\chi_a \in \mathbb{Z}^{Q_1}$  to  $\pi(\chi_a) = (\chi_{\text{h}(a)} - \chi_{\text{t}(a)}, \text{div}(a))$ . The projections onto the components are denoted  $\pi_1: \mathbb{Z}(Q) \rightarrow \text{Wt}(Q)$  and  $\pi_2: \mathbb{Z}(Q) \rightarrow \mathbb{Z}^{\Sigma(1)}$  respectively, and fit in to the commutative diagram

$$(5.2) \quad \begin{array}{ccc} \mathbb{Z}(Q) & \xrightarrow{\pi_1} & \text{Wt}(Q) \\ \downarrow \pi_2 & & \downarrow \text{pic} \\ \mathbb{Z}^{\Sigma(1)} & \xrightarrow{\text{deg}} & \text{Pic}(X) \end{array}$$

where  $\text{deg}: \mathbb{Z}^{\Sigma(1)} \rightarrow \text{Pic}(X)$  appears in (3.3), and where  $\text{pic}: \text{Wt}(Q) \rightarrow \text{Pic}(X)$  sends  $\sum_{i \in Q_0} \theta_i \chi_i$  to  $\bigotimes_{i \in Q_0} L_i^{\otimes \theta_i}$ . Write  $\mathbb{N}^{Q_1}$  and  $\mathbb{N}(Q)$  for the subsemigroups of  $\mathbb{Z}^{Q_1}$  and  $\mathbb{Z}(Q)$  generated by  $\{\chi_a : a \in Q_1\}$  and  $\{\pi(\chi_a) : a \in Q_1\}$  respectively. The coordinate ring  $\mathbb{k}[y_a : a \in Q_1]$  of  $\mathbb{A}_{\mathbb{k}}^{Q_1}$  is the semigroup algebra of  $\mathbb{N}^{Q_1}$ , and the map  $\pi$  induces a surjective  $\mathbb{k}$ -algebras homomorphism  $\pi_*: \mathbb{k}[y_a : a \in Q_1] \rightarrow \mathbb{k}[\mathbb{N}(Q)]$  with

$$I_Q := \text{Ker}(\pi_*) = (y^u - y^v \in \mathbb{k}[y_a : a \in Q_1] : u - v \in \text{Ker}(\pi));$$

this is the *toric ideal* for the labeled quiver  $Q$ . The incidence map factors through  $\mathbb{N}(Q)$ , so the action of  $T$  on  $\mathbb{A}_{\mathbb{k}}^{Q_1}$  restricts to an action on the affine toric variety  $\mathbb{V}(I_Q) = \text{Spec}(\mathbb{k}[\mathbb{N}(Q)])$  cut out by the prime  $I_Q$ , and we obtain

$$\mathbb{V}(I_Q) //_{\vartheta} T = \text{Proj} \left( \bigoplus_{j \geq 0} \mathbb{k}[\mathbb{N}(Q)]_{j\vartheta} \right).$$

The significance of this semiprojective toric variety is shown by the next result; the projective case is the first main result of Craw–Smith [31].

**Theorem 5.12.** *Let  $X$  be a semiprojective toric variety and let  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$  be a set of basepoint-free line bundles with complete quiver of sections  $Q$ . There is a morphism*

$$\varphi_{|\mathcal{L}|}: X \longrightarrow |\mathcal{L}| = \mathbb{A}_{\mathbb{k}}^{Q_1} //_{\vartheta} T$$

whose image  $\mathbb{V}(I_Q) //_{\vartheta} T$  is equal to the geometric quotient of  $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q)$  by  $T$ .

*Sketch proof.* The sheaf  $\bigoplus_{i \in Q_0} L_i$  on  $X$  defines a family of  $\vartheta$ -stable representations over  $X$ , and the morphism to the multilinear series is induced by the universal property of  $\mathcal{M}_{\vartheta}(Q) = |\mathcal{L}|$ . More concretely, the morphism  $\varphi_{|\mathcal{L}|}$  is induced by the  $\mathbb{k}$ -algebra homomorphism

$$\Phi_Q: \mathbb{k}[y_a : a \in Q_1] \rightarrow \mathbb{k}[x_{\rho} : \rho \in \Sigma(1)]$$

between the total coordinate rings of  $|\mathcal{L}|$  and  $X$  that sends  $y_a$  for  $a \in Q_1$  to the monomial  $x^{\text{div}(a)}$ . The subscheme of  $\mathbb{A}_{\mathbb{k}}^{Q_1}$  cut out by the kernel of  $\Phi_Q$  need not be invariant under the action of  $T$ , but by shrinking the defining ideal from  $\text{Ker}(\Phi_Q)$  to  $I_Q$ , we enlarge the subscheme and hence capture in  $\mathbb{V}(I_Q)$  the  $T$ -orbits of all points from  $\mathbb{V}(\text{Ker}(\Phi_Q))$ . The  $\text{Wt}(Q)$ -grading of the coordinate ring  $\mathbb{k}[\mathbb{N}(Q)]$  of  $\mathbb{V}(I_Q)$  and the  $A_{n-1}(X)$ -grading of the total coordinate ring

of  $X$  are each encoded by a horizontal map in Diagram (5.2). Commutativity of (5.2) shows that the map  $\mathbb{A}^{\Sigma(1)} \rightarrow \mathbb{V}(I_Q) \subseteq \mathbb{A}_{\mathbb{k}}^{Q_1}$  induced by  $\Phi_Q$  is equivariant with respect to the actions of  $G = \text{Hom}(A_{n-1}(X), \mathbb{k}^\times)$  on the domain and  $T = \text{Hom}(\text{Wt}(Q), \mathbb{k}^\times)$  on the target, and by the discussion above it is surjective after passing to group orbits. The statement of (i) follows after verifying that the preimage of the irrelevant subscheme  $\mathbb{V}(B_Q)$  is contained in  $\mathbb{V}(B_X)$ . This condition follows from the assumption that each line bundle in the sequence  $\mathcal{L}$  is basepoint-free. For details, see Craw–Smith [31, Corollary 4.1, Proposition 4.3].  $\square$

*Exercise 5.13.* For the sequence  $\mathcal{L} = (\mathcal{O}_X, L_1)$  with complete quiver of sections  $\mathcal{L}$  from Example 5.8, compute the generators of the semigroup  $\mathbb{N}(Q)$  and hence show that  $\varphi_{|\mathcal{L}|} = \varphi_{|L_1|}$ .

In order to reconstruct the toric variety  $X$  using Theorem 5.12 we must establish when the morphism to the multilinear series is a closed immersion. The following result provides the multigraded analogue of the classical result that relates very ample line bundles with closed immersions.

**Theorem 5.14.** *With the assumptions of Theorem 5.12, assume in addition that the multiplication map  $H^0(X, L_1) \otimes \cdots \otimes H^0(X, L_r) \rightarrow H^0(X, \bigotimes_{i \in Q_0} L_i)$  is surjective. Then*

- (i) *the morphism  $\varphi_{|\mathcal{L}|}$  is a closed immersion if and only if  $\bigotimes_{i \in Q_0} L_i$  is very ample; and*
- (ii) *for each  $i \in Q_0$ , the tautological line bundle  $\mathcal{W}_i$  on  $|\mathcal{L}| = \mathcal{M}_\vartheta(Q)$  satisfies  $\varphi_{|\mathcal{L}|}^*(\mathcal{W}_i) = L_i$ .*

*Sketch proof.* The proof of [31, Proposition 4.9] shows that the morphism  $\varphi_{|\mathcal{L}|}$  is a closed immersion if and only if the map to projective space (over the ring  $\mathbb{k}[\mathbb{N}^{Q_1} \cap \text{Cir}(Q)]$ ) determined by the line bundle  $L = \bigotimes_{i \in Q_0} L_i$  and the subspace  $\mathbb{k}\langle \pi_2(\mathbb{N}(Q) \cap \pi_1^{-1}(\vartheta)) \rangle \subseteq H^0(X, L)$  is a closed immersion. The sections  $\pi_2(\mathbb{N}(Q) \cap \pi_1^{-1}(\chi_i - \chi_0))$  that label the set of paths in  $Q$  from  $0 \in Q_0$  to  $i \in Q_0$  form a  $\mathbb{k}$ -vector space basis of  $H^0(X, L_i)$ . By taking the product of these sections over all vertices in  $Q$ , we see that the divisors labelling paths in the quiver encode the basis  $\pi_2(\mathbb{N}(Q) \cap \pi_1^{-1}(\vartheta))$  of  $\bigotimes_{i \in Q_0} H^0(X, L_i)$ . By the assumption, these sections span  $H^0(X, \bigotimes_{i \in Q_0} L_i)$ , and part (i) follows. For part (ii), see Craw–Smith [31, Theorem 4.15].  $\square$

**Example 5.15.** For a sequence  $\mathcal{L} = (\mathcal{O}_X, L_1)$  with  $L_1$  very ample of  $X$ , Theorem 5.14(ii) shows that the restriction of the tautological bundle  $\mathcal{W}_1 = \mathcal{O}_{|\mathcal{L}|}(1)$  on  $|L_1|$  to the image of  $X$  under the closed immersion  $\varphi_{|L_1|}: X \rightarrow |L_1|$  is equal to  $L_1$ .

**Remark 5.16.** Theorem 5.14 enables one to show that every projective toric variety  $X$  admits many collections  $\mathcal{L}$  for which  $\varphi_{|\mathcal{L}|}$  is a closed immersion. The key is the notion of *multigraded regularity* introduced by Maclagan–Smith [54]. Using an application of regularity due to Hering–Schenck–Smith [38], one can show that if  $L_1, \dots, L_{r-1}$  are basepoint-free line bundles that do not all lie in the same face of the nef cone of  $X$ , then there exists  $L_r \in \text{Pic}(X)$  that is  $\mathcal{O}_X$ -regular with respect to  $L_1, \dots, L_{r-1}$  such that  $\varphi_{|\mathcal{L}|}$  is a closed immersion for  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ .

**5.4. Toric varieties as fine moduli spaces.** We now use the morphism to the multilinear series  $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$  to give the fine moduli space construction of  $X$ . Recall from Section 4.6 that for any bound quiver  $(Q, \varrho)$ , the set of relations  $\varrho$  determine an ideal  $I_\varrho \subseteq \mathbb{k}[y_a : a \in Q_1]$  that cuts out the fine moduli space

$$\mathcal{M}_\vartheta(Q, \varrho) = \mathbb{V}(I_\varrho) //_\vartheta T$$

of  $\vartheta$ -stable bound representations of  $(Q, \varrho)$  of dimension vector  $(1, \dots, 1) \in \mathbb{Z}^{Q_0}$ . For a complete bound quiver of sections, the set  $\varrho$  of relations comprises path differences  $p - p' \in \mathbb{k}Q$ , so the

ideal  $I_\varrho$  is a binomial ideal. It is easy to see that the toric ideal of equations  $I_Q$  and the binomial ideal of relations  $I_\varrho$  satisfy  $I_\varrho \subseteq I_Q$ , and hence we obtain

$$(5.3) \quad \mathbb{V}(I_Q) \subseteq \mathbb{V}(I_\varrho) \subseteq \mathbb{A}_{\mathbb{k}}^{Q_1}.$$

It follows from Theorem 5.12 that the image  $\mathbb{V}(I_Q) //_{\vartheta} T$  of the morphism  $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$  is a subscheme of  $\mathcal{M}_{\vartheta}(Q, \varrho)$ . Both inclusions from (5.3) can be proper, both can be equality, and either one but not the other can be proper; in short, anything goes!

*Exercise 5.17.* For each sequence below, decide which of the inclusions from (5.3) is proper:

- (i)  $\mathcal{L} = (\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(3))$  on  $\mathbb{P}_{\mathbb{k}}^1$  defining the rational normal curve of degree three;
- (ii)  $\mathcal{L} = (\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$  on  $\mathbb{P}^2$  that defines the bound Beilinson quiver for  $\mathbb{P}_{\mathbb{k}}^2$ ;
- (iii)  $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(0, 1), \mathcal{O}_X(1, 0))$  on the 3-fold  $X$  from Example 5.10;

For a more involved example we augment the sequence  $\mathcal{L}$  on  $\mathbb{F}_1$  from Example 5.9 by adding an extra line bundle.

**Example 5.18.** For  $X = \mathbb{F}_1$  and  $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(1, 0), \mathcal{O}_X(0, 1), \mathcal{O}_X(1, 1))$ , the complete quiver of sections is shown in Figure 7: the sections from the total coordinate ring of  $X$  that determines the arrows are illustrated in Figure 7(a); the arrows are listed in Figure 7(b); and the section

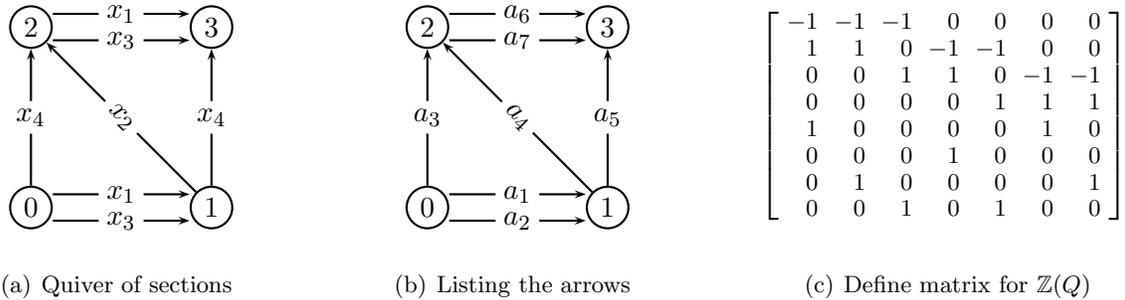


FIGURE 7. A tilting quiver on the Hirzebruch surface  $\mathbb{F}_1$

lattice  $\mathbb{Z}(Q)$  is generated by the columns of the matrix presented in Figure 7(c). The toric ideal is  $I_Q = (y_6y_3 - y_5y_1, y_7y_3 - y_5y_2, y_2y_6 - y_1y_7)$ , and  $\mathbb{V}(I_Q) \subset \mathbb{A}_{\mathbb{k}}^7$  is a normal affine toric variety of dimension 5. In this case, the ideal of relations is

$$I_\varrho = (y_3y_6 - y_1y_5, y_3y_7 - y_2y_5, y_2y_4y_6 - y_1y_4y_7) = I_Q \cap (y_3, y_4, y_5),$$

so  $I_Q$  is a primary component of  $I_\varrho$ . Geometrically, the toric variety  $\mathbb{V}(I_Q)$  is the unique irreducible component of the binomial subscheme  $\mathbb{V}(I_\varrho) \subseteq \mathbb{A}_{\mathbb{k}}^7$  that does not lie in the  $\vartheta$ -unstable locus  $\mathbb{V}(B_Q)$  in  $\mathbb{A}_{\mathbb{k}}^7$  for  $\vartheta = (-3, 1, 1, 1)$ . In particular, we have that  $\mathbb{V}(I_Q) //_{\vartheta} T \cong \mathbb{V}(I_\varrho) //_{\vartheta} T$  even although  $\mathbb{V}(I_Q) \neq \mathbb{V}(I_\varrho)$ .

For  $I_\varrho \subset \mathbb{k}[y_a : a \in Q_1]$ , the saturation of  $I_\varrho$  by  $B_Q$  is

$$(I_\varrho : B_Q^\infty) := \{f \in \mathbb{k}[y_a : a \in Q_1] : b^n f \in I_\varrho \text{ for some } b \in B_Q, n \geq 0\}.$$

The important point for our construction is that the ideal  $B_Q$  is given explicitly, so one can check when the image is a fine moduli space of stable representations of a bound quiver:

**Theorem 5.19.** *Let  $(Q, \varrho)$  be the complete bound quiver of sections for a sequence  $\mathcal{L}$  of basepoint-free line bundles on  $X$ , and let  $\vartheta \in \text{Wt}(Q)$  satisfy  $\vartheta_i > 0$  for  $i \neq 0$ . The following are equivalent:*

- (i) *the image of  $\varphi_{|\mathcal{L}|}$  is equal to the moduli space  $\mathcal{M}_{\vartheta}(Q, \varrho)$ ;*

(ii) the ideal  $I_Q$  equals the ideal quotient  $(I_\varrho : B_Q^\infty)$ .

Moreover, every projective toric variety  $X$  admits many sequences  $\mathcal{L}$  for which the morphism  $\varphi_{|\mathcal{L}|}$  is a closed immersion, and where one and hence both of the above conditions holds.

*Proof.* The image of  $\varphi_{|\mathcal{L}|}$  is  $\mathbb{V}(I_Q) //_{\vartheta} T$  by Theorem 5.12, while the moduli space  $\mathcal{M}_\vartheta(Q, \varrho)$  is defined to be  $\mathbb{V}(I_\varrho) //_{\vartheta} T$ . These GIT quotients coincide precisely when the  $\vartheta$ -stable loci in  $\mathbb{V}(I_Q)$  and  $\mathbb{V}(I_\varrho)$  coincide. The former is  $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Q)$ , the latter is  $\mathbb{V}(I_\varrho) \setminus \mathbb{V}(B_Q)$ , and since  $\mathbb{V}(I_Q) \subseteq \mathbb{V}(I_\varrho)$  holds by (5.3), these  $\vartheta$ -stable loci coincide precisely when  $I_Q = (I_\varrho : B_Q^\infty)$ .

The existence result is the hardest part of Craw–Smith [31]. To obtain appropriate sequences  $\mathcal{L}$ , we provide an explicit construction as follows: for basepoint-free line bundles  $L_1, \dots, L_{r-2}$  on  $X$ , if the subsemigroup of  $\text{Pic}(X)$  generated by  $L_1, \dots, L_{r-2}$  contains an ample line bundle, then there exist line bundles  $L_{r-1}$  and  $L_r$  such that  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$  satisfies the condition. The proof use multigraded regularity to show that a particular ideal has only quadratic generators, and then employs an efficient saturation technique.  $\square$

**Corollary 5.20.** *Every normal semiprojective toric variety  $X$  is isomorphic to a fine moduli space  $\mathcal{M}_\vartheta(Q, \varrho)$  for some complete bound quiver of sections and for  $\vartheta \in \text{Wt}(Q)$  satisfying  $\vartheta_i > 0$  for  $i \neq 0$ .*

**Example 5.21.** For the minimal resolution  $X$  of the quotient singularity of type  $\frac{1}{3}(1, 2)$ , the complete bound quiver of sections  $(Q, \varrho)$  of the sequence  $\mathcal{L} = (\mathcal{O}_X, L_1, L_2)$  on  $X$  from Example 5.4 is the bound McKay quiver shown in Figure 5(a). Using sequence (3.5), compute the lattice points of the polyhedra  $\text{conv}(\mathbb{N}^4 \cap \pi^{-1}(L_i))$  for  $i = 1, 2$  and compare  $\text{conv}(\mathbb{N}^4 \cap \pi^{-1}(L_1 \otimes L_2))$  from Figure 3(b). It follows that the multiplication map  $H^0(L_1) \otimes_{\mathbb{k}} H^0(L_2) \rightarrow H^0(L_1 \otimes L_2)$  is surjective. Since  $L_1 \otimes L_2$  is very ample, Theorem 5.14 implies that  $\varphi_{|\mathcal{L}|}$  is a closed immersion. The section lattice  $\mathbb{Z}(Q)$  is generated by the columns of the matrix

$$\begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and the toric ideal of equations  $I_Q = (y_1y_4 - y_2y_5, y_3y_6 - y_1y_4, y_2y_5 - y_3y_6)$  coincides with the binomial ideal  $I_\varrho$  cut out by the relations, so  $\mathbb{V}(I_Q) //_{\vartheta} T = \mathcal{M}_\vartheta(Q, \varrho)$  for the weight  $\vartheta = (-2, 1, 1) \in \text{Wt}(Q)$ . This establishes that the morphism  $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$  identifies the minimal resolution  $X$  with the fine moduli space  $\mathcal{M}_\vartheta(Q, \varrho)$ . Since  $(Q, \varrho)$  is the bound McKay quiver, Remark 4.20 shows that the minimal resolution  $X$  is isomorphic to the  $G$ -Hilbert scheme. This result is due originally to Ito–Nakamura [45]

**Remark 5.22.** For the bound McKay quiver  $(Q, \varrho)$  of a finite abelian subgroup  $G \subset \text{GL}(n, \mathbb{k})$ , Craw–Maclagan–Thomas [28, 29] constructed a toric ideal of equations  $I_Q \subset \mathbb{k}[y_a : a \in Q_1]$  using an especially simple matrix, namely, the incidence matrix of  $X$  augmented by  $|G|$  blocks of  $n \times n$  identity matrices. For any weight  $\vartheta \in \text{Wt}(Q)$  satisfying  $\vartheta_i > 0$  for  $i \neq 0$ , the GIT quotient  $\mathbb{V}(I_Q) //_{\vartheta} T \subseteq \mathcal{M}_\vartheta(Q, \varrho)$  is the irreducible component of the  $G$ -Hilbert scheme that contains free  $G$ -orbit. Some properties of this *coherent component* of  $\mathcal{M}_\vartheta(Q, \varrho)$  are described in Section 8.1.

## 6. TILTING BUNDLES AND EXCEPTIONAL COLLECTIONS

This section describes how the bounded derived category of coherent sheaves on a smooth projective toric variety can, in certain cases, be described via the category of finitely-generated representations of a bound quiver  $(Q, \varrho)$ . We establish the relation between tilting bundles and full strong exceptional collections, and describe how one might hope to extend some of the ideas to certain smooth toric DM stacks. We emphasise that, in light of the study of tilting bundles on rational surfaces by Hille–Perling [40, 41], tilting bundles appear to exist (on toric varieties) only in rather special cases.

**6.1. On derived categories of coherent sheaves.** The sections on derived categories that follow are based on lectures that were presented in parallel with a sequence of lectures by Andrei Căldăraru [19]. We omit in these notes the necessary introduction to derived categories, referring instead to [19] and to the excellent book by Huybrechts [43] for the background material. It is nevertheless appropriate to highlight one fundamental result that has special relevance for us.

To this end, let  $X$  be a smooth projective variety over  $\mathbb{k}$ . Write  $\text{coh}(X)$  for the abelian category of coherent sheaves on  $X$ , and  $D^b(\text{coh}(X))$  for the bounded derived category of coherent sheaves on  $X$ .

**Proposition 6.1.** *Let  $X$  be a smooth projective variety and write  $\pi_1, \pi_2: X \times X \rightarrow X$  for the first and second projections. Let  $\iota: \Delta \hookrightarrow X \times X$  denote the diagonal embedding. The functor*

$$\mathbf{R}(\pi_2)_*(\pi_1^*(-) \otimes^{\mathbf{L}} \mathcal{O}_\Delta): D^b(\text{coh}(X)) \rightarrow D^b(\text{coh}(X))$$

*is naturally isomorphic to the identity.*

*Proof.* For any object  $F \in D^b(\text{coh}(X \times X))$ , the projection formula for  $\iota$  gives

$$\mathbf{R}\iota_*(\mathbf{L}\iota^*(F) \otimes^{\mathbf{L}} \mathcal{O}_X) \cong F \otimes^{\mathbf{L}} \mathbf{R}\iota_*(\mathcal{O}_X) \cong F \otimes^{\mathbf{L}} \mathcal{O}_\Delta.$$

Therefore, for any object  $E \in D^b(\text{coh}(X))$  we have

$$\mathbf{R}(\pi_2)_*(\pi_1^*(E) \otimes^{\mathbf{L}} \mathcal{O}_\Delta) \cong \mathbf{R}(\pi_2)_*(\mathbf{R}\iota_*(\mathbf{L}\iota^*(\pi_1^*(E)) \otimes^{\mathbf{L}} \mathcal{O}_X)) \cong \mathbf{R}(\pi_2 \circ \iota)_*(\mathbf{L}(\pi_1 \circ \iota)^*(E) \otimes^{\mathbf{L}} \mathcal{O}_X)$$

This is isomorphic to  $E$  since both  $\pi_2 \circ \iota$  and  $\pi_1 \circ \iota$  coincide with the identity.  $\square$

Many other natural isomorphisms will be useful. The following exercise highlights a pair of these, the proofs of which can be found in Huybrechts [43] and Hartshorne [37].

*Exercise 6.2.* Establish the following natural isomorphisms of functors:

- (i) for any object  $E \in D^b(\text{coh}(X))$ ,  $\mathbf{R}\text{Hom}(E, \mathcal{O}_X) \cong \mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}om(E, \mathcal{O}_X)$ ;
- (ii)  $\mathbf{R}(\pi_2)_* \circ \pi_1^*(-) \cong \mathbf{R}\Gamma(-) \otimes \mathcal{O}_X$ ;

**6.2. Tilting sheaves.** Let  $F$  be a coherent sheaf on a smooth variety  $X$  that is projective over an affine variety  $\text{Spec}(R)$ . A coherent sheaf of the form  $T := \mathcal{O}_X \oplus F$  is *tilting* if

- (T1) the algebra  $A := \text{End}_{\mathcal{O}_X}(T)$  has finite global dimension, i.e., the maximal projective dimension of any object in  $\text{mod}(A)$  is finite;
- (T2) we have  $\text{Ext}_{\mathcal{O}_X}^k(T, T) = 0$  for all  $k > 0$ ; and
- (T3) the sheaf  $T$  classically generates  $D^b(\text{coh}(X))$ , i.e., the smallest triangulated subcategory of  $D^b(\text{coh}(X))$  containing  $T$  and all of its direct summands is  $D^b(\text{coh}(X))$ .

We call  $A = \text{End}_{\mathcal{O}_X}(T)$  the associated *tilting algebra*, and  $T$  a *tilting bundle* if it is locally-free (we assume  $\mathcal{O}_X$  is a summand, but one need not). The main result for tilting sheaves on smooth projective varieties is due independently to Baer [4] and Bondal [10]:

**Theorem 6.3.** *Let  $X$  be a smooth variety that is projective over an affine variety. For a tilting sheaf  $T$  on  $X$  with tilting algebra  $A = \text{End}(T)$ , the functor  $\text{Hom}_{\mathcal{O}_X}(T, -): \text{coh}(X) \rightarrow \text{mod}(A^{\text{op}})$  induces an equivalence of triangulated categories*

$$\mathbf{R}\text{Hom}_{\mathcal{O}_X}(T, -): D^b(\text{coh}(X)) \rightarrow D^b(\text{mod}(A^{\text{op}}))$$

with quasi-inverse  $(-)^{\mathbf{L}} \otimes_A T: D^b(\text{mod}(A^{\text{op}})) \rightarrow D^b(\text{coh}(X))$ .

*Proof.* To simplify notation, write the functors as  $F(-) := \text{Hom}_{\mathcal{O}_X}(T, -)$  and  $G(-) := - \otimes_A T$ . The first step is to construct the functors  $\mathbf{R}F$  and  $\mathbf{L}G$ . Let  $E$  be a quasicoherent sheaf on  $X$ . The vector space  $\text{Hom}_{\mathcal{O}_X}(T, E)$  becomes a right  $A$ -module by precomposition, that is, for  $a \in \text{Hom}_{\mathcal{O}_X}(T, T)$  and  $f \in \text{Hom}_{\mathcal{O}_X}(T, E)$  set  $a \cdot f = f \circ a \in \text{Hom}_{\mathcal{O}_X}(T, E)$ . Since  $\text{Hom}_{\mathcal{O}_X}(T, -)$  is a covariant left-exact functor and since the category of quasicoherent sheaves has enough injectives, one obtains a right-derived functor

$$\mathbf{R}F(-) = \mathbf{R}\text{Hom}_{\mathcal{O}_X}(T, -): D^b(Q\text{coh}(X)) \rightarrow D(\text{Mod}(A^{\text{op}}))$$

from the bounded derived category of quasicoherent sheaves on  $X$  to the derived category of the category  $\text{Mod}(A^{\text{op}})$  of (not necessarily finitely-generated) right  $A$ -modules. The cohomology modules of the image are

$$(6.1) \quad H^i(\mathbf{R}\text{Hom}_{\mathcal{O}_X}(T, E)) = \mathbf{R}^i \text{Hom}_{\mathcal{O}_X}(T, E) = \text{Ext}_{\mathcal{O}_X}^i(T, E).$$

Smoothness of  $X$  implies that  $\text{Ext}_{\mathcal{O}_X}^i(T, E) = 0$  for  $i < 0$  and  $i > \dim(X)$ , so the image of the functor  $\mathbf{R}F$  lies in  $D^b(\text{Mod}(A^{\text{op}}))$ . Since  $D^b(\text{coh}(X))$  is equivalent to the full subcategory of  $D^b(Q\text{coh}(X))$  whose objects have coherent cohomology sheaves, we may consider the restriction of  $\mathbf{R}F$  to  $D^b(\text{coh}(X))$ . We claim that the  $A$ -module  $\text{Ext}_{\mathcal{O}_X}^i(T, E)$  is finitely generated for any coherent sheaf  $E$ . Indeed,  $X$  is projective over an affine variety  $\text{Spec}(R)$  and  $T = \mathcal{O}_X \oplus F$ , so the centre of  $A$  contains  $\text{Hom}(\mathcal{O}_X, \mathcal{O}_X) \cong R$  as a subalgebra. The observation of Van den Bergh [65, proof of Corollary 3.2.8] now shows that the cohomology modules from (6.1) actually lie in  $\text{mod}(A)$  whenever  $E$  is coherent. Thus, we obtain by restriction a functor

$$\mathbf{R}F(-) = \mathbf{R}\text{Hom}_{\mathcal{O}_X}(T, -): D^b(\text{coh}(X)) \rightarrow D^b(\text{mod}(A^{\text{op}})).$$

Similarly, since the module category  $\text{Mod}(A^{\text{op}})$  has enough projectives and the functor  $- \otimes_A T$  is right-exact, we obtain a left-derived functor

$$\mathbf{L}G(-) = -^{\mathbf{L}} \otimes_A T: D^b(\text{Mod}(A^{\text{op}})) \rightarrow D(Q\text{coh}(X))$$

to the a priori unbounded derived category of quasicoherent sheaves. For an  $A$ -module  $B$ , the cohomology sheaves of the image are

$$\mathcal{H}^j(B^{\mathbf{L}} \otimes_A T) = \mathcal{T}\text{or}_{-j}^A(B, T),$$

and these vanish off a finite range since  $A$  has finite global dimension. Furthermore, restricting to finitely generated  $A$ -modules ensures that these cohomology sheaves are coherent, giving

$$\mathbf{L}G(-) = -^{\mathbf{L}} \otimes_A T: D^b(\text{mod}(A^{\text{op}})) \rightarrow D^b(\text{coh}(X)).$$

Since  $T$  satisfies property (T2), the composite functor satisfies

$$\mathbf{R}F \circ \mathbf{L}G(A) = \mathbf{R}F(A^{\mathbf{L}} \otimes_A T) = \mathbf{R}\text{Hom}_{\mathcal{O}_X}(T, T) = \text{Hom}_{\mathcal{O}_X}(T, T) = A,$$

so  $\mathbf{R}F \circ \mathbf{L}G$  is the identity on  $A$ . The smallest triangulated subcategory of  $D^b(\text{mod}(A^{\text{op}}))$  containing  $A$  and its direct summands contains finitely-generated free  $A$ -modules, and hence

finitely-generated projective  $A$ -modules. Since  $A$  has finite global dimension, every finitely generated  $A$ -module admits a finite projective resolution. This implies that smallest triangulated subcategory of  $D^b(\text{mod}(A^{\text{op}}))$  containing  $A$  and its direct summands is  $D^b(\text{mod}(A^{\text{op}}))$ . It follows that  $\mathbf{R}F \circ \mathbf{L}G$  is the identity on the whole of  $D^b(\text{mod}(A^{\text{op}}))$ , so  $\mathbf{L}G(-)$  identifies  $D^b(\text{mod}(A^{\text{op}}))$  with the triangulated subcategory of  $D^b(\text{coh}(X))$  generated by

$$\mathbf{L}G(A) = A \overset{\mathbf{L}}{\otimes}_A T = T.$$

Property (T3) now gives the derived equivalences.  $\square$

In the presence of a tilting sheaf, the derived equivalence from Theorem 6.3 provides a significant simplification, and is the most that one can hope for in a triangulated category. For example, the module category  $\text{mod}(A)$  has finite length whereas the abelian category of coherent sheaves on a projective variety does not.

**Example 6.4.** The trivial bundle always provides a tilting bundle on an affine variety  $X$ . Indeed,  $\mathcal{O}_X$  determines both the  $\mathbb{k}$ -algebra  $A = \text{Hom}(\mathcal{O}_X, \mathcal{O}_X) \cong H^0(\mathcal{O}_X)$  and the equivalence of abelian categories  $\Gamma(X, -): \text{coh}(X) \rightarrow \text{mod}(A)$ , where  $\text{mod}(A)$  is the category of finite dimensional  $H^0(\mathcal{O}_X)$ -modules. Clearly  $\mathbf{R}\Gamma(X, -)$  is a derived equivalence.

**Remark 6.5.** See Hille-Van den Bergh [42, Theorem 7.6] for a stronger form of Theorem 6.3.

**6.3. Strongly exceptional sequences.** The representation-theoretic notion of tilting sheaf on a smooth projective variety is closely related to the more algebro-geometric notion of a full strong exceptional sequence. First, recall from Huybrechts [43, §1.4] that an object  $E$  in a  $\mathbb{k}$ -linear triangulated category  $D$  is *exceptional* if

$$\text{Hom}_D(E, E) = \mathbb{k} \text{ and } \text{Hom}_D(E, E[\ell]) = 0 \text{ for } \ell \neq 0.$$

A sequence  $(E_0, E_1, \dots, E_m)$  of exceptional objects is *exceptional* if  $\mathbf{R}\text{Hom}_D(E_i, E_j) = 0$  for  $i > j$ , and it is *strongly exceptional* if in addition  $\text{Hom}_D(E_i, E_j[\ell]) = 0$  for  $i < j$  and  $\ell \neq 0$ . A sequence of objects is *full* if  $E_0, E_1, \dots, E_m$  classically generates  $D$ . If  $D$  is the bounded derived category of coherent sheaves on a variety  $X$  over  $\mathbb{k}$ , we have

$$\text{Hom}_{D^b(\text{coh}(X))}(E, F[\ell]) = \text{Ext}_{\mathcal{O}_X}^{\ell}(E, F).$$

**Proposition 6.6.** *Let  $(E_0 = \mathcal{O}_X, E_1, \dots, E_m)$  be a sequence of locally-free sheaves on a smooth projective variety  $X$  with  $\text{Hom}_{\mathcal{O}_X}(E_i, E_i) = \mathbb{k}$  for  $0 \leq i \leq m$ , e.g., each  $E_i$  is a line bundle.*

- (i) *If  $\bigoplus_{i=0}^m E_i$  satisfies conditions (T1) and (T2) then, by reordering if necessary, the sequence  $(E_0, E_1, \dots, E_m)$  is strongly exceptional; if in addition  $\bigoplus_{i=0}^m E_i$  satisfies (T3) then  $(E_0, E_1, \dots, E_m)$  is a full strongly exceptional sequence.*
- (ii) *Conversely, every such full strongly exceptional sequence defines a tilting bundle.*

*Proof.* For (i), since  $T := \bigoplus_{i=0}^m E_i$  satisfies (T2) we have

$$(6.2) \quad 0 = \text{Ext}_{\mathcal{O}_X}^{\ell}(T, T) = \bigoplus_{i,j} \text{Ext}_{\mathcal{O}_X}^{\ell}(E_i, E_j)$$

for  $\ell > 0$ , which gives the required higher Ext-vanishing. The assumption  $\text{Hom}_{\mathcal{O}_X}(E_i, E_i) = \mathbb{k}$  implies that each  $E_i$  is exceptional. For  $0 \leq i \neq j \leq m$ , one of the vector spaces  $\text{Hom}_{\mathcal{O}_X}(E_i, E_j)$ ,  $\text{Hom}_{\mathcal{O}_X}(E_j, E_i)$  must be trivial, otherwise  $\text{Hom}_{\mathcal{O}_X}(E_i, E_i) \neq \mathbb{k}$  which is absurd. We may relabel if necessary to ensure that  $i < j$  whenever  $\text{Hom}_{\mathcal{O}_X}(E_i, E_j) \neq 0$ . The resulting sequence is strongly exceptional. The second statement from part (i) is immediate.

For (ii), equation (6.2) guarantees that a strongly exceptional sequence  $(E_0, E_1, \dots, E_m)$  gives a locally free sheaf  $T := \bigoplus_i E_i$  satisfying (T2). Set  $A := \text{End}_{\mathcal{O}_X}(T)$ . Since  $X$  is projective, the  $\mathbb{k}$ -vector spaces  $\text{Hom}(E_i, E_j) \cong H^0(E_j \otimes E_i^{-1})$  each have finite dimension, and hence  $A$  is a finite dimensional  $\mathbb{k}$ -algebra. Then  $A$  is isomorphic to the quotient algebra  $\mathbb{k}Q/\langle \varrho \rangle$  of an acyclic bound quiver  $(Q, \varrho)$  with vertex set  $Q_0 = \{0, 1, \dots, m\}$ . Since  $\text{Hom}(E_i, E_j) = 0$  for  $i > j$ , this algebra can be realised as an algebra of lower triangular matrices, from which it follows that  $A$  has finite global dimension. Given (T1) and (T2), the fullness of the sequence then implies that  $T = \bigoplus_i E_i$  satisfies (T3).  $\square$

**6.4. Resolution of the diagonal.** Let  $T$  be a locally-free sheaf on  $X$  and set  $A = \text{End}_{\mathcal{O}_X}(T)$ . If we regard  $T$  as a sheaf of left  $A$ -modules then the (derived) dual  $T^\vee = \mathcal{H}om(T, \mathcal{O}_X)$  is a sheaf of right  $A$ -modules. By pulling back via the first and second projections  $\pi_i: X \times X \rightarrow X$ , the sheaves  $\pi_2^*(T)$  and  $\pi_1^*(T^\vee)$  are naturally sheaves of left and right  $A$ -modules respectively. Consider the object of  $D^b(\text{coh}(X \times X))$  given by

$$T^\vee \boxtimes_A T := \pi_1^*(T^\vee) \otimes_A \pi_2^*(T).$$

To understand this object, consider an object  $M \in \text{mod}(A)$  and let  $P_M^\bullet$  be a projective resolution of  $M$  in  $\text{mod}(A)$ . Then we have

$$\pi_1^*(T^\vee) \otimes_A M = \pi_1^*(T^\vee) \otimes_A P_M^\bullet.$$

Similarly, for an object  $N \in \text{mod}(A^{\text{op}})$  we calculate  $N \otimes_A \pi_2^*(T)$  by replacing  $N$  by a projective resolution of  $N$  in  $\text{mod}(A^{\text{op}})$ . Putting this together, if we let  $P_A^\bullet$  be a projective resolution of  $A$  in the category of left  $A \otimes_{\mathbb{k}} A^{\text{op}}$ -modules then the isomorphism  $M \otimes_A N \cong M \otimes_A A \otimes_A N$  enables us to compute

$$T^\vee \boxtimes_A T := \pi_1^*(T^\vee) \otimes_A P_A^\bullet \otimes_A \pi_2^*(T).$$

Armed with this observation, King [51] established the relation between tilting bundles on smooth projective varieties and the celebrated resolution of the diagonal by Beilinson [5].

**Proposition 6.7.** *Let  $T$  be a locally-free sheaf on  $X$  satisfying (T1) and (T2). If  $T^\vee \boxtimes_A T \rightarrow \mathcal{O}_\Delta$  is an isomorphism in  $D^b(\text{coh}(X \times X))$  then  $T$  is a tilting bundle.*

*Proof.* If (T1) and (T2) hold then the proof of Theorem 6.3 shows (T3) holds if and only if

$$\mathbf{R}\text{Hom}_{\mathcal{O}_X}(T, E) \otimes_A T \cong E \quad \text{for every object } E \in D^b(\text{coh}(X)).$$

Since  $T^\vee = \mathbf{R}\text{Hom}(T, \mathcal{O}_X)$ , Exercise 6.2 parts (i) and (ii) give

$$\mathbf{R}\text{Hom}(T, E) \otimes_A T \cong \mathbf{R}\Gamma(E \otimes_{\mathcal{O}_X} T^\vee) \otimes_A T \cong \mathbf{R}(\pi_2)_* \left( \pi_1^*(E \otimes_{\mathcal{O}_X} T^\vee) \right) \otimes_A T.$$

The projection formula (for sheaves of  $A$ -modules) and a standard result for derived tensor products under pullback gives

$$\begin{aligned} \mathbf{R}(\pi_2)_* \left( \pi_1^*(E \otimes_{\mathcal{O}_X} T^\vee) \right) \otimes_A T &\cong \mathbf{R}(\pi_2)_* \left( \pi_1^*(E \otimes_{\mathcal{O}_X} T^\vee) \otimes_A \pi_2^*(T) \right) \\ &\cong \mathbf{R}(\pi_2)_* \left( \pi_1^*(E) \otimes_{\mathcal{O}_{X \times X}} \pi_1^*(T^\vee) \otimes_A \pi_2^*(T) \right), \end{aligned}$$

giving  $\mathbf{R}\text{Hom}(T, E) \otimes_A T \cong \mathbf{R}(\pi_2)_* \left( \pi_1^*(E) \otimes_{\mathcal{O}_{X \times X}} T^\vee \boxtimes_A T \right)$ . Since  $T^\vee \boxtimes_A T$  is isomorphic to  $\mathcal{O}_\Delta$ , the result follows from Proposition 6.1.  $\square$

**Example 6.8** (Beilinson's resolution). For  $X = \mathbb{P}_{\mathbb{k}}^n$ , consider the sequence

$$\mathcal{L} = (\mathcal{O}_{\mathbb{P}_{\mathbb{k}}^n}, \mathcal{O}_{\mathbb{P}_{\mathbb{k}}^n}(1), \dots, \mathcal{O}_{\mathbb{P}_{\mathbb{k}}^n}(n))$$

and the locally free sheaf  $T = \bigoplus_{i=0}^n \mathcal{O}_{\mathbb{P}_{\mathbb{k}}^n}(i)$ . Proposition 5.2 and Exercise 5.3 establish that the algebra  $A = \text{End}_{\mathbb{P}_{\mathbb{k}}^n}(T)$  is isomorphic to the quotient algebra  $\mathbb{k}Q/\langle \varrho \rangle$  for the bound Beilinson quiver  $(Q, \varrho)$  from Example 4.5. Conditions (T1) and (T2) can both be verified explicitly. Condition (T3) also holds, but this is harder to verify; it follows from the celebrated result of Beilinson [5], but it also follows from Proposition 6.7. We now sketch this argument for  $\mathbb{P}_{\mathbb{k}}^2$ , leaving aside the (admittedly important) details of the maps below. Using Butler–King [18] we calculate the projective resolution of  $A$  in the category of left  $A^e := A \otimes_{\mathbb{k}} A^{\text{op}}$ -modules

$$(A^e(e_2 \otimes e_0^{\text{op}}))^{\oplus 3} \xrightarrow{\delta_2} \begin{array}{c} (A^e(e_0 \otimes e_1^{\text{op}}))^{\oplus 3} \\ \oplus \\ (A^e(e_2 \otimes e_1^{\text{op}}))^{\oplus 3} \end{array} \xrightarrow{\delta_1} \begin{array}{c} A^e(e_0 \otimes e_0^{\text{op}}) \\ \oplus \\ A^e(e_1 \otimes e_1^{\text{op}}) \\ \oplus \\ A^e(e_2 \otimes e_2^{\text{op}}) \end{array}$$

Having  $T = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$  gives  $T^\vee = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$ , and hence the complex  $\pi_1^*(T^\vee) \otimes_A P_A^\bullet \otimes_A \pi_2^*(T)$  is

$$(\mathcal{O}(-2) \boxtimes \mathcal{O})^{\oplus 3} \xrightarrow{\delta_2} \begin{array}{c} (\mathcal{O}(-1) \boxtimes \mathcal{O})^{\oplus 3} \\ \oplus \\ (\mathcal{O}(-2) \boxtimes \mathcal{O}(1))^{\oplus 3} \end{array} \xrightarrow{\delta_1} \begin{array}{c} \mathcal{O} \boxtimes \mathcal{O} \\ \oplus \\ \mathcal{O}(-1) \boxtimes \mathcal{O}(1) \\ \oplus \\ \mathcal{O}(-2) \boxtimes \mathcal{O}(2) \end{array}$$

One can verify that this complex provides a resolution of  $\mathcal{O}_\Delta$ , so (T3) holds by Proposition 6.7. Thus,  $T$  is tilting and the derived equivalences

$$D^b(\text{coh}(X)) \cong D(\text{mod}(A^{\text{op}})) \cong D(\text{rep}_{\mathbb{k}}(Q, \varrho))$$

hold by Theorem 6.3 and Proposition 4.4 (with the duality equivalence  $\text{mod}(A) \cong \text{mod}(A^{\text{op}})$ ).

**Remark 6.9.** To compare the resolution  $T^\vee \boxtimes_A^{\mathbf{L}} T \rightarrow \mathcal{O}_\Delta$  on  $\mathbb{P}_{\mathbb{k}}^2$  with that given by Beilinson, use the Euler sequences

$$0 \longrightarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \longrightarrow \mathcal{O}(-1) \boxtimes \mathcal{O}^{\oplus 3} \longrightarrow \mathcal{O}(-1) \boxtimes \mathcal{O}(1) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}(-2) \boxtimes \Omega^2(2) \longrightarrow \mathcal{O}(-2) \boxtimes \mathcal{O}^{\oplus 3} \longrightarrow \mathcal{O}(-2) \boxtimes \Omega^1(2) \longrightarrow 0$$

to see that  $T^\vee \boxtimes_A^{\mathbf{L}} T \rightarrow \mathcal{O}_\Delta$  is quasi-isomorphic to Beilinson's resolution

$$0 \longrightarrow \mathcal{O}(-2) \boxtimes \Omega^2(2) \longrightarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \longrightarrow \mathcal{O} \boxtimes \mathcal{O} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

The calculation for the tilting bundle on  $\mathbb{P}_{\mathbb{k}}^n$  is similar.

**6.5. Smooth toric Fanos.** The techniques introduced above were exploited by King [51] to construct tilting bundles on each of the 5 smooth toric del Pezzo surfaces (see Example 3.22). The bound Beilinson quiver that encodes  $D^b(\text{coh}(\mathbb{P}_k^2))$  is described in Example 6.8, and the bound quivers  $(Q, \varrho)$  encoding the derived categories for the other 4 examples are shown below.

We first note several factors that are common to each example. For every smooth toric del Pezzo surface  $X$ , King's tilting bundle is a direct sum of basepoint-free line bundles  $\bigoplus_{i=0}^r L_i$  on  $X$  with  $L_0 = \mathcal{O}_X$ . If we write  $(Q, \varrho)$  for the complete bound quiver of sections for the sequence  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$  as in Section 5.1, then Proposition 5.2 shows that the noncommutative algebra  $A = \text{End}_{\mathcal{O}_X}(T)$  is isomorphic to the quotient algebra  $kQ/\langle \varrho \rangle$ . Derived equivalences

$$(6.3) \quad D^b(\text{coh}(X)) \cong D(\text{mod}(A^{\text{op}})) \cong D(\text{rep}_k(Q, \varrho)).$$

follow from Theorem 6.3 and Proposition 4.4 (with the equivalence  $\text{mod}(A) \cong \text{mod}(A^{\text{op}})$  coming from duality). We refer to the bound quiver  $(Q, \varrho)$  as a *tilting quiver*.

**Example 6.10.** For  $X = \mathbb{P}^1 \times \mathbb{P}^1$ , consider the sequence  $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(1, 0), \mathcal{O}_X(0, 1), \mathcal{O}_X(1, 1))$ , where  $\mathcal{O}_X(1, 0)$  and  $\mathcal{O}_X(0, 1)$  are the pullbacks of  $\mathcal{O}_{\mathbb{P}^1}(1)$  via the first and second projections respectively. The complete quiver of sections  $Q$  for  $\mathcal{L}$  is shown in Figure 8(a); here we label each

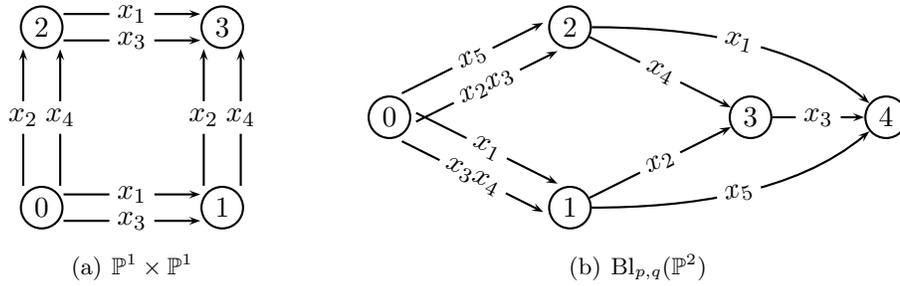


FIGURE 8. Tilting quivers on a pair of toric del Pezzo surfaces

arrow by the monomial  $x^{\text{div}(a)}$  in the Cox ring of  $X$  associated to the toric fan from Figure 2(a). This labelling makes it straightforward to read off the natural set of relations  $\varrho$  defined in (5.1).

**Example 6.11.** For  $X = \mathbb{F}_1$ , a tilting quiver  $(Q, \varrho)$  is provided by the complete bound quiver of sections of the sequence  $\mathcal{L}$  from Example 5.18.

**Example 6.12.** For  $X = \text{Bl}_{p,q}(\mathbb{P}^2)$ , list the  $T_X$ -invariant divisors on  $X$  according to the fan of Figure 2(c) and consider the sequence  $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(D_1), \mathcal{O}_X(D_5), \mathcal{O}_X(D_1 + D_2), \mathcal{O}_X(D_1 + D_5))$ . The complete quiver of sections  $Q$  for  $\mathcal{L}$  is shown in Figure 8(b), and the natural relations  $\varrho$  can be read off from the labelling by monomials using (5.1).

**Example 6.13.** For  $X = \text{Bl}_{p,q,r}(\mathbb{P}^2)$ , list the  $T_X$ -invariant divisors on  $X$  according to the fan in Figure 2(d) and consider the sequence  $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}_X(D_1 + D_2), \mathcal{O}_X(D_3 + D_4), \mathcal{O}_X(D_5 + D_6), \mathcal{O}_X(D_1 + D_2 + D_3), \mathcal{O}_X(D_4 + D_5 + D_6))$  of basepoint-free line bundles on  $X$ . The complete quiver of sections  $Q$  for  $\mathcal{L}$  is shown in Figure 9, and the relations  $\varrho$  can be read off from the labelling by monomials using (5.1).

*Exercise 6.14.* Use the results of Section 5 to show that each smooth toric del Pezzo surface  $X$  is isomorphic to  $\mathcal{M}_\vartheta(Q, \varrho)$  for the relevant bound quiver  $(Q, \varrho)$  and for the weight  $\vartheta \in \text{Wt}(Q)$  satisfying  $\vartheta_i > 0$  for  $i \neq 0$ . (The solution for  $X = \mathbb{F}_1$  is the content of Example 5.18).

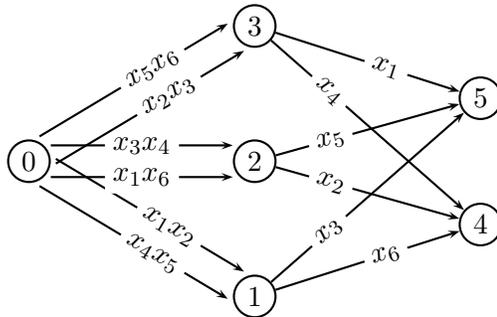


FIGURE 9. A tilting quiver on  $\text{Bl}_{p,q,r}(\mathbb{P}^2)$

Costa–Miró-Roig [22] extended King’s results to construct tilting bundles on 12 of the 18 smooth toric Fano threefolds, and this list was extended by Bondal to 16 of the 18. At about the same time, Greg Smith and I exhibited tilting bundles on all 18 smooth toric Fano threefolds and many of the 126 smooth toric Fano fourfolds. This suggests the following:

**Conjecture 6.15.** *Let  $X$  be a smooth toric Fano variety. There is a sequence of basepoint-free line bundles  $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$  on  $X$  with bound quiver of sections  $(Q, \varrho)$  such that*

- (i)  *$X$  is isomorphic to  $\mathcal{M}_\vartheta(Q, \varrho)$  for the weight  $\vartheta \in \text{Wt}(Q)$  satisfying  $\vartheta_i > 0$  for  $i \neq 0$ ;*
- (ii) *the tautological bundle  $\bigoplus_{0 \leq i \leq r} L_i$  on  $X$  is tilting, giving the derived equivalences (6.3).*

**Remark 6.16.** A couple of important remarks are in order:

- (i) The Fano condition is not necessary. For example, Costa–Miró-Roig [22] constructed tilting bundles on several families of smooth toric varieties, none of which consists entirely of Fano varieties. Nevertheless, some restriction is required, since the example of Hille–Perling [40] shows that even smooth projective toric surfaces need not admit a tilting bundle obtained as a direct sum of line bundles.
- (ii) If Conjecture 6.15(ii) holds, then one expects part (i) to follow. Bergman–Proudfoot [6] show that if  $X$  admits a tilting bundle for which the corresponding bound quiver  $(Q, \varrho)$  admits a character ‘great’  $\theta$ , then  $X$  is isomorphic to a connected component of  $\mathcal{M}_\theta(Q, \varrho)$ . On the other hand, getting the natural moduli construction correct first can also help enormously (see the  $G$ -Hilbert scheme and Theorem 7.6).

**6.6. Smooth toric DM stacks.** Since these lectures on derived categories were presented at the Utah summer school, Borisov–Hua [12] extended Conjecture 6.15(ii) further to include all nef-Fano toric DM stacks with trivial generic stabiliser.

That derived category questions for a projective variety  $X$  with at worst orbifold singularities should be tackled using a stack  $\mathcal{X}$  whose coarse moduli space is  $X$  is due largely to a sequence of papers by Kawamata including [47, 48]. He generalised many fundamental results on derived categories of smooth projective varieties due to Orlov and Bondal–Orlov by replacing  $\text{coh}(X)$  by the abelian category  $\text{coh}(\mathcal{X})$  of coherent orbifold sheaves on  $\mathcal{X}$ . By applying these results with the minimal model programme for toric varieties, Kawamata [49] established the strongest results known at present for the derived category of a general class of toric varieties.

To describe the result, let  $X$  be a projective toric variety whose fan  $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q}$  is simplicial, so  $X$  has at worst orbifold singularities by Remark 3.21. Recall from Theorem 3.23 that  $X$  is obtained as the geometric quotient of  $U := \mathbb{A}_{\mathbb{k}}^{\Sigma(1)} \setminus \mathbb{V}(B_X)$  by an action of  $\text{Hom}(A_{n-1}(X), \mathbb{k}^\times)$ . Borisov–Chen–Smith [11] define a collection of smooth Deligne–Mumford stacks each with coarse

moduli space  $X$  as follows. Let  $\beta: (\mathbb{Z}^{\Sigma(1)})^\vee \rightarrow N$  be a  $\mathbb{Z}$ -linear map that picks out a lattice point  $v_\rho = \beta(e_\rho)$  on each ray  $\rho \in \Sigma(1)$ , and let  $\text{Gale}(N)$  denote the cokernel of the dual map  $\beta^\vee: N^\vee \rightarrow \mathbb{Z}^{\Sigma(1)}$ . Note that  $\text{Gale}(N) = A_{n-1}(X)$  if each  $v_\rho$  is the primitive generator of  $\rho \cap N$ . The diagonalisable algebraic group  $G := \text{Hom}(\text{Gale}(N), \mathbb{k}^\times)$  acts on  $U$ , and the *smooth toric DM stack* of  $(\Sigma, \beta)$  is defined to be the stack quotient  $\mathcal{X}(\Sigma, \beta) := [U/G]$ . Our assumption that the abelian group  $N$  is free implies that  $\mathcal{X}(\Sigma, \beta)$  has trivial generic stabiliser.

The main result of Kawamata [49] can be stated as follows (compare Proposition 6.6).

**Theorem 6.17.** *Let  $X$  be a projective toric variety with simplicial fan  $\Sigma$ , choose  $\beta$  as above and set  $\mathcal{X} := \mathcal{X}(\Sigma, \beta)$ . Then  $D^b(\text{coh}(\mathcal{X}))$  admits a full exceptional sequence of sheaves.*

The sheaves in the exceptional sequence need not be line bundles, and the sequence need not a priori be strong, so we do not in general obtain the derived equivalences (6.3).

*Exercise 6.18.* This exercise exhibits a simple link between the minimal model programme and derived categories. We start with  $\text{Bl}_{p,q,r}(\mathbb{P}^2)$  whose fan is shown in Figure 2(d).

- (i) Consider the blow-down of the  $(-1)$ -curve labelled  $D_6$  in  $\text{Bl}_{p,q,r}(\mathbb{P}^2)$ . In terms of the tilting quiver  $Q$  from Figure 9, remove  $x_6$  whenever it appears in the label on an arrow; this forces us to contract the arrow from  $1 \in Q_0$  to  $4 \in Q_0$  and hence identify vertices 1 and 4. After removing the arrows in the resulting quiver that are not irreducible (see Section 5.1), show that we obtain the tilting quiver on  $\text{Bl}_{p,q}(\mathbb{P}^2)$  from Figure 8.
- (ii) Continue by contracting successively the  $(-1)$ -curves to produce the tilting quiver from Figures 7 and then the bound Beilinson quiver for  $\mathbb{P}_{\mathbb{k}}^2$  (up to a relabelling of the divisors).

**Remark 6.19.** Kawamata [49] defines his stack  $\mathcal{X}$  in terms of pairs  $(X, B)$ , where  $B$  is a  $T_X$ -invariant  $\mathbb{Q}$ -divisor of the form  $\sum_{\rho \in \Sigma(1)} (1 - \frac{1}{r_\rho}) D_\rho$  for some positive integers  $\{r_\rho : \rho \in \Sigma(1)\}$ . This choice of  $B$  is equivalent to choosing the map  $\beta: (\mathbb{Z}^{\Sigma(1)})^\vee \rightarrow N$  for which  $v_\rho$  is precisely  $r_\rho$  times the primitive generator of  $\rho \cap N$ . Indeed, if we let  $\mathcal{D}_\rho$  denote the prime divisor in  $\mathcal{X}$  corresponding to  $\rho \in \Sigma(1)$ , then the morphism to the coarse moduli space  $\pi: \mathcal{X} \rightarrow X$  satisfies  $\pi^*(D_\rho) = r_\rho \mathcal{D}_\rho$  in each case. In particular,  $B = 0$  if and only if  $\beta$  chooses the primitive lattice generator in each ray, in which case  $\mathcal{X}(\Sigma, \beta)$  is the canonical covering stack of  $X$ .

Borisov–Hua [12] propose that an analogue of Conjecture 6.15(ii) should hold for toric DM stacks  $\mathcal{X}(\Sigma, \beta)$  that are *nef-Fano*, i.e., a positive power of the anticanonical bundle on  $\mathcal{X}$  is the pullback of a nef and big line bundle on  $X$ . As evidence, they construct full strong exceptional collections of line bundles on all smooth Fano toric DM stacks of dimension two and Picard rank at most three, as well as on all smooth Fano toric DM stacks of Picard number at most two (generalising the result for toric varieties of Costa–Miró-Roig [22]).

## 7. THE DERIVED MCKAY CORRESPONDENCE FOR $G$ -Hilb

In this section we apply derived category methods to provide an elegant explanation for the McKay correspondence in dimension  $n \leq 3$ . Rather than pursue further the construction of tilting bundles explicitly (see Van den Bergh [65]), we describe in some detail the original method of Bridgeland–King–Reid [16] that uses the universal sheaf on the  $G$ -Hilbert scheme to construct an equivalence of derived categories as a Fourier–Mukai transform. The varieties that we consider in this section are all normal semiprojective toric varieties if and only the finite subgroup  $G \subset \text{SL}(n, \mathbb{k})$  is abelian. Imposing this restriction does not simply the proof greatly, so we do not restrict to the toric case here.

**7.1. A few words on the McKay correspondence.** Let  $G \subset \mathrm{SL}(n, \mathbb{k})$  be a finite subgroup. We may assume that  $G$  acts without quasireflections, in which case the quotient  $X := \mathbb{A}_{\mathbb{k}}^n/G$  is Gorenstein, i.e., the canonical sheaf  $\omega_X$  is locally free. The globally defined nonvanishing form  $dx_1 \wedge \cdots \wedge dx_n$  on  $\mathbb{A}_{\mathbb{k}}^n$  is  $G$ -invariant and hence it descends to  $X$ , forcing  $\omega_X$  to be trivial. A resolution of singularities  $\tau: Y \rightarrow X$  is *crepant* if  $\tau^*(\omega_X) = \omega_Y$ .

**Example 7.1.** The motivating examples of crepant resolutions are the minimal resolutions of Kleinian surface singularities, i.e., quotients  $\mathbb{A}_{\mathbb{k}}^2/G$  for a finite subgroup  $G \subset \mathrm{SL}(2, \mathbb{k})$ . These crepant resolutions provide an ADE classification for Kleinian singularities; the resolution of the  $A_2$ -singularity is described in Example 3.25.

In higher dimensions, crepant resolutions need not exist a priori, and when they do they are typically nonunique. Nevertheless, by introducing the  $G$ -Hilbert scheme  $Y := G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^n)$ , Ito–Nakamura [45] and Nakamura [58] provided a candidate for a crepant resolution, at least in dimension two and three. Recall from Remark 4.20 that the  $G$ -Hilbert scheme parametrises (coordinate rings of)  $G$ -homogeneous ideals  $I \subset \mathbb{k}[x_1, \dots, x_n]$  for which the quotient module  $\mathbb{k}[x_1, \dots, x_n]/I$  is isomorphic as a  $\mathbb{k}[G]$ -module to  $\mathbb{k}[G]$ . The subschemes  $\mathbb{V}(I) \subset \mathbb{A}_{\mathbb{k}}^n$  defined by such ideals are called  *$G$ -clusters*. Examples include any free  $G$ -orbit, all of which define points in a single irreducible component of  $Y$  (see [28, §5]). The map sending a  $G$ -cluster to its supporting  $G$ -orbit defines a projective and surjective Hilbert–Chow morphism  $\tau: Y \rightarrow \mathbb{A}_{\mathbb{k}}^n/G$  and hence, irrespective of whether  $Y$  is a crepant resolution of  $X = \mathbb{A}_{\mathbb{k}}^n/G$ , the map  $\tau$  fits into a commutative diagram of the form

$$(7.1) \quad \begin{array}{ccc} & Y \times \mathbb{A}_{\mathbb{k}}^n & \\ \pi_Y \swarrow & & \searrow \pi_V \\ Y & & \mathbb{A}_{\mathbb{k}}^n \\ & \searrow \tau & \swarrow \pi \\ & X & \end{array}$$

where  $\pi: V := \mathbb{A}_{\mathbb{k}}^n \rightarrow X$  is the quotient morphism and  $\pi_Y$  and  $\pi_V$  are the projections to the first and second factors. Let  $G$  act trivially on both  $Y$  and  $X$ , so that all morphisms in the above diagram are  $G$ -equivariant. The universal sheaf on  $Y \times \mathbb{A}_{\mathbb{k}}^n$  is the structure sheaf  $\mathcal{O}_{\mathcal{Z}}$  of the universal closed subscheme  $\mathcal{Z} \subset Y \times \mathbb{A}_{\mathbb{k}}^n$  whose restriction to the fibre over a point  $[I] \in Y$  is the subscheme  $\mathbb{V}(I) \subset \mathbb{A}_{\mathbb{k}}^n$ . The fibre of  $\pi_Y$  over each closed point is isomorphic to  $\mathbb{k}[G]$  as a  $\mathbb{k}[G]$ -module. If  $\mathrm{Irr}(G)$  denotes the set of isomorphism classes of the irreducible representations of  $G$ , then  $(\pi_Y)_*(\mathcal{O}_{\mathcal{Z}})$  is a  $G$ -equivariant locally free sheaf on  $Y$  that decomposes into a sum of locally free sheaves

$$\mathcal{W} = \bigoplus_{\rho \in \mathrm{Irr}(G)} \mathcal{W}_{\rho}^{\oplus \dim(\rho)}$$

according to the irreducible decomposition of  $\mathbb{k}[G]$ , where  $\mathrm{rank}(\mathcal{W}_{\rho}) = \dim(\rho)$ .

**Remark 7.2.** Recall from Remark 4.20 (compare Remark 4.8) that the tautological bundle  $\mathcal{W}$  on the  $G$ -Hilbert scheme  $Y$  is isomorphic a a  $G$ -equivariant locally free sheaf to the tautological sheaf on the fine moduli space  $\mathcal{M}_{\vartheta}(Q, \varrho)$  of  $\vartheta$ -stable representations of the bound McKay quiver. An analogous statement holds if  $G$  is not abelian, see Ito–Nakajima [44, Section 2].

*Exercise 7.3.* Read Reid [60, Theorem 4.11] and deduce that the cyclic quotient singularity of type  $\frac{1}{2}(1, 1, 1, 1)$  does not admit a crepant resolution. Thus, crepant resolutions of Gorenstein quotient singularities need not exist in dimension 4 and higher.

Motivated by the geometric explanation of the McKay correspondence in dimension two by Gonzalez–Sprinberg and Verdier [36], Reid [61, 62] formulated a derived category version of the McKay correspondence (see Aspinwall et.al. [1] for background on the McKay correspondence):

**Conjecture 7.4.** *Let  $G \subset \mathrm{SL}(n, \mathbb{k})$  be a finite subgroup, and let  $G\text{-coh}(\mathbb{A}_{\mathbb{k}}^n)$  denote the abelian category of  $G$ -equivariant coherent sheaves on  $\mathbb{A}_{\mathbb{k}}^n$ . If a crepant resolution  $\tau: Y \rightarrow \mathbb{A}_{\mathbb{k}}^n/G$  exists then there is an equivalence of triangulated categories*

$$(7.2) \quad \Phi: D^b(\mathrm{coh}(Y)) \rightarrow D^b(G\text{-coh}(\mathbb{A}_{\mathbb{k}}^n)).$$

**7.2. The statement for the  $G$ -Hilbert scheme.** A coherent sheaf  $\mathcal{F}$  on  $\mathbb{A}_{\mathbb{k}}^n$  is  $G$ -equivariant if and only if the space of sections  $\Gamma(\mathcal{F})$  is a finitely-generated  $\mathbb{k}[x_1, \dots, x_n]$ -module that carries an action by  $G$  such that  $g \cdot (sm) = (g \cdot s)(g \cdot m)$  for all  $g \in G, m \in \Gamma(\mathcal{F})$  and  $s \in \mathbb{k}[x_1, \dots, x_n]$ . Combining this with Proposition 5.2 and Example 4.7 (or Remark 4.8) establishes the next result:

**Lemma 7.5.** *The following abelian categories are equivalent: the category*

- (i)  $G\text{-coh}(\mathbb{A}_{\mathbb{k}}^n)$  of  $G$ -equivariant coherent sheaves on  $\mathbb{A}_{\mathbb{k}}^n$ ;
- (ii)  $\mathrm{mod}(A)$  of left-modules over the skew group algebra  $A := \mathbb{k}[x_1, \dots, x_n] * G$ ;
- (iii)  $\mathrm{rep}_{\mathbb{k}}(Q, \varrho)$  of finite-dimensional representations of the bound McKay quiver.

Given this result, one approach to Conjecture 7.4 is to construct a tilting bundle  $T$  on a crepant resolution  $Y$  for which  $\mathrm{End}_{\mathcal{O}_Y}(T)$  is isomorphic to the algebra  $A$ . Reid [61] proposed the tautological sheaf  $\mathcal{W}$  on the  $G$ -Hilbert scheme as a candidate, at least in low dimensions, and King began to investigate whether the resolution of the diagonal from Proposition 6.7 could be established explicitly in such cases. However, with the timely completion of Bridgeland’s thesis in the summer of 1998, Bridgeland, King and Reid together realised that an alternative approach is to establish the equivalence by using the universal sheaf for the  $G$ -Hilbert scheme to construct a  $G$ -equivariant version of a Fourier–Mukai transform (the same observation was made independently and carried out in dimension two by Kapranov–Vasserot [46]).

Define a functor  $\Phi^{\mathcal{O}_Z}: D^b(\mathrm{coh}(Y)) \rightarrow D^b(\mathrm{mod}(A))$  by setting

$$(7.3) \quad \Phi^{\mathcal{O}_Z}(-) = \mathbf{R}(\pi_Y)_* \left( \mathcal{O}_Z^{\mathbf{L}} \otimes (\pi_Y)^*(- \otimes \rho_0) \right).$$

The tensor product with the trivial representation acknowledges that  $G$  acts trivially on  $Y$ , enabling us to take the  $G$ -equivariant pullback via  $\pi_Y$ . Therefore,  $\Phi^{\mathcal{O}_Z}$  is an equivariant version of an integral functor (see Huybrechts [43]). The main result of Bridgeland–King–Reid [16], also known as “Mukai  $\implies$  McKay”, can be stated as follows.

**Theorem 7.6.** *Let  $G \subset \mathrm{SL}(n, \mathbb{k})$  be a finite subgroup and let  $A = \mathbb{k}[x_1, \dots, x_n] * G$  denote the skew group algebra. If the irreducible component  $Y \subseteq G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^n)$  containing the free  $G$ -orbits satisfies  $\dim(Y \times_X Y) \leq n + 1$ , then:*

- (i) *the morphism  $\tau: Y \rightarrow X$  is a crepant resolution; and*
- (ii) *the functor  $\Phi^{\mathcal{O}_Z}: D^b(\mathrm{coh}(Y)) \rightarrow D^b(\mathrm{mod}(A))$  is an equivalence of derived categories.*

**Corollary 7.7.** *Conjecture 7.4 holds for surfaces, and for 3-folds when  $Y = G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^3)$ .*

*Proof.* The dimension of  $Y \times_X Y$  is at most twice the dimension of the exceptional locus; this is two for  $n = 2$  and four for  $n = 3$ . Since crepant resolutions of canonical surfaces are unique, Theorem 7.6 settles completely the case  $n = 2$  (this is due to Kapranov–Vasserot [46]).  $\square$

*Exercise 7.8.* Show that the condition on the dimension of the fibre product fails to hold for the unique crepant resolution of the singularity of type  $\frac{1}{4}(1, 1, 1, 1)$ .

**Remark 7.9.** Bridgeland–King–Reid [16] also show that  $G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^3)$  is connected for a finite subgroup  $G \subset \text{SL}(3, \mathbb{k})$ , giving  $Y \cong G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^3)$ .

**7.3. First steps of the proof.** In this section and the next we describe much of the proof of Theorem 7.6. The original proof from [16] is very well written. My aim is simply to present a roadmap for the construction as I learned it from the authors while the paper was being written (and to shed more light on the rather tightly written [16, §6, Step 5]). As in Section 6, we refer to [19, 43] for the necessary triangulated category definitions.

To simplify the discussion, we make a couple of rather outrageous assumptions: assume

- (A1) that  $Y$  is projective and that Serre duality holds on  $\mathbb{A}_{\mathbb{k}}^n$ . This is patently false (!);
- (A2) that  $\tau: Y \rightarrow X$  is a crepant resolution. While this forms half of the result (!), it was known prior to [16] for finite abelian subgroups  $G \subset \text{SL}(3, \mathbb{k})$  by Nakamura [58].

*Sketch proof.* The strategy of the proof is to invoke a key result that gives a criteria for an integral functor to be an equivalence. Since  $Y$  is smooth by assumption (A2), the skyscraper sheaves  $\{\mathcal{O}_y : y \in Y\}$  form a spanning class in  $D^b(\text{coh}(Y))$ . If we accept that the triangulated category  $D^b(\text{mod}(A))$  is indecomposable [16, Lemma 4.2], then assumption (A1) enables us to state the result [16, Theorem 2.4] in the following form:

**Theorem 7.10.** *Let  $Y$  be a smooth projective variety and  $\Phi: D^b(\text{coh}(Y)) \rightarrow D^b(\text{mod}(A))$  an exact functor that admits a left adjoint. Then  $\Phi$  is fully faithful if and only if*

$$\text{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_{y'})[i]) \cong \text{Hom}(\mathcal{O}_y, \mathcal{O}_{y'}[i]) \quad \text{for all } i \in \mathbb{Z} \text{ and } y, y' \in Y.$$

*If, in addition,  $\Phi$  commutes with the Serre functors on the triangulated categories  $D^b(\text{coh}(Y))$  and  $D^b(\text{mod}(A))$ , then  $\Phi$  is a derived equivalence.*

**Remark 7.11.** This result is motivated by earlier results of Mukai and Bondal–Orlov, though note that the form of the result used here requires that the Hom groups be tested for all  $i \in \mathbb{Z}$  and all  $y, y' \in Y$  (compare Huybrechts [43, Remark 7.3]).

The first step is to ensure that the exact functor  $\Phi^{\mathcal{O}^Z}$  admits a left adjoint. This follows from the projectivity assumption (A1), since Neeman’s  $G$ -equivariant version of Grothendieck duality enables us to repeat the original construction of adjoints due to Mukai (see [43, Proposition 5.9]) to deduce that the functor  $\Psi: D^b(\text{mod}(A)) \rightarrow D^b(\text{coh}(Y))$  defined by

$$\Psi(-) := \left[ \mathbf{R}(\pi_Y)_* \left( \mathcal{O}_Z^{\vee} \otimes \pi_V^*(-)[n] \right) \right]^G$$

is left-adjoint to  $\Phi^{\mathcal{O}^Z}$  (we use the fact that  $G \subset \text{SL}(n, \mathbb{k})$  to deduce that  $\pi_V^*(\omega_V)$  is trivial as a  $G$ -equivariant sheaf). Note that taking the  $G$ -invariant part is adjoint to the functor obtained by taking the tensor product with the trivial representation of  $G$ .

*Exercise 7.12.* For  $\rho \in \text{Irr}(G)$ , let  $\rho^*$  denote the contragredient representation and  $\mathcal{O}_{\mathbb{A}_{\mathbb{k}}^n} \otimes \rho^*$  the  $G$ -equivariant sheaf on  $\mathbb{A}_{\mathbb{k}}^n$  for which  $\Gamma(\mathcal{O}_{\mathbb{A}_{\mathbb{k}}^n} \otimes \rho^*)$  is the  $\mathbb{k}[x_1, \dots, x_n]^G$ -module consisting of the  $\rho^*$ -semiinvariant functions. Use Grothendieck duality to show that  $\Psi(\mathcal{O}_{\mathbb{A}_{\mathbb{k}}^n} \otimes \rho^*) = \mathcal{W}_{\rho}^{\vee}$ .

As a second step, applying Theorem 7.10 shows that  $\Phi^{\mathcal{O}^Z}$  is fully faithful if and only if

$$(7.4) \quad \text{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_{y'})[i]) = \begin{cases} 0 & \text{if } y \neq y' \text{ or } i \notin [0, n] \\ \text{Ext}_{\mathcal{O}_Y}^i(\mathcal{O}_y, \mathcal{O}_{y'}) & \text{otherwise,} \end{cases}$$

for all  $i \in \mathbb{Z}$  and  $y, y' \in Y$ . Using formula (7.3) for the Fourier–Mukai transform, we obtain  $\Phi(\mathcal{O}_y) = \mathcal{O}_{Z_y}$  where  $Z_y \subset \mathbb{A}_{\mathbb{k}}^n$  is the  $G$ -cluster corresponding to the point  $y \in Y$ . Moreover, under the equivalence  $\text{mod}(A) \cong G\text{-coh}(\mathbb{A}_{\mathbb{k}}^n)$ , the Hom-groups in  $D^b(\text{mod}(A))$  correspond to the  $G$ -invariant part of Ext-groups, so we work with the groups

$$\text{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_{y'})[i]) = G\text{-Ext}_{\mathcal{O}_{\mathbb{A}_{\mathbb{k}}^n}}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}).$$

The third step is to show that these groups vanish for  $y \neq y'$ ; this requires the assumption on the dimension on the fibre product. The  $G$ -clusters  $Z_y$  and  $Z_{y'}$  define different points of the classical Hilbert scheme of points, giving  $\text{Hom}(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}) = 0$  and hence  $G\text{-Hom}(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}) = 0$ . The  $G$ -clusters  $Z_y$  and  $Z_{y'}$  are disjoint when their supporting  $G$ -orbits  $\tau(y), \tau(y') \in \mathbb{A}_{\mathbb{k}}^n$  satisfy  $\tau(y) \neq \tau(y')$ , so

$$(7.5) \quad (y, y') \in Y \times Y \text{ with } \tau(y) \neq \tau(y') \implies G\text{-Ext}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

Otherwise  $y \neq y'$  and  $\tau(y) = \tau(y')$ , so the pair  $y, y' \in Y$  satisfies  $(y, y') \in Y \times_X Y \setminus \Delta$ , where  $\Delta$  is the diagonal. Our spurious assumption (A1) that Serre duality holds on  $V = \mathbb{A}_{\mathbb{k}}^n$ , combined with the fact that  $\omega_V$  is trivial as a  $G$ -equivariant sheaf implies that

$$G\text{-Ext}^n(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}) \cong G\text{-Hom}(\mathcal{O}_{Z_{y'}}, \mathcal{O}_{Z_y}) = 0$$

(omitting the dual). Since the global dimension of the skew group algebra is  $n$ , we obtain

$$(7.6) \quad (y, y') \in Y \times_X Y \text{ with } y \neq y' \implies G\text{-Ext}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}) = 0 \quad \text{unless } 1 \leq i \leq n-1.$$

We're now in a position to make the following important claim:

**Claim 7.13.** *The assumption  $\dim(Y \times_X Y) \leq n+1$  in Theorem 7.6 is imposed to ensure that the groups  $G\text{-Ext}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}})$  vanish for the remaining values  $1 \leq i \leq n-1$ .*

**7.4. On the Intersection theorem.** To justify Claim 7.13 and hence complete step three of our discussion, we first collect a few results from Bridgeland–Maciocia [17]. The *support* of an object  $E \in D^b(\text{coh}(Y))$ , denoted  $\text{supp}(E)$ , is the closed subset of  $Y$  obtained as the union of the supports of the cohomology sheaves  $\mathcal{H}^i(E)$ . The proof of the following lemma is a simple spectral sequence argument [17, Lemma 5.3].

**Lemma 7.14.** *Let  $E \in D^b(\text{coh}(Y))$  be an object and  $y \in Y$  a point. Then*

$$y \in \text{supp}(E) \iff \exists i \in \mathbb{Z} \text{ such that } \text{Hom}(E, \mathcal{O}_y[i]) \neq 0.$$

The *homological dimension* of a nonzero object  $E \in D^b(\text{coh}(Y))$  is the smallest integer  $\text{homdim}(E) := d$  such that  $E$  is quasi-isomorphic to a complex of locally-free sheaves of length  $d$ . An upper bound on the homological dimension was provided by Bridgeland–Maciocia [17]:  $\text{homdim}(E) \leq d$  if and only if there exists  $j \in \mathbb{Z}$  such that for all  $y \in Y$ ,

$$(7.7) \quad \text{Hom}(E, \mathcal{O}_y[i]) = 0 \quad \text{unless } j \leq i \leq j+d;$$

The next result, however, is the geometric interpretation of a deep result from commutative algebra called the Intersection Theorem.

**Theorem 7.15.** *Let  $E \in D^b(\text{coh}(Y))$  be a nonzero object.*

- (i) *We have  $\text{codim}(\text{supp}(E)) \leq \text{homdim}(E)$ .*

(ii) Suppose that  $H^0(E) \cong \mathcal{O}_y$  for some closed point  $y \in Y$ . Suppose further that for any point  $y' \in Y$  and integer  $i \in \mathbb{Z}$  we have

$$\mathrm{Hom}(E, \mathcal{O}_{y'}[i]) = 0 \quad \text{unless } y = y' \text{ and } 0 \leq i \leq n.$$

Then  $E \cong \mathcal{O}_y$ , and  $Y$  is smooth at  $y$ .

*Proof.* See Bridgeland–Maciocia [17, Section 5] (compare also Bridgeland–Iyengar [15]).  $\square$

**7.5. Conclusion of the proof.** We now return to the discussion of the third step.

*Sketch proof (continued).* To describe the logic in the rest of the proof, suppose that we can find an object  $\mathcal{Q} \in D^b(\mathrm{coh}(Y \times Y))$  such that

$$(7.8) \quad \mathrm{Hom}_{D^b(\mathrm{coh}(Y \times Y))}(\mathcal{Q}, \mathcal{O}_{(y,y')}[i]) = G\text{-Ext}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}).$$

In light of the vanishing from (7.5), applying Lemma 7.14 to the object  $\mathcal{Q}|_{Y \times Y \setminus \Delta}$  shows that the support of  $\mathcal{Q}|_{Y \times Y \setminus \Delta}$  is contained in  $Y \times_X Y \setminus \Delta$ . The assumption on the dimension of the fibre product from Theorem 7.6 now implies that  $\dim(\mathrm{supp}(\mathcal{Q}|_{Y \times Y \setminus \Delta})) \leq n + 1$ , i.e.,

$$\mathrm{codim}(\mathrm{supp}(\mathcal{Q}|_{Y \times Y \setminus \Delta})) \geq n - 1.$$

On the other hand, if the object  $\mathcal{Q}|_{Y \times Y \setminus \Delta}$  is nonzero, then substituting the vanishing from (7.6) into formula (7.7) gives

$$\mathrm{homdim}(\mathcal{Q}|_{Y \times Y \setminus \Delta}) \leq n - 2.$$

This contradicts the inequality from Theorem 7.15(i) unless  $\mathcal{Q}|_{Y \times Y \setminus \Delta} \cong 0$ . Now chase the logic backwards: since  $\mathcal{Q}$  is supported on  $\Delta$ , Lemma 7.14 together with (7.8) imply that

$$G\text{-Ext}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}) = 0 \quad \text{for all } (y, y') \in Y \times_X Y \setminus \Delta \text{ and } i \in \mathbb{Z}.$$

This completes the required  $G$ -Ext-vanishing in the case  $y \neq y'$  and hence completes step three.

The fourth step, and by far the hardest, is to establish the equality from (7.4) for the case  $y = y'$ . Our goal is to show that for all  $y \in Y$  and  $i \in \mathbb{Z}$ , we have

$$G\text{-Ext}_{\mathcal{O}_{\mathbb{A}_{\mathbb{k}}^n}}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_y}) = \mathrm{Ext}_{\mathcal{O}_Y}^i(\mathcal{O}_y, \mathcal{O}_y).$$

Note that both vanish for  $i \notin [0, n]$ . As for  $0 \leq i \leq n$ , we add a little to [16, §6, Step 5]. For a point  $y \in Y$ , the adjunction  $\Psi \dashv \Phi$  gives a canonical map  $\Psi(\Phi(\mathcal{O}_y)) \rightarrow \mathcal{O}_y$  and hence a triangle

$$(7.9) \quad C \rightarrow \Psi(\Phi(\mathcal{O}_y)) \rightarrow \mathcal{O}_y \rightarrow C[1]$$

for some object  $C \in D^b(\mathrm{coh}(Y))$ . Using the adjunction  $\Psi \dashv \Phi$  again, we claim that the long exact sequence obtained by applying  $\mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(-, \mathcal{O}_y)$  to the triangle (7.9) is

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(C, \mathcal{O}_y[-1]) \longrightarrow \mathrm{Hom}(\mathcal{O}_y, \mathcal{O}_y) \xrightarrow{\alpha} \mathrm{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) \longrightarrow \\ \mathrm{Hom}(C, \mathcal{O}_y) \longrightarrow \mathrm{Hom}(\mathcal{O}_y, \mathcal{O}_y[1]) \xrightarrow{\varepsilon} \mathrm{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)[1]) \longrightarrow \dots \end{aligned}$$

Indeed, both  $\mathrm{Hom}(\mathcal{O}_y, \mathcal{O}_y[i]) = \mathrm{Ext}^i(\mathcal{O}_y, \mathcal{O}_y)$  and  $\mathrm{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)[i]) = G\text{-Ext}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_y})$  vanish for  $i < 0$ . Moreover,  $G\text{-Hom}(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_y}) \cong \mathbb{k}$  because  $G$ -invariance forces the generator  $1 \in H^0(\mathcal{O}_{Z_y}) \cong \mathbb{k}[G]$  to be sent to a scalar multiple of itself. This forces  $\alpha$  to be injective, hence  $\mathrm{Hom}(C, \mathcal{O}_y[-1]) = 0$ . Since  $Y$  is the fine moduli space of  $G$ -clusters, Bridgeland [13, Lemma 5.3] shows that the Kodaira–Spencer map  $\varepsilon$  for the universal sheaf  $\mathcal{O}_{\mathcal{Z}}$  over the product  $Y \times \mathbb{A}_{\mathbb{k}}^n$  is injective (see Huybrechts [43, Example 5.4(vii)]). This gives  $\mathrm{Hom}(C, \mathcal{O}_y) = 0$ . Now that we

have  $\mathrm{Hom}(C, \mathcal{O}_y[i]) = 0$  for all  $i \leq 0$ , the spectral sequence argument from [13, Example 2.2] gives  $H^0(C) = 0$ . Taking cohomology sheaves in the triangle (7.9) implies

$$\mathcal{H}^0(\Psi(\Phi(\mathcal{O}_y))) = \mathcal{O}_y.$$

Therefore  $\Psi(\Phi(\mathcal{O}_y))$  satisfies the first condition of Theorem 7.15(ii). Moreover, applying adjunction again, our third step shows that the groups

$$\mathrm{Hom}(\Psi(\Phi(\mathcal{O}_y)), \mathcal{O}_{y'}[i]) = \mathrm{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_{y'})[i]) = G\text{-Ext}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}})$$

vanish unless  $y = y'$  and  $0 \leq i \leq n$ . This vanishing is the second condition of Theorem 7.15(ii), so we conclude  $\Psi(\Phi(\mathcal{O}_y)) \cong \mathcal{O}_y$  and hence that the object  $C$  in the triangle (7.9) is zero. Substituting  $\mathrm{Hom}(C, \mathcal{O}_y[i]) = 0$  in to the long exact sequence above gives the required isomorphisms

$$\mathrm{Hom}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)[i]) \cong \mathrm{Hom}(\mathcal{O}_y, \mathcal{O}_y[i])$$

for  $0 \leq i \leq n$ . This clears up the remaining cases in (7.4) and concludes step four.

Our fifth and final step is to construct an object  $\mathcal{Q}$  satisfying (7.8), after which we conclude that  $\Phi^{\mathcal{O}^z}$  is fully faithful. In fact, since  $\omega_Y$  is trivial as a  $G$ -equivariant sheaf,  $\Phi^{\mathcal{O}^z}$  commutes with the Serre functors and hence Theorem 7.10 implies that  $\Phi^{\mathcal{O}^z}$  is a derived equivalence. We conclude, then, by constructing the crucial object  $\mathcal{Q}$ . As with the functor arising in the proof of Theorem 6.3, we compose  $\Phi$  with its left-adjoint  $\Psi$  to obtain the composite

$$(\Psi \circ \Phi)(-) = \mathbf{R}(\pi_2)_*(\mathcal{Q} \otimes^{\mathbf{L}} \pi_1^*(-)),$$

where  $\pi_1, \pi_2: Y \times Y \rightarrow Y$  are the first and second projections and where  $\mathcal{Q} \in D^b(\mathrm{coh}(Y \times Y))$  is obtained by composition of correspondences. For the closed immersion  $\iota_y: \{y\} \times Y \hookrightarrow Y \times Y$  we have  $\mathbf{L}\iota_y^*(\mathcal{Q}) = \Psi(\Phi(\mathcal{O}_y)) = \mathcal{O}_y$ . Thus, for  $i \in \mathbb{Z}$ , we have

$$\begin{aligned} \mathrm{Hom}_{D^b(\mathrm{coh}(Y \times Y))}(\mathcal{Q}, \mathcal{O}_{(y, y')}[i]) &\cong \mathrm{Hom}_{D^b(\mathrm{coh}(Y \times Y))}(\mathcal{Q}, (\iota_y)_*(\mathcal{O}_{y'})[i]) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\mathbf{L}\iota_y^*(\mathcal{Q}), \mathcal{O}_{y'}[i]) && \text{by adjunction} \\ &= \mathrm{Hom}_{D^b(\mathrm{coh}(Y))}(\Psi(\Phi(\mathcal{O}_y)), \mathcal{O}_{y'}[i]) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{mod}(A))}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_{y'})[i]) && \text{by adjunction} \\ &\cong G\text{-Ext}^i(\mathcal{O}_{Z_y}, \mathcal{O}_{Z_{y'}}) \end{aligned}$$

as required. This completes the proof given our outrageous assumptions (A1) and (A2).

**Remark 7.16.** We describe briefly how [16] proves the result without (A1) and (A2).

- (A1) The trick is to consider a projective closure of  $\mathbb{A}_{\mathbb{k}}^n$  and hence a projective closure of the  $G$ -Hilbert scheme  $Y$ . One then restricts the resulting derived equivalence to a functor on compactly supported objects. An alternative approach to the lack of projectivity was given more recently by Logvinenko [53].
- (A2) The statement of Theorem 7.10 uses the fact that the skyscraper sheaves  $\{\mathcal{O}_y : y \in Y\}$  form a spanning class in  $D^b(\mathrm{coh}(Y))$ , but this follows only from smoothness of  $Y$ . In fact, smoothness follows when we apply the Intersection Theorem in Step 4 above, after which one can apply Theorem 7.10 in the given form. Finally, that  $\tau$  is crepant is proven only after establishing the derived equivalence. A local argument shows that triviality of the Serre functor on  $D^b(\mathrm{mod}(A))$  carries across the equivalence to gives  $\omega_Y \cong \mathcal{O}_Y$ .

**Remark 7.17.** Van den Bergh [65] observes that the object  $\mathcal{Q} \in D^b(\mathrm{coh}(Y \times Y))$  which plays the crucial role in the proof of Theorem 7.6 satisfies

$$\mathcal{Q} = \mathcal{W}^{\vee} \boxtimes_A^{\mathbf{L}} \mathcal{W}$$

(compare Proposition 6.7). This links the Fourier–Mukai construction of Bridgeland, King and Reid with the tilting approach to derived equivalences described in Section 6.

## 8. DERIVED MCKAY BY VARIATION OF GIT QUOTIENT

The derived McKay Correspondence was proposed by Reid [62] (see Conjecture 7.4) for an arbitrary crepant resolution  $Y$  of  $\mathbb{A}_{\mathbb{k}}^n/G$  for a finite subgroup  $G \subset \mathrm{SL}(n, \mathbb{k})$ , assuming one exists. For  $n = 3$ , Theorem 7.6 establishes that  $G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^3)$  always provides a crepant resolution, but what if  $\mathbb{A}_{\mathbb{k}}^3/G$  admits more than one crepant resolution? And what if  $G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^n)$  is singular or the Hilbert–Chow morphism fails to be crepant for  $n \geq 4$ ? Can a crepant resolution exist and if so, does the McKay correspondence still hold? This section answers these questions for abelian group actions using toric techniques.

**8.1. Variation of GIT quotient.** Let  $G \subset \mathrm{SL}(n, \mathbb{k})$  be a finite abelian subgroup with group algebra  $\mathbb{k}[G]$  and skew group algebra  $A = \mathbb{k}[x_1, \dots, x_n] * G$ . The vertices of the bound McKay quiver  $(Q, \varrho)$  (see Example 4.7) correspond to elements of the character group  $G^* = \mathrm{Hom}(G, \mathbb{k}^\times)$ . Recall from Example 4.19 that a  $G$ -constellation is a left  $A$ -module that is isomorphic as a  $\mathbb{k}[G]$ -module to  $\mathbb{k}[G]$ . For any generic weight  $\theta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ , the GIT quotient

$$\mathcal{M}_\theta := \mathcal{M}_\theta(Q, \varrho) = \mathbb{V}(I_Q) //_\theta T$$

constructed in Section 4.6 is the fine moduli space of  $\theta$ -stable  $G$ -constellations (it represents the functor from Exercise 4.18). Thus,  $\mathcal{M}_\theta$  carries a tautological bundle  $\mathcal{W}_\theta := \bigoplus_{\rho \in G^*} (\mathcal{W}_\theta)_\rho$  obtained as a direct sum of line bundles with  $(\mathcal{W}_\theta)_{\rho_0}$  trivial and, moreover, there is a natural homomorphism of  $\mathbb{k}$ -algebras  $A \rightarrow \mathrm{End}(\mathcal{W}_\theta)$ .

As with the  $G$ -Hilbert scheme, we will restrict to the irreducible component of  $\mathcal{M}_\theta$  containing the  $G$ -constellations arising as the structure sheaves of the free  $G$ -orbits in  $\mathbb{A}_{\mathbb{k}}^n$ . This *coherent component* was constructed explicitly by Craw–Maclagan–Thomas [28, 29] in terms of a toric ideal  $I_Q$  in the coordinate ring of  $\mathbb{A}_{\mathbb{k}}^{Q_1}$  (compare Remark 5.22):

**Theorem 8.1.** *The binomial scheme  $\mathbb{V}(I_Q)$  has a unique irreducible component  $\mathbb{V}(I_Q)$  that lies in no coordinate hyperplane of  $\mathbb{A}_{\mathbb{k}}^{Q_1}$ . Moreover, the affine toric variety  $\mathbb{V}(I_Q)$  is such that*

- (i) *the semiprojective toric variety  $Y_\theta := \mathbb{V}(I_Q) //_\theta T$  is the unique irreducible component of  $\mathcal{M}_\theta$  containing the free  $G$ -orbits;*
- (ii) *if we write  $\overline{\mathcal{M}}_0$  for the categorical quotient of the 0-semistable points of  $\mathbb{V}(I_Q)$  by the action of  $T$ , then there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_\theta & \xrightarrow{\tau} & \overline{\mathcal{M}}_0 \\ \uparrow & & \uparrow \\ Y_\theta & \xrightarrow{\tau|_{Y_\theta}} & X = \mathbb{A}_{\mathbb{k}}^n/G \end{array}$$

*where the vertical maps are closed immersions and the horizontal maps are projective morphisms obtained by variation of GIT quotient sending  $\theta \rightsquigarrow 0$ .*

**Corollary 8.2.** *For any weight  $\vartheta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  such that  $\mathcal{M}_\vartheta \cong G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^n)$ , the coherent component  $Y_\vartheta$  is isomorphic to the irreducible scheme  $\mathrm{Hilb}^G(\mathbb{A}_{\mathbb{k}}^n)$  introduced by Nakamura.*

**Remark 8.3.** Recall from Example 4.19 that the  $G$ -Hilbert scheme can be nonnormal. In fact, it is the coherent component  $Y_\vartheta = \mathrm{Hilb}^G(\mathbb{A}_{\mathbb{k}}^n)$  that is not normal in that example. Thus, even if one hopes to work only with normal toric varieties (defined by fans), this example forces one to drop the normality assumption.

As we vary the stability parameter  $\theta$ , the geometry of  $Y_\theta$  may change as  $\theta$  varies between GIT chambers. Even if the variety does not change, the tautological bundle  $\mathscr{W}_\theta$  on  $\mathcal{M}_\theta$  and its restriction to  $Y_\theta$  does vary between chambers.

**Example 8.4.** For the action of type  $\frac{1}{3}(1, 2)$ , the McKay quiver  $Q$  is shown in Figure 5.4. For the weight  $\vartheta = (-2, 1, 1) \in \text{Wt}(Q)$ , Example 5.21 shows directly that  $\mathcal{M}_\vartheta$  is isomorphic to the minimal resolution  $Y$  of the  $A_2$ -singularity  $\mathbb{A}_{\mathbb{k}}^2/G$ . The GIT chamber decomposition of  $\text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  coincides with the Weyl chamber decomposition of type  $A_2$  by Kronheimer [52] (see also Cassens–Slodowy [20]). Here we compute this directly.

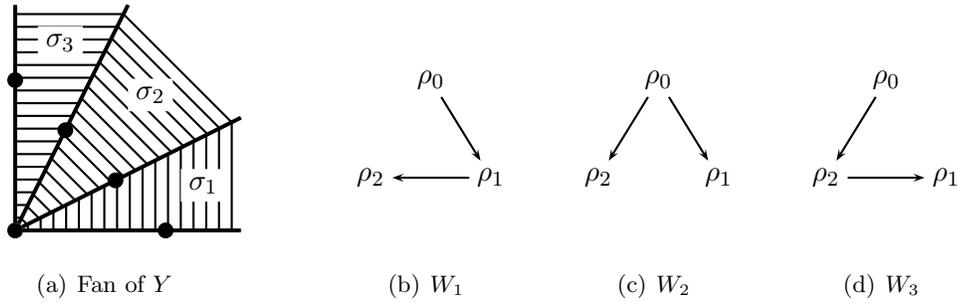


FIGURE 10.  $T_Y$ -invariant  $\vartheta$ -stable quiver representations

The three  $\vartheta$ -stable representations of  $(Q, \varrho)$  that correspond to the  $T_Y$ -invariant points of  $Y$  are shown in Figure 10; here, the representation  $W_i$  corresponds to the origin in the toric chart  $\text{Spec}(\mathbb{k}[\sigma_i^Y \cap M])$  in  $Y$  for  $i = 1, 2, 3$ . One can perform this calculation either by computing the  $T_Y$ -invariant  $G$ -clusters in  $\mathbb{A}_{\mathbb{k}}^2$ , or directly using Craw–Maclagan–Thomas [28, Theorem 1.3]. Notice that the representation  $W_2$  is  $\vartheta$ -stable for all parameters in the cone

$$C = \{(\theta_0, \theta_1, \theta_2) \in \mathbb{Q}^3 : \theta_0 + \theta_1 + \theta_2 = 0, \theta_1 > 0, \theta_2 > 0\};$$

this is the cone from Example 4.12. Now perform variation of GIT quotient, pushing  $\theta$  through the wall  $\theta_1 = 0$  to produce a weight  $\vartheta'$  in an adjacent chamber

$$C' = \{(\theta_0, \theta_1, \theta_2) \in \mathbb{Q}^3 : \theta_0 + \theta_1 + \theta_2 = 0, \theta_1 + \theta_2 > 0, \theta_1 < 0\}.$$

The representation  $W_1$  remains  $\vartheta'$ -stable since  $\vartheta'_1 + \vartheta'_2 > 0$ . However, both  $W_2$  and  $W_3$  admit submodules supported on vertex  $\rho_1$ , so they become  $\vartheta'$ -unstable for  $\vartheta'_1 \leq 0$ . In their place we obtain the new pair of  $\vartheta'$ -stable representations  $W'_2$  and  $W'_3$  shown in Figure 11. If we now pass

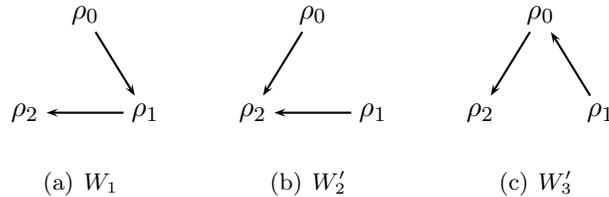


FIGURE 11.  $T_Y$ -invariant  $\vartheta'$ -stable quiver representations

through the wall  $\theta_1 + \theta_2 = 0$ , it's clear that  $W'_3$  remains  $\theta$ -stable as long as  $\theta_2 > 0$ , while both  $W_1$  and  $W'_2$  becomes  $\theta$ -unstable for  $\theta_1 + \theta_2 \leq 0$ .

*Exercise 8.5.* Continue the calculation from Example 8.4 to produce all six GIT chambers. The result coincides precisely<sup>1</sup> with the fan from Figure 2(d). The symmetry in the  $A_2$ -decomposition is lost only because we project  $\mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathbb{Q}^3$  onto the plane ( $\theta_0 = 0$ ).

**8.2. Derived equivalences from universal bundles.** The projective morphism constructed in Theorem 8.1(ii) can be used to construct a commutative diagram of the form (7.1), and hence by using the universal  $\theta$ -stable  $G$ -constellation  $\mathcal{U}_\theta$  on the product  $Y_\theta \times \mathbb{A}_{\mathbb{k}}^n$  we define a functor  $\Phi^{\mathcal{U}_\theta}: D^b(\mathrm{coh}(Y_\theta)) \rightarrow D^b(\mathrm{mod}(A))$  via

$$(8.1) \quad \Phi^{\mathcal{U}}(-) = \mathbf{R}(\pi_V)_* \left( \mathcal{U}_\theta \otimes^{\mathbf{L}} (\pi_{Y_\theta})^*(- \otimes \rho_0) \right).$$

The results of Bridgeland, King and Reid [16] described in Section 7 require only that the scheme  $Y$  is the coherent component of a fine moduli space of  $\theta$ -stable  $G$ -constellations for some generic weight  $\theta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ , so the results extend from  $G$ -Hilb( $\mathbb{A}_{\mathbb{k}}^n$ ) to  $\mathcal{M}_\theta$ . Stability preserves the vanishing of  $G$ -Hom( $\mathcal{F}_y, \mathcal{F}_{y'}$ ) for distinct points  $y, y' \in Y_\theta$ , since  $\theta$ -stable  $G$ -constellations  $\mathcal{F}_y$  are simple objects in the full category of  $\theta$ -semistable  $A$ -modules. Thus, we obtain the following (the abelian assumption is not necessary here, see [26, Theorem 2.5]):

**Theorem 8.6.** *Let  $G \subset \mathrm{SL}(n, \mathbb{k})$  be a finite abelian subgroup and let  $\theta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  be generic. If the coherent component  $Y_\theta \subseteq \mathcal{M}_\theta$  satisfies  $\dim(Y_\theta \times_X Y_\theta) \leq n + 1$ , then:*

- (1) *the morphism  $\tau: Y_\theta \rightarrow X$  is a crepant resolution; and*
- (2) *the functor  $\Phi^{\mathcal{U}_\theta}: D^b(\mathrm{coh}(Y_\theta)) \rightarrow D^b(\mathrm{mod}(A))$  is an equivalence of derived categories.*

**Corollary 8.7.** *The  $\mathbb{k}$ -algebra  $\mathrm{End}(\mathcal{W}_\theta)$  is isomorphic to the skew group algebra  $A$ .*

*Proof.* To simplify notation, write  $\mathcal{W} = \bigoplus_{\rho \in G^*} \mathcal{W}_\rho$  thereby omitting the dependence on  $\theta$ . Then

$$\begin{aligned} A &\cong \mathrm{Hom}_A(A, A) \\ &\cong G\text{-Hom}_{\mathcal{O}_{\mathbb{A}_{\mathbb{k}}^n}} \left( \bigoplus_{\rho \in G^*} \mathcal{O}_{\mathbb{A}_{\mathbb{k}}^n} \otimes \rho, \bigoplus_{\rho \in G^*} \mathcal{O}_{\mathbb{A}_{\mathbb{k}}^n} \otimes \rho \right) && \text{by the equivalence from Lemma 7.5} \\ &\cong \mathrm{Hom}_{Y_\theta} \left( \bigoplus_{\rho \in G^*} \mathcal{W}_{\rho^*}^{-1}, \bigoplus_{\rho \in G^*} \mathcal{W}_{\rho^*}^{-1} \right) && \text{by applying the equivalence } \Phi^{\mathcal{U}_\theta} \end{aligned}$$

which is isomorphic to  $\mathrm{End}_{Y_\theta}(\mathcal{W})$  as claimed. □

**8.3. The 3-fold case.** In dimension 3, the  $G$ -Hilbert scheme is a projective crepant resolution of  $\mathbb{A}_{\mathbb{k}}^3/G$ , but in general there may be more than one. We now describe how to establish the derived equivalence from Conjecture 7.4 for the other projective crepant resolutions when the finite subgroup  $G \subset \mathrm{SL}(3, \mathbb{k})$  is abelian.

First, we describe how to construct crepant resolutions of  $\mathbb{A}_{\mathbb{k}}^3/G$  using toric geometry. The abelian quotient singularity  $\mathbb{A}_{\mathbb{k}}^3/G$  is the normal affine toric variety  $\mathrm{Spec}(\mathbb{k}[\sigma^\vee \cap M])$  from Example 3.17. The *junior simplex* is the lattice triangle in  $N \otimes_{\mathbb{Z}} \mathbb{Q}$  obtained as the intersection of the positive octant with the hyperplane normal to the vector  $(1, 1, 1)$ . Every basic triangulation of the junior simplex determines uniquely a fan  $\Sigma$  supported on the positive orthant such that every three-dimensional cone  $\sigma \in \Sigma$  is spanned by a lattice basis of  $N$ . If the triangulation is *coherent* (see Sturmfels [63]) then the smooth toric variety  $Y$  defined by the fan  $\Sigma$  is projective over  $\mathbb{A}_{\mathbb{k}}^3/G$ . Moreover, the resolution  $Y \rightarrow \mathbb{A}_{\mathbb{k}}^3/G$  is crepant and, conversely, every projective crepant resolution arises in this way.

<sup>1</sup>I cannot resist pointing out the the toric del Pezzo surface  $\mathrm{Bl}_{p,q,r}(\mathbb{P}^2)$  and the crepant resolution  $\mathcal{M}_\theta$  of the  $A_2$ -singularity are linked by the Duality Theorem for affine toric quotients of Craw–Maclagan [27, Theorem 1.2].

**Example 8.8.** Consider the subgroup  $G \cong \mathbb{Z}/2 \times \mathbb{Z}/2$  of  $\mathrm{SL}(3, \mathbb{k})$ , where the action is of type  $\frac{1}{2}(1, 1, 0) \oplus \frac{1}{2}(1, 0, 1)$ . Following the toric recipe described in Example 3.17, the junior simplex is shown in Figure 12(a). In this case there are four basic triangulations, all of which are coherent. The junior simplex for two such fans are shown in Figure 12(b) and (c).

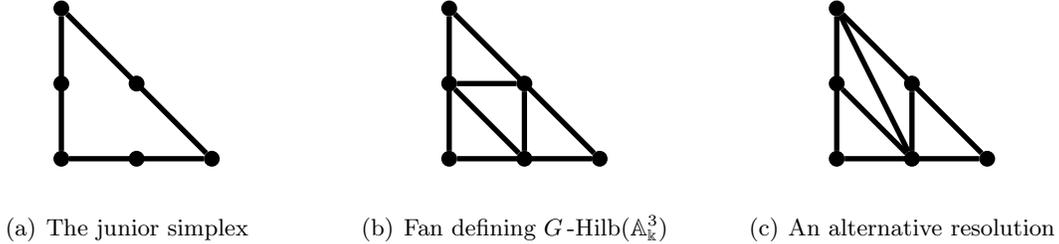


FIGURE 12. Resolving the quotient by  $\mathbb{Z}/2 \times \mathbb{Z}/2$  in  $\mathrm{SL}(3, \mathbb{k})$

**Remark 8.9.** Using the description of the toric fan of  $G$ -Hilb( $\mathbb{A}_{\mathbb{k}}^3$ ) from Craw–Reid [30], one can show that  $\mathbb{A}_{\mathbb{k}}^3/G$  admits a unique crepant resolution if and only if the action of  $G$  is either (i) of type  $\frac{1}{r}(1, r-1, 0)$  for some  $r \geq 1$ ; (ii) of type  $\frac{1}{r}(1, 1, r-2)$  for some  $r \geq 1$ ; or (iii) of type  $\frac{1}{7}(1, 2, 4)$ . In particular, quotients  $\mathbb{A}_{\mathbb{k}}^3/G$  that admit a unique crepant resolution are very rare.

The following result of Craw–Ishii [26] addresses the question of the McKay correspondence for quotients  $\mathbb{A}_{\mathbb{k}}^3/G$  that admit more than one projective crepant resolution.

**Theorem 8.10.** *Let  $G \subset \mathrm{SL}(3, \mathbb{k})$  be a finite abelian subgroup. For every projective crepant resolution  $\tau: Y \rightarrow \mathbb{A}_{\mathbb{k}}^3/G$ , there exists a generic weight  $\theta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  such that  $Y \cong \mathcal{M}_{\theta}$ .*

Together with Theorem 8.6, this implies the derived McKay correspondence Conjecture 7.4 for the class of projective crepant resolutions of  $\mathbb{A}_{\mathbb{k}}^3/G$ . Moreover, by considering the compositions

$$(\Phi^{\mathcal{W}_{\theta'}})^{-1} \circ \Phi^{\mathcal{W}_{\theta}}: D^b(\mathrm{coh}(Y_{\theta})) \rightarrow D^b(\mathrm{coh}(Y_{\theta'}))$$

arising from any pair of generic weights  $\theta, \theta' \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ , we see that any two projective crepant resolutions of  $\mathbb{A}_{\mathbb{k}}^3/G$  have equivalent derived categories.

**Remark 8.11.** For any finite subgroup  $G \subset \mathrm{SL}(3, \mathbb{k})$ , Bridgeland [14] constructs derived equivalences between the bounded derived categories of coherent sheaves on any pair of projective crepant resolutions of  $\mathbb{A}_{\mathbb{k}}^3/G$ . This establishes Conjecture 7.4 in dimension  $n = 3$  without having to restrict to abelian subgroups. More recently, Logvinenko [53] established Conjecture 7.4 for a specific nonprojective crepant resolution.

**8.4. Sketch of the proof.** The  $G$ -Hilbert scheme provides one crepant resolution of  $\mathbb{A}_{\mathbb{k}}^3/G$ , and every other crepant resolution is obtained from  $G$ -Hilb( $\mathbb{A}_{\mathbb{k}}^3$ ) by a finite sequence of flops. Since  $G$ -Hilb( $\mathbb{A}_{\mathbb{k}}^3$ ) =  $\mathcal{M}_{\vartheta}$  for the weight  $\vartheta$  from Remark 4.20, it is enough to prove that if  $\mathcal{M}_{\theta}$  is a crepant resolution for generic  $\theta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ , and if  $Y'$  is another crepant resolution that is obtained from  $\mathcal{M}_{\theta}$  by the flop of a toric curve, then  $Y' = \mathcal{M}_{\theta'}$  for some other generic  $\theta'$ . Thus, we choose one such flop  $\mathcal{M}_{\theta} \dashrightarrow Y'$  and must show that we can induce the flop by variation of GIT quotient in the weight space  $\mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ , thereby obtaining  $Y' = \mathcal{M}_{\theta'}$ .

Before explaining the logic of the proof we first describe the link between the space of weights and birational geometry of the moduli spaces. For generic  $\theta \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ , write  $C$  for the open GIT chamber containing  $\theta$  (see Section 2.6). For simplicity, write  $\mathcal{W} = \bigoplus_{\rho \in G^*} \mathcal{W}_{\rho}$  for the

tautological bundle on the fine moduli space  $Y = \mathcal{M}_\theta$ . It follows tautologically from the GIT construction of  $\mathcal{M}_\theta$  that the map

$$(8.2) \quad L_\theta: \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

sending  $\eta = (\eta_\rho)_{\rho \in G^*} \in \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  to the (fractional) line bundle  $\bigotimes_{\rho \in G^*} \mathcal{W}'_\rho^{\eta_\rho}$  sends every weight in  $C$  to an ample bundle. Thus, there are two options for a weight  $\theta_0$  in a codimension-one face  $W$  of the boundary of the closure  $\overline{C}$ : the line bundle  $L := L_\theta(\theta_0)$  either lies in the interior of the ample cone, in which case it is ample; or it lies on the boundary of the closure, in which case it is nef but not ample. If we could move  $L$  freely around the ample cone of  $Y$ , then we would simply procede towards the codimension-one face of the nef cone that defines our flop  $\mathcal{M}_\theta = Y \dashrightarrow Y'$  and pass  $L$  through the wall, thereby inducing the flop.

However, while we may move freely in  $\mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ , this does not mean we may move the corresponding line bundle freely in  $\mathrm{Pic}(Y)$ . To see this, suppose that passing  $\theta$  through the wall  $W$  defines an isomorphism rather than the desired flop (so  $L_\theta(\theta_0)$  was in the interior of the ample cone). We have  $\mathcal{M}_\theta = Y \cong \mathcal{M}_{\theta'}$  for  $\theta'$  in the adjacent chamber  $C'$ , but the new tautological bundle is  $\mathcal{W}' = \bigoplus_{\rho \in G^*} (\mathcal{W}')_\rho$ . Thus, the line bundle associated to  $\theta' \in C'$  is determined by

$$L_{\theta'}: \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where  $L_{\theta'}(\eta) = \bigotimes_{\rho \in G^*} (\mathcal{W}')_\rho^{\eta_\rho}$ . The maps  $L_\theta$  and  $L_{\theta'}$  are  $\mathbb{Q}$ -linear, but together they form only a piecewise  $\mathbb{Q}$ -linear map on the union  $C \cup C'$ . Thus, while we may move  $L_\theta(\theta)$  to the boundary of  $L_\theta(C)$  and beyond, we do not know whether the line bundles  $L_{\theta'}(\theta')$  for  $\theta' \in C'$  lie any closer to the face of the nef cone of  $Y$  that gives our chosen flop. As Example 8.4 shows, crossing walls in  $\mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  may even define a reflection in  $\mathrm{Pic}(Y)$ . The majority of Craw–Ishii [26] is devoted to a careful analysis of how wall crossings affect  $\mathcal{M}_\theta$  and  $\mathcal{W}_\theta$ .

*Exercise 8.12.* For the action of type  $\frac{1}{2}(1, 1, 0) \oplus \frac{1}{2}(1, 0, 1)$  from Example 8.8. Compute explicitly the quiver representations that define the torus-invariant points of the  $G$ -Hilbert scheme, and hence show that  $G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^3) = \mathcal{M}_\vartheta$  if and only if  $\vartheta$  lies in the cone

$$C = \{(\theta_0, \theta_1, \theta_2, \theta_3) \in \mathbb{Q}^3 : \theta_0 + \theta_1 + \theta_2 + \theta_3 = 0, \theta_1 > 0, \theta_2 > 0, \theta_3 > 0\};$$

(compare Example 8.4). Show that all three flops of  $G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^3)$  can be produced by crossing a single wall of this chamber.

We now sketch the proof the Theorem 8.10. The key to understanding the wall crossings is to interpret geometrically the weight space  $\mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  using the Fourier–Mukai transform. For a generic weight  $\theta \in C$  with  $Y = \mathcal{M}_\theta$ , restricting  $\Phi^{\mathcal{W}_\theta}$  to the full subcategory  $D_0^b(\mathrm{coh}(Y))$  consisting of objects supported on the subscheme  $\tau^{-1}(\pi_V(0))$  induces a  $\mathbb{Z}$ -linear isomorphism

$$\varphi_C: K_0(Y) \longrightarrow K_0(\mathrm{mod}(A)) \cong \bigoplus_{\rho \in G^*} \mathbb{Z}\rho$$

between the Grothendieck groups of coherent sheaves supported on  $\tau^{-1}(\pi_V(0))$  and finitely generated left nilpotent  $A$ -modules. There is a natural isomorphism

$$\mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathrm{mod}(A)), \mathbb{Q}) : \theta(\mathbb{k}[G]) = 0\},$$

and hence for any compactly supported sheaf  $\mathcal{F}$  on  $Y$  we may compute  $\theta(\varphi_C(\mathcal{F})) \in \mathbb{Q}$ . The geometric interpretation of the walls of any given chamber can be described thus:

**Proposition 8.13.** *Let  $C \subset \mathrm{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  be a GIT chamber. Then  $\theta \in C$  if and only if*

- (i) *for every exceptional (e.g., flopping) curve  $\ell$  in  $Y$  we have  $\theta(\varphi_C(\mathcal{O}_\ell)) > 0$ ; and*

(ii) for every compact reduced divisor  $D$  in  $Y$  and  $\rho \in G^*$  we have

$$\theta(\varphi_C(\mathcal{R}_\rho^{-1} \otimes \omega_D)) < 0 \quad \text{and} \quad \theta(\varphi_C(\mathcal{R}_\rho^{-1}|_D)) > 0.$$

The inequalities listed in (i), i.e., those of the form  $\theta(\varphi_C(\mathcal{O}_\ell)) > 0$  for curves  $\ell$  in  $Y$ , are good in the sense that they lift via the map  $L_\theta$  from (8.2) inequalities defining the walls of the ample cone of  $Y$ . In particular, when we pass through these walls then we induce the corresponding birational change in  $Y$ . However, those listed in (ii), dubbed *walls of Type 0* in [26], are a priori problematic since they are not present in  $\text{Pic}(Y)$  and hence, as described above, we may lose track completely of the polarising line bundle as  $\theta$  passes through such a wall. It is important therefore to understand what happens to the variety  $\mathcal{M}_\theta$  and its tautological bundle  $\mathcal{W}_\theta$  as  $\theta$  passes through a wall of type 0.

To describe the changes, we recall that an object  $E \in D^b(\text{coh}(Y))$  that is supported on the subscheme  $\tau^{-1}(\pi_V(0))$  is *spherical* if

$$\text{Hom}_{D_0^b(\text{coh}(Y))}(E, E[j]) = \begin{cases} \mathbb{k} & \text{if } j = 0, 3; \\ 0 & \text{otherwise} \end{cases}$$

(see Huybrechts [43, Section 8.1]; we require no additional condition since  $\omega_Y \cong \mathcal{O}_Y$ ). The *twist* along a spherical object  $E$  is defined via the distinguished triangle

$$\mathbf{R}\text{Hom}_{\mathcal{O}_Y}(E, F) \otimes_{\mathbb{C}} E \xrightarrow{\text{ev}} F \longrightarrow T_E(F)$$

for any object  $F \in D^b(\text{coh}(Y))$ , where  $\text{ev}$  is the evaluation morphism. The corresponding *spherical twist functor*  $T_E: D^b(\text{coh}(Y)) \rightarrow D^b(\text{coh}(Y))$  is an autoequivalence.

**Lemma 8.14.** *Let  $C, C'$  be adjacent chambers in  $\text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  separated by a wall of type 0, and choose  $\theta \in C$  and  $\theta' \in C'$ . Then  $\mathcal{M}_\theta$  is isomorphic to  $\mathcal{M}_{\theta'}$ , and the composition*

$$(\Phi^{\mathcal{W}_{\theta'}})^{-1} \circ \Phi^{\mathcal{W}_\theta}: D^b(\text{coh}(\mathcal{M}_\theta)) \rightarrow D^b(\text{coh}(\mathcal{M}_{\theta'}))$$

*is a spherical twist functor (up to tensoring by a line bundle). Moreover, the changes in the tautological bundles  $\mathcal{W}_\theta$  can be tracked across the equivalence.*

We now complete the description of the proof of Theorem 8.10. For  $Y = \mathcal{M}_\theta$ , let  $C$  denote the chamber containing  $\theta$ . If  $C$  has a wall that defines the desired flop according to Proposition 8.13, then we pass through the wall, inducing the flop as required. Otherwise, move  $\theta$  into an adjacent chamber  $C'$  separated from  $C$  by a wall of type 0. A property of twists reveals that, roughly speaking, the images  $L_\theta(C)$  and  $L_{\theta'}(C')$  for  $\theta' \in C'$  are adjacent cones in the Picard group. We proceed in this way towards the desired wall of the ample cone, passing through at most finitely many such type 0 walls. By crossing the final wall we induce the desired flop  $\mathcal{M}_\theta = Y \dashrightarrow Y' = \mathcal{M}_{\theta'}$ .

**8.5. McKay in higher dimensions.** It is natural to ask whether new examples of the McKay correspondence can be constructed using this moduli space approach in dimension  $n \geq 4$ , when the singularity  $X = \mathbb{A}_{\mathbb{k}}^n/G$  may admit crepant resolutions even though  $G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^n)$  is singular or discrepant. This final section studies one such example.

Consider the quotient of  $\mathbb{A}_{\mathbb{k}}^4$  by the action of type  $\frac{1}{2}(1, 1, 1, 0) \oplus \frac{1}{2}(1, 1, 0, 1) \oplus \frac{1}{2}(1, 0, 1, 1)$ , that is, the action by the maximal diagonal subgroup  $G = (\mathbb{Z}/2)^{\oplus 3} \subset \text{SL}(4, \mathbb{k})$  of exponent two. The resolution  $Y = G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^4) \rightarrow X$  has one exceptional divisor  $E \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  of discrepancy one that can be blown down to  $\mathbb{P}^1 \times \mathbb{P}^1$  in three different ways, giving rise to crepant resolutions  $Y_i \rightarrow X$  with exceptional loci  $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$  for  $i = 1, 2, 3$ . All four resolutions of

$X$  are toric morphisms, and the 3-dimensional cross-sections of the 4-dimensional fans defining these resolutions are shown in Figure 8.5. (In fact, the junior simplex is a tetrahedron, and the octahedra below are obtained only after chopping off all four corners; see Chiang–Roan [21]): We

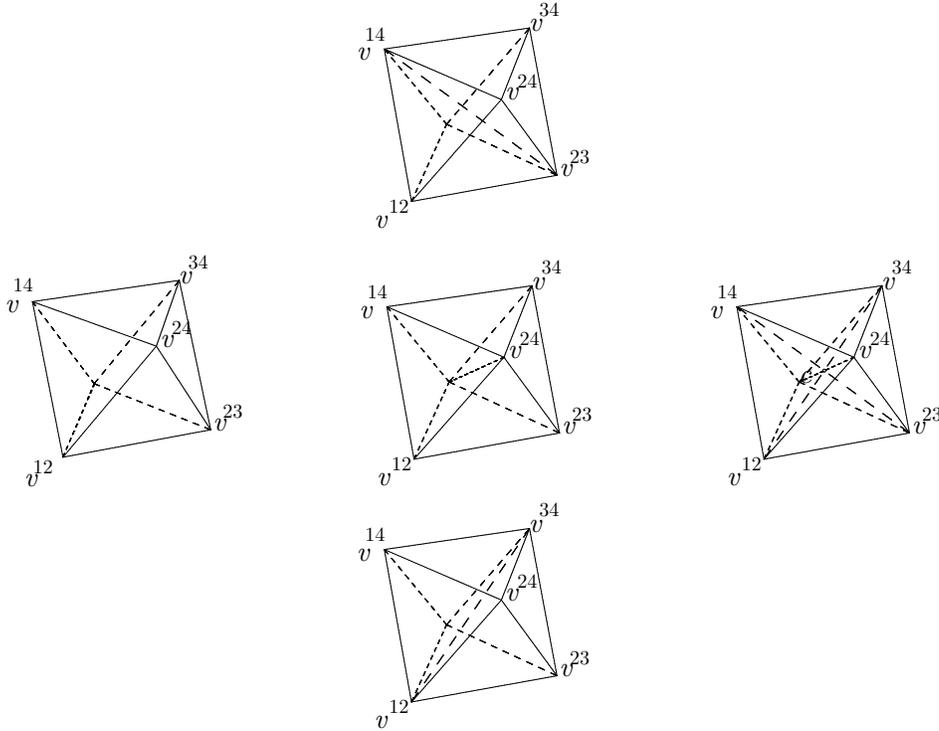


FIGURE 13. Toric picture as drawn by Chiang–Roan [21, Figure 3]

now show that each crepant resolution  $Y_i$  is a fine moduli space  $\mathcal{M}_\theta$  of  $\theta$ -stable representations of the bound McKay quiver  $(Q, \varrho)$  for some generic  $\theta \in \text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Note that these are not the only ones, there are another 189 distinct crepant resolutions (!).

The chamber containing the weights defining  $\mathcal{M}_\theta = G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^4)$  is

$$C_0 := \{ \theta = (\theta_\rho)_{\rho \in G^*} \in \text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q} : \theta_\rho > 0 \text{ for all } \rho \neq \rho_0 \}.$$

Write  $\mathbb{k}[x, y, z, w]$  for the coordinate ring of  $\mathbb{A}_{\mathbb{k}}^4$ . The variables  $x, y, z, w$  lie in distinct eigenspaces of the  $G$ -action that we denote  $\rho_{100}, \rho_{010}, \rho_{001}, \rho_{111} \in G^* \cong (\mathbb{Z}/2)^{\oplus 3}$ . To simplify the notation, we write  $\rho_0 = \rho_{000}, \rho_1 := \rho_{011}, \rho_2 := \rho_{101}, \rho_3 := \rho_{110}$  for the complementary set of nontrivial irreducible representations. For  $i = 1, 2, 3$ , let  $W_i := \{ \theta \in \overline{C_0} : \theta_{\rho_i} = 0 \}$  denote the codimension-one face of the closure of the chamber  $C_0$  that is contained in the indicated coordinate hyperplane.

**Lemma 8.15.** *For  $1 \leq i \leq 3$ , let  $C_i$  denote the unique GIT chamber lying adjacent to  $C_0$  that satisfies  $W_i = \overline{C_i} \cap \overline{C_0}$ . Then  $Y_i = \mathcal{M}_\theta$  for all  $\theta \in C_i$ .*

*Proof.* As in Example 8.4, we follow the fate of the  $\vartheta$ -stable  $G$ -constellations defined by the twelve torus-invariant points of  $Y$  as  $\vartheta$  passes through the wall  $W_i$ . There are three cases.

First, consider the four  $G$ -clusters defining the torus-invariant points corresponding to the four simplices that were omitted in the right-most drawing in Figure 8.5. One such  $G$ -cluster  $W_1$  is shown in Figure 14(a), while the other three are obtained by cyclically permutting  $x, y, z, w$ . These remain  $\theta$ -stable with respect to parameters  $\theta \in C_i$ . Of the remaining eight torus-invariant  $G$ -clusters, four are of the type shown in Figure 14(b), each obtained by cyclically permutting

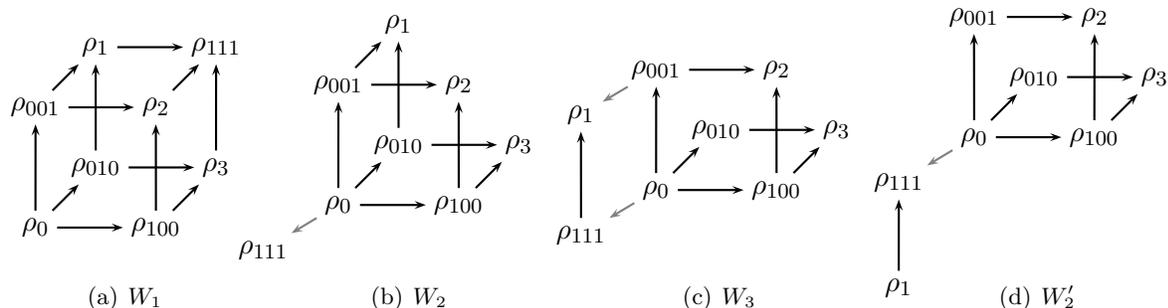


FIGURE 14.  $T_Y$ -invariant  $\vartheta'$ -stable quiver representations

$x, y, z, w$ , while the remaining four are of the type shown in Figure 14(c). All 8  $G$ -clusters in these latter two cases become unstable as  $\vartheta \in C_0$  moves to the chamber  $C_i$ . In addition, for  $\theta \in C_i$ , one new family of four  $\theta$ -stable  $G$ -constellations appears, one of which is shown in Figure 14(d). An explicit deformation calculation shows that for each  $i = 1, 2, 3$ , the four new  $G$ -constellations define the four torus-invariant points of  $Y_i$  corresponding to the four innermost simplices in the pictures that run down the centre of Figure 8.5. This shows that for each  $i = 1, 2, 3$ , the birational map  $G\text{-Hilb}(\mathbb{A}_{\mathbb{k}}^4) = \mathcal{M}_{\vartheta} \rightarrow \mathcal{M}_{\theta}$  induced by variation of GIT quotient through the wall  $W_i$  coincides with the contraction  $Y \rightarrow Y_i$ .  $\square$

Applying Theorem 8.6 carefully leads to the following result:

**Theorem 8.16.** *For  $i = 1, 2, 3$ , there is a chamber  $C_i \subset \text{Wt}(Q) \otimes_{\mathbb{Z}} \mathbb{Q}$  such that  $Y_i = \mathcal{M}_{\theta}$  for  $\theta \in C_i$ . Moreover, in each case the universal sheaf  $\mathcal{U}_{\theta}$  on  $Y_i \times \mathbb{A}_{\mathbb{k}}^4$  defines an equivalence of derived categories  $\Phi^{\mathcal{U}_{\theta}} : D^b(\text{coh}(Y_{\theta})) \rightarrow D^b(\text{mod}(A))$ .*

*Proof.* It remains to prove the second statement. Write  $\tau_i : \mathcal{M}_{\theta_i} \rightarrow X$  for the crepant resolution with  $\theta_i \in C_i$  for  $i = 1, 2, 3$ . The fibre  $\tau_i^{-1}(\pi(0))$  over the point  $\pi(0) \in \mathbb{A}_{\mathbb{k}}^4/G$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ , so it satisfies the dimension condition required to apply Theorem 8.10. This is not yet enough since the images under  $\pi : \mathbb{A}_{\mathbb{k}}^4 \rightarrow \mathbb{A}_{\mathbb{k}}^4/G$  of all four coordinate axes are also singular and hence have been resolved in creating the resolution. Nevertheless, these singularities arise along a subvariety of dimension one, so the dimension of the fibre product over any such point  $\pi(x) \in \mathbb{A}_{\mathbb{k}}^4/G$  is  $2 \cdot 3 - 1$ , which equals  $n + 1$  in this case. Theorem 8.10 now applies.  $\square$

**Remark 8.17.** Conjecture 7.4 has also been established as an equivalence of derived categories for finite subgroups  $G \subset \text{Sp}(n, \mathbb{C})$  by Kaledin–Bezrukavnikov [7] and for finite abelian subgroups  $G \subset \text{SL}(n, \mathbb{C})$  by Kawamata [48].

## REFERENCES

- [1] P. Aspinwall, T. Bridgeland, A. Craw, M. Douglas, M. Gross, A. Kapustin, G. Moore, G. Segal, B. Szendrői, and P. Wilson. *Dirichlet Branes and Mirror Symmetry*, 2008. Clay Mathematics Institute, to appear.
- [2] M. Atiyah and I. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
- [3] M. Auslander, I. Reiten, and S. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. C.U.P., Cambridge, 1997.
- [4] D. Baer. Tilting sheaves in representation theory of algebras. *Manuscripta Math.*, 60(3):323–347, 1988.
- [5] A. Beilinson. Coherent sheaves on  $\mathbb{P}^n$  and problems in linear algebra. *Funktsional. Anal. i Prilozhen.*, 12(3):68–69, 1978.
- [6] A. Bergman and N. Proudfoot. Moduli spaces for Bondal quivers, 2008. to appear in *Pacific J. Math.*, [arXiv:math/0512166](https://arxiv.org/abs/math/0512166).

- [7] R. Bezrukavnikov and D. Kaledin. McKay equivalence for symplectic resolutions of quotient singularities. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):20–42, 2004.
- [8] R. Bocklandt, T. Schedler, and M. Wemyss. *Superpotentials and Higher Order Derivations*, 2008. [arXiv:0802.0162](#).
- [9] A. Boer and A. Quintero Velez. *Noncommutative resolutions of ADE fibered Calabi-Yau threefolds*, 2008. [arXiv:0806.3177](#).
- [10] A. Bondal. Helices, representations of quivers and Koszul algebras. In *Helices and vector bundles*, volume 148 of *London Math. Soc. Lecture Note Ser.*, pages 75–95. Cambridge Univ. Press, Cambridge, 1990.
- [11] L. Borisov, L. Chen, and G.G. Smith. The orbifold Chow ring of toric Deligne-Mumford stacks. *J. Amer. Math. Soc.*, 18(1):193–215 (electronic), 2005.
- [12] L. Borisov and Z. Hua. *On the conjecture of King for smooth toric Deligne-Mumford stacks*, 2008. [arXiv:0801.2812](#).
- [13] T. Bridgeland. Equivalences of triangulated categories and Fourier-Mukai transforms. *Bull. London Math. Soc.*, 31(1):25–34, 1999.
- [14] T. Bridgeland. Flops and derived categories. *Invent. Math.*, 147(3):613–632, 2002.
- [15] T. Bridgeland and S. Iyengar. A criterion for regularity of local rings. *C. R. Math. Acad. Sci. Paris*, 342(10):723–726, 2006.
- [16] T. Bridgeland, A. King, and M. Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001.
- [17] T. Bridgeland and A. Maciocia. Fourier-Mukai transforms for  $K3$  and elliptic fibrations. *J. Algebraic Geom.*, 11(4):629–657, 2002.
- [18] M. Butler and A. King. Minimal resolutions of algebras. *J. Algebra*, 212(1):323–362, 1999.
- [19] A. Căldăraru. Derived categories of sheaves: a skimming. In *Snowbird lectures in algebraic geometry*, volume 388 of *Contemp. Math.*, pages 43–75. AMS, Providence, RI, 2005.
- [20] H. Cassens and P. Slodowy. On Kleinian singularities and quivers. In *Singularities (Oberwolfach, 1996)*, volume 162 of *Progr. Math.*, pages 263–288. Birkhäuser, Basel, 1998.
- [21] L. Chiang and S. Roan. Crepant resolutions of  $\mathbb{C}^n/A_1(n)$  and flops of  $n$ -folds for  $n = 4, 5$ . In *“Calabi-Yau varieties and mirror symmetry”*, eds. N. Yui and J. Lewis, *Fields Inst. Comm.* 38, AMS, pages 27–41, 2003.
- [22] L. Costa and R. M. Miró-Roig. Tilting sheaves on toric varieties. *Math. Z.*, 248(4):849–865, 2004.
- [23] D. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.
- [24] A. Craw. *The McKay correspondence and representations of the McKay quiver*. PhD thesis, University of Warwick, 2001.
- [25] A. Craw. *The Special McKay correspondence as an equivalence of derived categories*, 2007. [arXiv:0704.3627](#).
- [26] A. Craw and A. Ishii. Flops of  $G$ -Hilb and equivalences of derived categories by variation of GIT quotient. *Duke Math. J.*, 124(2):259–307, 2004.
- [27] A. Craw and D. Maclagan. Fiber fans and toric quotients. *Discrete Comput. Geom.*, 37(2):251–266, 2007.
- [28] A. Craw, D. Maclagan, and R.R. Thomas. Moduli of McKay quiver representations. I. The coherent component. *Proc. Lond. Math. Soc. (3)*, 95(1):179–198, 2007.
- [29] A. Craw, D. Maclagan, and R.R. Thomas. Moduli of McKay quiver representations. II. Gröbner basis techniques. *J. Algebra*, 316(2):514–535, 2007.
- [30] A. Craw and M. Reid. How to calculate  $A$ -Hilb  $\mathbb{C}^3$ . In *Geometry of toric varieties*, volume 6 of *Sémin. Congr.*, pages 129–154. Soc. Math. France, Paris, 2002.
- [31] A. Craw and G.G. Smith. Projective toric varieties as fine moduli spaces of quiver representations, 2008. To appear in *Amer. J. Math.*, [math.AG/0608183](#).
- [32] I. Dolgachev. *Lectures on Invariant Theory*, volume 296 of *London Math. Soc. Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
- [33] I. Dolgachev and Y. Hu. Variation of geometric invariant theory quotients. *Inst. Hautes Études Sci. Publ. Math.*, (87):5–56, 1998. With an appendix by Nicolas Ressayre.
- [34] D. Eisenbud and B. Sturmfels. Binomial ideals. *Duke Math. J.*, 84(1):1–45, 1996.
- [35] W. Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
- [36] G. Gonzalez-Sprinberg and J.-L. Verdier. Construction géométrique de la correspondance de McKay. *Ann. Sci. École Norm. Sup. (4)*, 16(3):409–449 (1984), 1983.

- [37] R. Hartshorne. *Residues and duality*. Lecture notes of a seminar on the work of A. Grothendieck. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer-Verlag, Berlin, 1966.
- [38] M. Hering, H. Schenck, and G.G. Smith. Syzygies, multigraded regularity and toric varieties. *Compos. Math.*, 142(6):1499–1506, 2006.
- [39] L. Hille. Toric quiver varieties. In *Algebras and modules, II (Geiranger, 1996)*, volume 24 of *CMS Conf. Proc.*, pages 311–325. Amer. Math. Soc., Providence, RI, 1998.
- [40] L. Hille and M. Perling. A counterexample to King’s conjecture. *Compos. Math.*, 142(6):1507–1521, 2006.
- [41] L. Hille and M. Perling. *Strongly exceptional sequences of line bundles on toric surfaces*, 2008. Work in progress described in an E.G.A. seminar in Aberdeen.
- [42] L. Hille and M. Van den Bergh. Fourier-Mukai transforms. In *Handbook of tilting theory*, volume 332 of *London Math. Soc. Lecture Note Ser.*, pages 147–177. Cambridge Univ. Press, Cambridge, 2007.
- [43] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, Oxford, 2006.
- [44] Y. Ito and H. Nakajima. McKay correspondence and Hilbert schemes in dimension three. *Topology*, 39(6):1155–1191, 2000.
- [45] Y. Ito and I. Nakamura. Hilbert schemes and simple singularities. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Notes*, pages 151–233. CUP, Cambridge, 1999.
- [46] M. Kapranov and E. Vasserot. Kleinian singularities, derived categories and Hall algebras. *Math. Ann.*, 316:565–576, 2000.
- [47] Y. Kawamata. Equivalences of derived categories of sheaves on smooth stacks. *Amer. J. Math.*, 126(5):1057–1083, 2004.
- [48] Y. Kawamata. Log crepant birational maps and derived categories. *J. Math. Sci. Univ. Tokyo*, 12(2):211–231, 2005.
- [49] Y. Kawamata. Derived categories of toric varieties. *Michigan Math. J.*, 54(3):517–535, 2006.
- [50] A. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.
- [51] A. King. Tilting bundles on some rational surfaces. <http://www.maths.bath.ac.uk/~masadk/papers>, 1997.
- [52] P. Kronheimer. The construction of ALE spaces as hyper-Kähler quotients. *J. Differential Geom.*, 29(3):665–683, 1989.
- [53] T. Logvinenko. Derived McKay correspondence via pure-sheaf transforms. *Math. Ann.*, 341(1):137–167, 2008.
- [54] D. Maclagan and G.G. Smith. Multigraded Castelnuovo-Mumford regularity. *J. Reine Angew. Math.*, 571:179–212, 2004.
- [55] J. Milne. Algebraic groups and arithmetic groups, 2005. Lecture notes available from [www.jmilne.org/math](http://www.jmilne.org/math).
- [56] S. Mukai. *An introduction to invariants and moduli*, volume 81 of *Cambridge Studies in Advanced Mathematics*. CUP, Cambridge, 2003. Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury.
- [57] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2)*. Springer-Verlag, Berlin, 1994.
- [58] I. Nakamura. Hilbert schemes of abelian group orbits. *J. Algebraic Geom.*, 10(4):757–779, 2001.
- [59] T. Oda. *Convex bodies and algebraic geometry*, volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1988. Translated from the Japanese.
- [60] M. Reid. Young person’s guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 345–414. AMS, Providence, RI, 1987.
- [61] M. Reid. McKay correspondence. In *Proc. of algebraic geometry symposium (Kinosaki, Nov 1996)*, T. Katsura (ed.), pages 14–41, (1997).
- [62] M. Reid. La correspondance de McKay. *Astérisque*, (276):53–72, 2002. Séminaire Bourbaki, Vol. 1999/2000.
- [63] B. Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.
- [64] M. Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996.
- [65] M. Van den Bergh. Three-dimensional flops and noncommutative rings. *Duke Math. J.*, 122(3):423–455, 2004.
- [66] M. Wemyss. Reconstruction algebras of type A, 2007. [arXiv:0704.3693](https://arxiv.org/abs/0704.3693).