ORBIFOLD COHOMOLOGY FOR GLOBAL QUOTIENTS

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Abstract. Let X be an orbifold which is a global quotient of a manifold Y by a finite group G. We construct a noncommutative ring $H^*(Y,G)$ with a G action such that $H^*(Y,G)^G$ is the orbifold cohomology ring of X defined by Chen and Ruan. When $Y = S^n$, with S a surface with trivial canonical class and $G = \mathfrak{S}_n$, we prove that (a small modification of) the orbifold cohomology of X is naturally isomorphic to the cohomology ring of the Hilbert scheme $S^{[n]}$, computed by Lehn and Sorger.

Introduction

If a finite group G acts on a complex manifold Y, the quotient Y/G has a natural structure of smooth orbifold, which we denote by $[Y/G]$. Originating in physics (see [D*1], [D*2], [Z]) cohomological invariants, in particular orbifold Euler number and orbifold Hodge numbers of the orbifold $[Y/G]$ have been defined and studied, with the idea that they should coincide with the invariants of a nice (crepant) resolution of singularities. In [BB] it was shown that this is indeed true. Recently the cohomology ring for the resolution of symplectic singularities was determined in Thm. 1.8(ii) of [EG].

One specially interesting case which was often used to test the physicists' predictions before a general proof was available [G], [GS], [HH], is that of the Hilbert scheme of points on a surface; namely, S is a complex surface, $Y = S^n$ and $G = \mathfrak{S}_n$, the symmetric group on n letters acting in the obvious way. A crepant resolution of the symmetric product is then provided by the Hilbert scheme $S^{[n]}$ (or, in case S is not algebraic, by the corresponding Douady space).

Chen and Ruan [CR1], [CR2] have introduced an orbifold cohomology ring for any orbifold, whose Hodge numbers coincide with the orbifold Hodge numbers. In the special case of the Hilbert scheme, they conjectured that the orbifold cohomology ring should be isomorphic to the cohomology ring of the Hilbert scheme if the surface S has trivial canonical class.

Since the first version of [CR1] appeared, the cohomology ring of $S^{[n]}$ has been explicitly computed by Lehn and Sorger [LS2] for a surface S with trivial $c_1(T_S)$ (building on previous results of Nakajima [N1], [N2], Grojnowski [Gr], Lehn [L] and Li, Qin and Wang [LQW]).

This paper started as an attempt to verify Chen and Ruan's conjecture using the result of Lehn and Sorger. In section 3 we show that the conjecture is essentially true, i.e. by introducing suitable signs into the ring structure of the orbifold cohomology ring, one gets a canonical ring isomorphism respecting also the duality pairing.

In order to prove this, we first (in section 1) introduce, for a manifold Y with the action of a finite group G, a cohomology ring $H^*(Y,G)$. The

construction is inspired by Chen and Ruan's definition of the orbifold cohomology. The ring $H^*(Y,G)$ is not commutative, on the other hand it is often simpler then the orbifold cohomology ring.

The group G acts naturally on the ring $H^*(Y,G)$, and the G invariant subring is naturally isomorphic to the orbifold cohomology of the quotient, as proven in section 2; there we also make some considerations on the more general version of Chen and Ruan's conjecture on the relationship between orbifold cohomology and ordinary cohomology of a crepant resolution in the Hyperkähler case.

In the fourth section we compute the orbifold cohomology of Beauville's generalized Kummer varieties; we expect that also in this case the orbifold cohomology ring (again up to an explicit sign change) will be isomorphic to the ordinary cohomology of the crepant resolution.

We collect in the appendix a few elementary results about certain Galois covers of smooth and nodal curves.

This paper started from conversations with K.S. Narain, G. Thompson and M.S. Narasimhan. We would like to thank M. Lehn and C. Sorger for sharing and discussing with us the results of their paper [LS2]. A particular thank goes to M.S. Narasimhan for very useful discussions, in particular about parabolic bundles and the relation to the shift in cohomology degree.

We were informed that a result close to Theorem 3.8, namely the existence of a ring isomorphism between $H^*_o([S^n/\mathfrak{S}_n])[nd]$ and the $\mathfrak{S}_n$ invariant part of $H^*(S)[n]\{\mathfrak{S}_n\}$, has been obtained independently by Uribe [U].

This paper is dedicated to Susanna [PB].

1. Orbifold cohomology for a quotient orbifold

Notation and conventions. A symbol := means that the left hand side is defined by the right hand side.

All manifolds will be complex (although one could consider almost complex ones with no major changes). All vector bundles will be complex holomorphic. By dimension (or codimension) we will always mean complex dimension (or codimension).

For elements $g_1, \dots, g_n$ in a group G we write $\langle g_1, \dots, g_n\rangle$ for the subgroup they generate.

All group actions will be left group actions. For a manifold T with the action of a finite group H, denote by T^H the H-invariant locus (which is always a closed submanifold); for every $g \in H$ let T^g be the g-invariant locus. Write $T^{g_1,\dots,g_n}$ for $T^{\langle g_1,\dots,g_n\rangle}$.

For a topological space T let $H^i(T)$ denote $H^i(T,\mathbb{Q})$. A morphism $f : T \to S$ of manifolds induces via pullback a degree-preserving ring homomorphism $f^* : H^*(S) \to H^*(T)$. If T is a submanifold of S and $\alpha \in H^*(S)$, we write $\alpha|_T$ instead of $i^*\alpha$, where $i : T \to S$ is the inclusion.

If f is proper, then the pushforward $f_* : H^*(T) \to H^*(S)$ on cohomology is given as follows: to every cohomology class α, associate its Poincaré dual homology class $\bar{\alpha}$; if T is not compact, this is a class in the Borel-Moore homology group. Then, as f is proper, f_* is defined on Borel-Moore homology, and the Poincaré dual of $f_*(\bar{\alpha})$ in cohomology is defined to be $f_*(\alpha)$.

Note that if $f : T \to S$ is an isomorphism, then $f_*\alpha = (f^{-1})^*\alpha$. We will denote the cup product of cohomology classes α, β just by $\alpha \cdot \beta$ or $\alpha\beta$.

The ambient ring as vector space with G-action. Let Y be a complex manifold with the action of a finite group G.

Definition 1.1. The vector space $H^*(Y, G)$ is defined as follows:

$$H^*(Y, G) := \bigoplus_{g \in G} H^*(Y^g).$$

For $g \in G$ and α in $H^*(Y^g)$, denote by α_g the corresponding element in the g-th direct summand of $H^*(Y, G)$.

Definition 1.2. If g, h are two elements of G, then $h(Y^g) = Y^{hgh^{-1}}$. Hence G acts on $H^*(Y, G)$ by

$$h(\alpha_g) := (h_*\alpha)_{hgh^{-1}}.$$

Remark 1.3. There is an alternative way to define $H^*(Y, G)$ together with the G action. Let $(p, a) : G \times Y \to Y \times Y$ be the map defined by projection and action, and let $\bar{Y} = (p, a)^{-1}(\Delta_Y)$. Then (p, a) is G-equivariant with respect to the action $h(g, y) = (hgh^{-1}, hy)$ on $G \times Y$; as Δ_Y is G-invariant, so is $\bar{Y}$. The vector space $H^*(Y, G)$ can be viewed as as $H^*(\bar{Y})$, with the induced G-action.

Remark 1.4. The invariant subspace $H^*(Y, G)^G$ under the action of G is isomorphic to

$$\bigoplus_{g \in T} H^*(Y^g)/C(g)$$

where $T \subset G$ is a set of representatives of the conjugacy classes of G (in particular $T = G$ if and only if G is commutative) and $C(g)$ is the centralizer of g in G. Hence as a vector space $H^*(Y, G)$ coincides with the orbifold cohomology of the quotient orbifold of Y by G as defined in [CR1].

The grading. The following definition [R] is now becoming a standard, sometimes under different names such as fermionic shift number [Z] or degree shifting number [CR1].

Definition 1.5. Let Y be a manifold of dimension D with the action of a finite group G. For $g \in G$ and $y \in Y^g$, let $\lambda_1, \dots, \lambda_D$ be the eigenvalues of the action of g on $T_{Y,y}$; note that they are roots of unity. Write $\lambda_j = e^{2\pi i r_j}$ where r_j is a rational number in the interval $[0, 1[$. The *age* of g in y is the rational number $a(g, y) := \sum_{j=1}^{D} r_j$.

The age $a(g, y)$ is a nonnegative rational number, and it is zero if and only if g acts as the identity in a neighborhood of y; it is an integer if the action of g near Y preserves the canonical class (i.e., the induced automorphism of $T_{Y,y}$ has determinant 1).

Remark 1.6. The age $a(g, y)$ only depends on the connected component Z of Y^g in which y lies; we can therefore denote it by $a(g, Z)$. It is easy to check that

$$a(g, Z) + a(g^{-1}, Z) = \operatorname{codim}(Z \subset Y).$$

Definition 1.7. We define a (rational) grading on $H^*(Y,G)$ as follows. Let $g \in G$ and let Z be a connected component of Y^g, and $j : Z \to Y^g$ the inclusion. Let $\alpha \in H^i(Z)$; we assign to $j_*\alpha_g$ the degree $i + 2a(g,Z)$.

Note that $H^*(Y,G)$ is integrally graded if the age of every element of G at every point in its fixed locus is an integer (or in fact a half-integer, as will be the case in the section 3).

Definition 1.8. For later use, we also define a splitting of $H^*(Y,G)$ into even and odd part, as follows:

$$H^{ev}(Y,G) = \bigoplus_{g\in G} H^{ev}(Y^g)$$

and analogously for Y^{odd}.

Note that $H^{ev}(Y,G)$ coincides with the even-graded part if and only if for every $g \in G$ and for every $y \in Y^g$ the age of g in y is an integer.

Note also that the G action on $H^*(Y,G)$ preserves both the splitting into even and odd parts and the grading.

Definition of the classes $c(g,h)$. We construct cohomology classes $c(g,h) \in H^*(Y^{g,h})$ which will be used in defining the multiplication in $H^*(Y,G)$.

We use the following convention: a vector bundle on a disjoint union of manifolds is the datum of a vector bundle on each connected component, possibly having different ranks on different components; its top Chern class is the cohomology class restricting to the top Chern class on each connected component.

We also use the following fact: let H be a finite group. Assume that E is a bundle with H action on a manifold M on which H acts trivially (i.e., we are given a homomorphism $H \to End(E)$); we say that E is an H-bundle. Then the representation of H defined by a fiber of E is locally constant on M. In particular the H invariant part of E is also a vector bundle.

Construction 1.9. Let g and h be elements of G, and let H be the subgroup they generate; H contains $(gh)^{-1}$. Let $C = C(\mathbb{P}^1, g, h, (gh)^{-1}, H)$ be the induced Galois cover of $\mathbb{P}^1$, with Galois group H, branched over the three points $0, 1, \infty$ (see the definition in the appendix).

The bundle T_Y is G-equivariant over Y, hence $T_Y|_{Y^H}$ is an H bundle.

Let E on $Y^H \times C$ be the H-equivariant pullback $E = \pi^* T_Y|_{Y^H}$, where π is the projection of $Y^H \times C$ to Y^H (which is H-equivariant): this means that an element $f \in H$ maps the triple $(y, c, v) \in U \times C \times E_{(y,c)}$ to $(y, f(c), f(v))$.

Define $F(Y,g,h)$ to be $R^1\pi_*^H(E)$, where $R^1\pi_*^H$ is the derived functor of the invariant pushforward. We will write just $F(g,h)$ when Y is clear from the context. We define $c(g,h)$ to be the top Chern class of $F(g,h)$. Note that $F(g,h)$ is the H invariant part of the H-bundle $R^1\pi_*(E) = T_Y|_{Y^H} \otimes R^1\pi_*(\mathcal{O}_{C\times M})$, hence it is a vector bundle on Y^H.

Lemma 1.10. *(1) Let $g_1, g_2 \in G$, and let $H = \langle g_1, g_2\rangle$ and $g_3 = (g_1g_2)^{-1}$. The bundle $F(g_1,g_2)$ is isomorphic to $F(g_2,g_1)$ and to $F(g_2,g_3)$.*

(2) Let $v \in G$, then $v : Y^{g,h} \to Y^{g',h'}$ where a prime denotes conjugation by v (e.g., $g' = vgv^{-1}$). Then $v^*F(g',h') = F(g,h)$.

Proof. (1) The curve $C(\mathbb{P}^1, g_1, g_2, g_3, H)$ is isomorphic, as a curve with H action, to $C(\mathbb{P}^1, g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}, H)$ for any permutation σ of the indices $1, 2, 3$.

(2) Conjugation by v defines an isomorphism φ between H and $H' = \langle g', h' \rangle$. There is clearly a natural φ equivariant isomorphism between the curves $C(\mathbb{P}^1, g, h, (gh)^{-1}, H)$ and $C(\mathbb{P}^1, g, h, (gh)^{-1}, H')$, and from this the result follows. □

Lemma 1.11. *The same $F(Y, g, h)$ can be obtained by replacing in its definition E with $\tilde{E} = \pi^* N_{Y^H/Y}$.*

Proof. The subbundle T_{Y^H} is the H-invariant part of $T_Y|_{Y^H}$; since the group G is finite, it is a direct summand and $T_Y|_{Y^H} = T_{Y^H} \oplus N_{Y^H/Y}$ as H-bundle.

On the other hand, $R^1\pi_*^H(\pi^* T_{Y^H})$ is zero, since $H^1(C, \mathcal{O}_C)^H$ is equal to $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ which is zero. □

One of the fundamental facts that allows to define the orbifold cohomology ring, is that the age $a(g)$ can be interpreted in terms of parabolic bundles.

Let $u \in Y^H$. Then $E_u := E|_{\{u\}\times C}$ is an H-equivariant bundle on C, giving rise to a parabolic bundle on the quotient $\mathbb{P}^1 = C/H$. The formula of [Gro] for the parabolic degree $c^1(E_u, H)$ implies

$$c^1(E_u, H) = deg(p_*^H(E_u)) + a(g, U) + a(h, U) + a((gh)^{-1}, U), \tag{1}$$

where $p_*^H(E_u)$ is the H-equivariant pushforward via the natural projection $p : C \to \mathbb{P}^1$.

The following is the analogue of Lemma 4.2.2 in [CR1].

Lemma 1.12. *Let U be a connected component of Y^H. Then the coherent sheaf $F(Y, g, h)|_U$ defined in Construction 1.9 is a vector bundle of rank*

$$a(g, U) + a(h, U) - a(gh, U) - \operatorname{codim}(U \subset Y^{gh}). \tag{2}$$

Proof. Let $u \in U$. It is enough to show that the fiber F_u of $F(Y, g, h)$ at u has dimension given by (2). The H-equivariant bundle $E_u := E|_{\{u\}\times C}$ on C is defined by a representation of H. By the principal theorem of [Gro] it follows that the parabolic degree $c^1(E_u, H)$ is 0.

Applying the Riemann-Roch theorem to (1) gives

$$\begin{aligned}\dim(\pi_*^H(E_u)) - \dim(F_u) &= \chi(p_*^H(E_u)) \\ &= \dim(Y) - a(g, U) - a(h, U) - a((gh)^{-1}, U).\end{aligned}$$

The result follows by observing that $\pi_*^H(E_u) = T_{Y^H,u}$, and by definition $a((gh)^{-1}, U) = \dim(Y) - \dim(T_{Y^{gh},u}) - a(gh, U)$. □

The following Lemma will be needed in Section 3.

Lemma 1.13. *Assume that there is a product decomposition $Y = Y_1 \times Y_2$ such that H acts on each factor separately. Let $Y^H = Y_1^H \times Y_2^H$ be the induced product decomposition: then $F(g, h) = p_1^* F(Y_1, g, h) \oplus p_2^* F(Y_2, g, h)$.*

Proof. The bundle $T_Y|_{Y^H}$ splits, as a vector bundle with H-action, as direct sum $p_1^* T_{Y_1}|_{Y_1^H} \oplus p_2^* T_{Y_2}|_{Y_H^2}$. Pulling back to $Y^H \times C$ and applying $R^1\pi_*^H$ both respect this direct sum decomposition; this proves the result. □

The product.

Definition 1.14. Define a bilinear map

$$\mu : H^*(Y,G) \times H^*(Y,G) \to H^*(Y,G)$$

by

$$\mu(\alpha_g, \beta_h) := \gamma_{gh}$$

where

$$\gamma = i_*(\alpha|_{Y^{g,h}} \cdot \beta|_{Y^{g,h}} \cdot c(g,h))$$

and $i : Y^{g,h} \to Y^{gh}$ is the natural inclusion.

Lemma 1.15. *The bilinear map μ sends $H^i(Y,G) \otimes H^j(Y,G)$ to $H^{i+j}(Y,G)$. Hence, it defines a graded multiplication (in general non-commutative: its associativity will be proven in Theorem 1.18) on $H^*(Y,G)$.*

Proof. Basically this follows from Lemma 1.12. $H^i(Y,G)$ is the direct sum, over $g \in G$ and over connected components Z of Y^g, of $H^{i-2a(g,Z)}(Z)$. It is therefore enough to prove the following: given two elements g_1, g_2 in G, connected components Z_i in Y^{g_i} and elements $\alpha_i \in H^{d_i}(Z_i)$, for every connected component U of $Z_1 \cap Z_2$ the element

$$\iota_*(\alpha_1|_U \cdot \alpha_2|_U \cdot c(g_1,g_2)|_U)$$

is in $H^{i+j-2a(g_1g_2,U)}(Y^{g_1g_2})$. Here $\iota : U \to Y^{g_1g_2}$ is the natural inclusion. As restriction preserves the grading, and ι_* raises the degree by $2\,\mathrm{codim}\,(U \subset Y^{g_1g_2})$, the only thing left to prove is indeed Lemma 1.12. □

Remark 1.16. The map μ is equivariant with respect to the action of G. In fact, this follows directly from Lemma 1.10 (2). The multiplication μ is commutative if the group G is abelian, or more generally if every point of Y has abelian stabilizer: this follows from Lemma 1.10 (1).

We recall here a particularly simple special case of the excess intersection formula. This is certainly well known in many contexts: it is proven for instance in [Q], Proposition 3.3.

Let S be a manifold, S_1 and S_2 closed submanifolds, and assume that $U := S_1 \cap S_2$ is also a submanifold of S; let $j_i : S_i \to S$ and $\iota_i : U \to S_i$ be the natural inclusions. The excess bundle $E(S,S_1,S_2)$ of U as intersection of S_1 and S_2 in S is a bundle on U "measuring" how much the intersection of S_1 and S_2 fails to be transverse along U. It is defined as the cokernel of the natural map $N_{U/S_1} \to N_{S_2/S}|_U$ or, equivalently, of $N_{U/S_2} \to N_{S_1/S}|_U$. In particular it is equivalent, in the Grothendieck group of vector bundles on U, to $T_S|_U + T_U - T_{S_1}|_U - T_{S_2}|_U$. We denote the top Chern class of $E(S,S_1,S_2)$ by $e(S,S_1,S_2)$.

For any cohomology class $\alpha \in H^*(S_1)$, the following excess intersection formula holds in the cohomology ring of S_2:

$$j_2^* j_{1*}(\alpha) = \iota_{2*}(e(S,S_1,S_2) \cdot \iota_1^*(\alpha)).$$

Lemma 1.17. *A sufficient condition for the map μ to define an associative product on $H^*(Y,G)$ is that, for every ordered triple of elements $(g_1,g_2,g_3)\in G$, the following relation hold in the cohomology ring of $W=Y^{g_1}\cap Y^{g_2}\cap Y^{g_3}$:*

$$c(g_1,g_2)|_W\cdot c(g_1g_2,g_3)|_W\cdot e_{12}=c(g_1,g_2g_3)|_W\cdot c(g_2,g_3)|_W\cdot e_{23}, \tag{3}$$

where $e_{12}=e(Y^{g_1g_2},Y^{g_1,g_2},Y^{g_1g_2,g_3})$ and $e_{23}=e(Y^{g_2g_3},Y^{g_1,g_2g_3},Y^{g_2,g_3})$.

Proof. Write $i:W\to Y^{g_1g_2g_3}$ for the natural inclusion. Using the excess intersection formula, it is a straightforward computation to check that the product $(\alpha_{g_1}\cdot\beta_{g_2})\cdot\gamma_{g_3}$ is equal to $i_*\lambda_{g_1g_2g_3}$ where $\lambda\in H^*(W)$ is

$$\lambda=\alpha|_W\cdot\beta|_W\cdot\gamma|_W\cdot c(g_1,g_2)|_W\cdot c(g_1g_2,g_3)|_W\cdot e_{12}.$$

The only thing one needs to remember is that the assumption that all the $c(g,h)$ have even degree allows one to move them around in the product freely. Analogously, $\alpha_{g_1}\cdot(\beta_{g_2}\cdot\gamma_{g_3})$ is equal to $i_*\lambda'_{g_1g_2g_3}$ where $\lambda'\in H^*(W)$ is

$$\alpha|_W\cdot\beta|_W\cdot\gamma|_W\cdot c(g_1,g_2g_3)|_W\cdot c(g_2,g_3)|_W\cdot e_{23}.$$

□

Proof of associativity.

Theorem 1.18. *The bilinear map μ defines a graded, G-equivariant, associative multiplication on $H^*(Y,G)$.*

Proof. The only thing left to prove is associativity: we will check that the sufficient condition of Lemma 1.17 is verified. We will construct on W two vector bundles F_L and F_R such that the left hand side of (3) is $c_{top}(F_L)$ (Lemma 1.23) and the right hand side of (3) is $c_{top}(F_R)$ (Remark 1.24); then (Proposition 1.25) we will prove that F_L and F_R are isomorphic.

□

Notation 1.19. From now on we will just write $\alpha\cdot\beta$ or $\alpha\beta$ instead of $\mu(\alpha,\beta)$ for the product in $H^*(Y,G)$.

Write g_4 for the unique element of the group G such that $g_1g_2g_3g_4$ is the identity. Note that $Y^{g_1g_2}=Y^{g_3g_4}$ and that $F(g_1g_2,g_3)$ is isomorphic to $F(g_3,g_4)$ by Lemma 1.10 (1). Note also that $\langle g_1g_2,g_3\rangle=\langle g_3,g_4\rangle$.

Lemma 1.20. *Let F_L be a bundle on W. Then its top Chern class is equal to the left hand side of (3) if F_L is equivalent, in the Grothendieck group of vector bundles on W, to*

$$F(g_1,g_2)+F(g_3,g_4)+T_{Y^{g_1g_2}}-T_{Y^{g_1,g_2}}-T_{Y^{g_3,g_4}}+T_W$$

where we suppress the $|_W$ from the notation. Analogously, a vector bundle F_R has as top Chern class the right hand side of (3) if it is equivalent to

$$F(g_2,g_3)+F(g_4,g_1)+T_{Y^{g_2g_3}}-T_{Y^{g_2,g_3}}-T_{Y^{g_4,g_1}}+T_W$$

Proof. The left hand side is the top Chern class of the bundle $F(g_1,g_2)\oplus F(g_3,g_4)\oplus E(Y^{g_1g_2},Y^{g_1,g_2},Y^{g_1g_2,g_3})$. As equivalent vector bundles in the Grothendieck group have the same Chern classes the result follows. □

Note that the second formula in the Lemma above can be obtained from the first by a cyclic permutation of the indices $(1,2,3,4)$ to $(2,3,4,1)$.

Let H be the finite subgroup of G generated by g_1, g_2, g_3. Note that $T_Y|_W$ is a H-bundle. We now construct a nodal curve C with an H action such that the bundle $F_L = (T_Y|_W \otimes H^1(C, \mathcal{O}_C))^H$ has the properties claimed in Lemma 1.20. A similar construction, replacing (g_1, g_2, g_3) by (g_2, g_3, g_4) will yield a vector bundle F_R as in the Lemma.

Construction 1.21. Let D be the union of two smooth rational curves D' and D'' meeting transversally at a point p. Choose distinct marked points p_1, p_2 on D' and p_3, p_4 on D''. Let $C = C(D, g_i, H)$ be the associated connected Galois H cover of D (see the appendix for a definition), branched over the p_i's and the node p. Let C' be the inverse image of D' and C'' the inverse image of D''. Let $Z = C' \cap C''$. Write C_{12} (resp. C_{34}) for a connected component of C' (resp. C'') invariant under $\langle g_1, g_2\rangle$ (resp. $\langle g_3, g_4\rangle$) and isomorphic to $C(D', g_1, g_2, (g_1g_2)^{-1}, \langle g_1, g_2\rangle)$ (resp. $C(D'', g_3, g_4, (g_3g_4)^{-1}, \langle g_3, g_4\rangle)$).

We use the following notation. For a finite group K, we denote by $\mathbf{1}_K$ the trivial one dimensional representation of K; if K' is a subgroup of K and U is a representation of K', we denote by $Ind_{K'}^{K}(U)$ the induced representation of K (see e.g. [FH] page 32).

Lemma 1.22. *In the Grothendieck group of representations of H, the vector space $H^1(C, \mathcal{O}_C)$ is equivalent to*

$$Ind_{\langle g_1,g_2\rangle}^{H} H^1(C_{12}, \mathcal{O}_{C_{12}}) + Ind_{\langle g_3,g_4\rangle}^{H} H^1(C_{34}, \mathcal{O}_{C_{34}}) + \mathbf{1}_H +$$
$$Ind_{\langle g_1g_2\rangle}^{H} \mathbf{1}_{\langle g_1g_2\rangle} - Ind_{\langle g_1,g_2\rangle}^{H} \mathbf{1}_{\langle g_1,g_2\rangle} - Ind_{\langle g_3,g_4\rangle}^{H} \mathbf{1}_{\langle g_3,g_4\rangle}.$$

Proof. Let $Z = C' \cap C''$. We have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C'} \oplus \mathcal{O}_{C''} \to \mathcal{O}_Z \to 0;$$

as all sheaves involved are H-sheaves and the maps are H-equivariant, the long exact sequence of cohomology is an exact sequence of finite dimensional H representations. As an H representation, $H^0(C, \mathcal{O}_C)$ is equal to $\mathbf{1}_H$, since C is connected. As the connected component C_{12} of C' has stabilizer $\langle g_1, g_2\rangle$, the space $H^0(C, \mathcal{O}_{C'})$ is equal to $Ind_{\langle g_1,g_2\rangle}^{H} \mathbf{1}_{\langle g_1,g_2\rangle}$ (see the appendix for details); analogously $H^0(C, \mathcal{O}_{C''})$ is the representation $Ind_{\langle g_3,g_4\rangle}^{H} \mathbf{1}_{\langle g_3,g_4\rangle}$ and $H^0(C, \mathcal{O}_Z)$ is $Ind_{\langle g_1g_2\rangle}^{H} \mathbf{1}_{\langle g_1g_2\rangle}$. Moreover $H^1(C', \mathcal{O}_{C'})$ is equal to $Ind_{\langle g_1,g_2\rangle}^{H} H^1(C_{12}, \mathcal{O}_{C_{12}})$ and analogously for $H^1(C'', \mathcal{O}_{C''})$. □

Corollary 1.23. *The top Chern class of F_L defined above is the left hand side of (3) in Lemma 1.17.*

Proof. For any H bundle T, any subgroup K of H and any representation V of K, we have $(T \otimes Ind_K^H V)^H = (T \otimes V)^K$. The result follows immediately by applying this remark to the induced representations in the previous Lemma, and taking $T = T_Y|_W$. □

Remark 1.24. It is clear how to construct analogously the bundle F_R: just permute cyclically everywhere in the definition of F_L the indices $(1,2,3,4)$ to $(2,3,4,1)$, obtaining thus a curve $\bar{C}$ such that $F_R = (T_Y|_W \otimes H^1(\bar{C}, \mathcal{O}_{\bar{C}}))^H$.

It follows that the top Chern class of F_R is the right hand side of (3) in Lemma 1.17.

Proposition 1.25. *The bundles F_L and F_R are isomorphic.*

Proof. It is of course enough to prove that $H^1(C, \mathcal{O}_C)$ is isomorphic as representation of H to $H^1(\bar{C}, \mathcal{O}_{\bar{C}})$. Consider the natural map $Ext^1(\Omega_C, \mathcal{O}_C)^H \to \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$. It is easy to prove that it is surjective. As deformations of C are unobstructed, this proves that there exists a smoothing of C preserving the H action; therefore there is a flat family of curves $f : \mathcal{C} \to B$ over a small disk such that H acts fiberwise, the central fiber is C and the other fibers are smooth. Hence the general fiber $\tilde{C}$ of $\mathcal{C}$ is isomorphic to the Galois H cover $C(\mathbb{P}^1, g_1, g_2, g_3, g_4, H)$ of $\mathbb{P}^1$ branched over four points with stabilizers g_1, g_2, g_3 and g_4 (defined in the appendix); the H representation $H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})$ does not depend on the particular branch points chosen. On the other hand, since $R^1 f_* \mathcal{O}_{\mathcal{C}}$ is an H vector bundle with fiber the H^1 of the fiber, $H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})$ is isomorphic as a representation to $H^1(C, \mathcal{O}_C)$. The same argument also applies to $\bar{C}$, completing the proof. □

The duality pairing.

Definition 1.26. Let $e \in G$ be the neutral element, assume that Y is compact. Define a degree map $\int_{Y,G} : H^*(Y, G) \to \mathbb{Q}$ by $\int_{Y,G} \alpha_e = \int_Y \alpha$ for $\alpha \in H^*(Y)$ and $\int_{Y,G} \beta_g = 0$ for $\beta \in H^*(Y^g)$ and $g \neq 0$. Define a duality pairing on $H^*(X, G)$ by

$$\langle \alpha_g, \beta_h \rangle = \int_{Y,G} \alpha_g \cdot \beta_h.$$

Lemma 1.27.
1. *$\langle \alpha_g, \beta_h \rangle = 0$ if $h \neq g^{-1}$ and $\langle \alpha_g, \beta_{g^{-1}} \rangle = \int_{Y^g} \alpha\beta$, is just the Poincaré duality pairing on the compact manifold Y^g.*
2. *The pairing $\langle\, , \,\rangle$ on $H^*(Y, G)$ is nondegenerate.*

Proof. By definition $\alpha_g \cdot \beta_h = \gamma_{gh}$ for suitable γ, so $\langle \alpha_g, \beta_h \rangle = 0$ if $h \neq g^{-1}$. By Lemma 1.12 combined with Remark 1.6 we get that $F(Y, g, g^{-1})$ has rank 0 and therefore $c(g, g^{-1}) = 1$. Therefore $\alpha_g \cdot \beta_{g^{-1}} = i_*(\alpha\beta)$, where $i : Y^g \to Y$ is the inclusion and $\alpha\beta$ is the cup product on Y^g. Therefore $\langle \alpha_g, \beta_{g^{-1}} \rangle = \int_{Y^g} \alpha\beta$. This shows (1). (2) follows immediately from (1). □

Orbifold cohomology.

Definition 1.28. The *orbifold cohomology* of the orbifold $[Y/G]$ is the graded ring

$$H_o^*([Y/G]) := H^*(Y, G)^G.$$

It is a rationally graded associative ring.

Theorem 1.29. *The orbifold cohomology is skew commutative with respect to the decomposition in even and odd part introduced in Definition 1.8.*

Proof. Let $\tilde{g}, h \in G$; choose $\tilde{\alpha} \in H^n(Y^{\tilde{g}})$ and $\beta \in H^m(Y^h)$. Define

$$\gamma := \sum_{f \in G} f^{-1}(\tilde{\alpha}_{\tilde{g}});$$

such classes γ generate $H^{ev}(Y,G)^G$ for n even and $H^{odd}(Y,G)^G$ for n odd. We will prove that $\gamma \cdot \beta_h = (-1)^{mn}\beta_h \cdot \gamma$; this proves the stronger statement that every element of $H^*(Y,G)^G$ skew commutes with every element of $H^*(Y,G)$.

It is enough to check that, for any fixed $f \in G$,

$$f^{-1}(\tilde{\alpha}_{\tilde{g}}) \cdot \beta_h = (-1)^{mn}\beta_h \cdot v^{-1}(\tilde{\alpha}_{\tilde{g}})$$

for $v = hf$. Let $g = f^{-1}\tilde{g}f$, $\alpha = f^*(\tilde{\alpha}) \in H^n(Y^g)$, and $H = \langle g, h\rangle$. Using this new notation, we have to check that

$$\alpha_g \cdot \beta_h = (-1)^{mn}\beta_h (h^*\alpha)_{h^{-1}gh}.$$

Let $H = \langle g, h\rangle = \langle h, h^{-1}gh\rangle$. By definition

$$\alpha_g \cdot \beta_h = i_*(\alpha|_{Y^H} \cdot \beta|_{Y^H} \cdot c(g,h))_{gh},$$

where $i : Y^H \to Y^{gh}$ is the natural inclusion; analogously

$$\beta_h \cdot (h^*\alpha)_{h^{-1}gh} = i_*(\beta|_{Y^H} \cdot (h^*\alpha)|_{Y^H} \cdot c(h,g))_{gh}$$

Note that $c(g,h) = c(h,g)$ in view of Lemma 1.10 part (2); on the other hand, $(h^*\alpha)|_{Y^H} = \alpha|_{Y^H}$, and this completes the proof. □

Definition 1.30. In case Y is compact, define a degree map on $H^*_o([X/G]) = H^*(X,G)^G$ by $\int_{[X/G]} := \frac{1}{|G|}\int_{X,G}$.

In the same assumption, define a duality pairing on $H^*_o([X/G])$ by letting, for $\alpha, \beta \in H^*_o([X/G])$:

$$\langle \alpha, \beta\rangle_{[X/G]} = \int_{[X/G]} \alpha \cdot \beta = \frac{1}{|G|}\langle \alpha, \beta\rangle.$$

Remark 1.31. Remark 1.27 immediately implies that $\langle\ ,\ \rangle_{X/G}$ is a nondegenerate pairing. It is also easy to see that this pairing coincides with the pairing defined in [CR1].

We include here a few final remarks on the ring $H^*(Y,G)$.

Remark 1.32. Let Y be a complex manifold with an action by a finite group G. Let H be a subgroup of G with the induced action. Then $H^*(Y,H) = \sum_{h\in H} H^*(Y^h)$ is a subvector space of $H^*(Y,G) = \sum_{g\in G} H^*(Y^g)$, and from the definition it follows immediately that $H^*(Y,H)$ is also a subring of $H^*(Y,G)$.

In particular the results in section 3 give a very explicit description of $H^*(S^n,H)$ where S is a smooth manifold and H is any subgroup of $\mathfrak{S}_n$.

Remark 1.33. Let Y and Z be two manifolds with the action of the same group G, $\varphi : Y \to Z$ an étale G equivariant map. Then φ induces a natural, degree preserving ring homomorphism $H^*(Z,G) \to H^*(Y,G)$ which is functorial.

In fact, the same functoriality property (pullback exists for étale maps) is true for the orbifold cohomology of arbitrary orbifolds, while pullback under general morphisms seems difficult to define. Thus orbifold cohomology has properties in between those of ordinary cohomology and those of quantum chomology (being, indeed, the degree zero quantum cohomology).

As another instance of the closeness of orbifold cohomology to usual cohomology, the definition given in this paper can be modified to yield an orbifold Chow ring and a corresponding noncommutative ring $A^*(Y,G)$ (see [F]).

2. Orbifold cohomology and crepant resolutions

Comparison with Chen and Ruan's definition. In their paper [CR1] Chen and Ruan define orbifold cohomology for an almost complex orbifold. In case the orbifold X is a global quotient of a complex manifold Y by a finite group G, their definition goes as follows.

We start by remarking that, if in G we have a relation $h = vgv^{-1}$, the element v defines an isomorphism $v : Y^g \to Y^h$, hence a ring isomorphism $v^* : H^*(Y^h) \to H^*(Y^g)$. If h is also equal to ugu^{-1}, then v^* differs from u^* by z^*, where $z = uv^{-1}$ commutes with h. Let $C(g)$ be the centralizer of g; then the induced isomorphism $v^* : H^*(Y^h)^{C(h)} \to H^*(Y^g)^{C(g)}$ only depends on g and h, and not on the choice of v. We denote it by $\iota_{h,g}$.

Therefore it makes sense to define $H^*(Y^g)^{C(g)}$ for any g in a given conjugacy class $[g]$: a different choice of g leads to a canonically isomorphic ring.

Chen and Ruan's orbifold cohomology is, as a vector space,

$$H^*_{CR}(X) = \bigoplus_{[g]\in T} H^*(Y^g)^{C(g)},$$

where T is the set of all conjugacy classes of G.

Define a linear map $\psi : H^*_{CR}(X) \to H^*(Y,G)^G$ by sending $\alpha \in H^*(Y^g)^{C(g)}$ to $\sum_{h\in[g]}(\iota_{g,h}\alpha)_h$. It is easy to check that $\psi(\alpha)$ is indeed G invariant and ψ is a linear isomorphism; it is also grade preserving, since our definition of grading is the same as in [CR2].

The product is defined via the pairing, as follows: for any three conjugacy classes $[g_1],[g_2],[g_3]$ and elements $\gamma_i \in H^*(Y^{g_i})^{C(g_i)}$ the triple pairing is

$$\langle\gamma_1,\gamma_2,\gamma_3\rangle := \sum_{[h_1,h_2]\in S} \frac{1}{|C(h_1,h_2)|}\int_{X^{h_1,h_2}} \gamma_1'\cdot\gamma_2'\cdot\gamma_3'\, c(h_1,h_2)$$

where S is the set of pairs $(h_1,h_2) \in [g_1]\times[g_2]$ such that $h_3 = (h_1h_2)^{-1} \in [g_3]$ modulo simultaneous conjugation, $C(h_1,h_2) = C(h_1)\cap C(h_2)$ and $\gamma_i' = (\iota_{g_i,h_i}(\gamma_i))|_{Y^{h_1,h_2}}$. Here $c(h_1,h_2)$ is the same cohomology class which was introduced in section 1.

The pairing $\langle\gamma_1,\gamma_2\rangle$ is defined to be $\langle\gamma_1,\gamma_2,\gamma_3\rangle$ where g_3 is be the identity of G and γ_3 the identity in $H^*(Y)$ (i.e., the fundamental class of Y). The datum of the pairing and of the triple pairing defines a product by requiring that $\langle\gamma_1\gamma_2,\gamma_3\rangle = \langle\gamma_1,\gamma_2,\gamma_3\rangle$.

We want to prove that the two ring definitions coincide. To do this, let us compute $\int_{[X/G]}\psi(\gamma_1)\psi(\gamma_2)\psi(\gamma_3)$ and prove it coincides with $\langle\gamma_1,\gamma_2,\gamma_3\rangle$.

By definition,

$$\int_{[X/G]} \psi(\gamma_1)\psi(\gamma_2)\psi(\gamma_3) = \frac{1}{|G|} \sum_{(h_i)\in B} \int_{Y^{h_1,h_2}} \tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_3 c(h_1,h_2)$$

where $\tilde{\gamma}_i = \gamma_i'|_{Y^{h_1,h_2}}$ and $B = \{(h_1,h_2,h_3) \mid h_i \in [g_i],\, h_3 = (h_1h_2)^{-1}\}$.

The set S above is equal to B/G, where G acts by simultaneous conjugation. For any $(h_1,h_2,h_3) \in B$, its stabilizer in G is $C(h_1,h_2)$. Moreover, conjugation by elements of G doesn't affect the product or the integral, so the sum over $(h_i) \in B$ is the same as the sum over $(h_i) \in S$ if we multiply the result by $|G|/|C(h_1,h_2)|$. Hence the two formulas give the same result.

When we started working on this paper, the multiplication in the orbifold cohomology of Chen and Ruan differed from ours by a numerical factor. In the new version of their paper the multiplication is changed by changing the definition of the integral.

The conjecture. Let S be a complex surface with trivial canonical class (in other words, complex symplectic or Hyperkähler). Let X be the orbifold quotient of S^n by the obvious action of the symmetric group $\mathfrak{S}_n$. The first part of Conjecture 6.3 in [CR2] states that $H^*_{CR}(X)$ coincides with the cohomology of the Hilbert scheme of n points on S. Since the appearance of the first version of [CR2], the latter ring has been computed by Lehn and Sorger. We will prove in section 3 that the conjecture holds modulo a certain sign change in the definition of the product.

We want to make here some remarks on the second part of Conjecture 6.3 in [CR2], namely that if X is an orbifold, Z a crepant resolution of (the singular space associated to) X, such that both X and Z carry a Hyperkähler structure, then the orbifold cohomology ring of X coincides with the cohomology of Z.

Notation 2.1. Until the end of this section Y will be a complex manifold with the action of a finite group G, $X = [Y/G]$ the quotient orbifold, and Z a crepant resolution of singularities of Y/G. We will write G' for $G \setminus \{e\}$, where e is the identity of G.

Remark 2.2. Both $H^*(Z)$ and $H^*_o(X)$ have naturally a subring isomorphic to $H^*(Y/G)$; for $H^*(Z)$ it is the pullback, and for the orbifold cohomology it is $H^*(Y)^G_e$. If Y is compact, the pairings also coincide. When we discuss the existence of an isomorphism between $H^*(Z)$ and $H^*_o(X)$, we will always require that it induces the identity on $H^*(Y/G)$.

We compute the rings $H^*(Y,G)$ and $H^*_o([Y/G])$ in the following special case. In the rest of this section, Y will be a complex surface with the faithful Gorenstein action of a finite group G. Then Y/G has rational double points as singularities, and its minimal resolution Z is crepant. If Y is a torus or a $K3$, then Y, $[Y/G]$ and Z are Hyperkähler.

Rational double points come in two series A_n and D_n, plus three exceptional kinds E_6, E_7 and E_8. We describe in detail the A_n case. Let $y \in Y$ be a point whose stabilizer G_y is cyclic of order $n+1$ and let g_y be a generator of the stabilizer acting (in local coordinates s,t on Y) by $g_y(s,t) = (\omega s, \omega^{-1}t)$, where $\omega = e^{2\pi i/(n+1)}$. Let $E(y) \in H^0(y)$ be the natural generator. Clearly

given $g, h \in G_y$, in $H^*(Y, G)$ we have $E(y)_g \cdot E(y)_h = 0$ unless $gh = e$, the identity of G; $E(y)_g \cdot E(y)_{g^{-1}} = p_e$, where $p \in H^4(Y)$ is the class of a point. In particular if Y is compact $\langle E(y)_g, E(y)_{g^{-1}} \rangle = 1$ and $\langle E(y)_g, E(y)_h \rangle = 0$ if $h \neq -1$. The quotient Y/G has an A_n singularity at the image of y.

Assume that G is cyclic of order $n+1$ and that Y^g is the same for every $g \in G'$. Then the orbifold cohomology of $[Y/G]$ is isomorphic to

$$H^*(Y)^G \oplus \bigoplus_{y \in Y^G, g \in G'} \mathbb{Q} E(y)_g.$$

The cohomology of the resolution Z is canonically isomorphic to

$$H^*(Y)^G \oplus \bigoplus_{g \in G', y \in Y^g} \mathbb{Q} F(y)_g,$$

for suitable classes $F(y)_g \in H^2(Z)$, corresponding to the (-2) curves.

At first sight the isomorphism between $H^*(Z)$ and $H^*_o([Y/G])$ is easy to construct. However, in the cohomology of Z the pairing $\langle F(y)_g, F(y)_h \rangle$ is equal to -2 if $g = h$, to 1 if $h = gk$ or $g^{-1}k$ and is zero otherwise; in $H^*_o([Y/G])$ the pairing $\langle E(y)_g, E(y)_h \rangle$ is equal to $1/(n+1)$ if $h = h^{-1}$ and 0 otherwise.

If $n = 1$ we can make $H^*(Z)$ isomorphic to a modified ring structure on $H^*_o([Y/G], *)$; it is enough to map $F(y)_g$ to $2E(y)_g$ and change $c(g, g^{-1})$ from 1 to (-1). A similar change of sign is required in the case of the Hilbert scheme of n points on a surface, as we shall see in the next section.

However, no such generalization can be found if there is a point where $n \geq 2$; in fact, in $H^*(Z)$ the intersection pairing on the subspace generated by the $F(y)_g$ is negative definite, while each $E(y)_g$ is isotropic (i.e., $\langle E(y)_g, E(y)_g \rangle = 0$). Hence even changing each class $c(g, h)$ by a sign or more generally by a rational scalar will yield a non-isomorphic $\mathbb{Q}$-algebra. Note that the isomorphism must map the vector subspace generated by the $F(y)_g$'s to the vector subspace generated by the $E(y)_g$'s since in either case it's the orthogonal to the subspace $H^*(Y/G)$.

The same argument applies even if X is not a global quotient, but just the smooth orbifold associated to a Gorenstein surface with rational double points. Of course there are many $K3$ surfaces with an A_n singularity with $n > 1$, thus providing some kind of counterexample to the more general form of the conjecture of Chen and Ruan.

We give an elementary example which is also a global quotient. Let E be the elliptic curve which is a Galois triple cover of $\mathbb{P}^1$ branched over 3 points, and let f be a generator of the Galois group. Let $Y = E \times E$ and G the automorphism group of Y generated by (f, f^{-1}). Then Y is Hyperkähler, G respects the Hyperkähler structure, and the quotient Y/G is a $K3$ surface with 9 singularities of type A_2.

3. The case of the symmetric product

In this section, fix a smooth complex manifold S of dimension d, and a positive integer n. Let $Y = S^n$, and $\mathfrak{S}_n$ the group of permutations of the set $\{1, \dots, n\}$; $\mathfrak{S}_n$ acts on Y by $\sigma(s)_i = s_{\sigma(i)}$. We prove that the orbifold cohomology $H^*_o([Y/\mathfrak{S}_n])$ is naturally isomorphic up to a degree shifting to $H^*(S)^{[n]}$ as defined in [LS2]. There $H^*(S)^{[n]}$ is also shown, up to additional

signs to be isomorphic to $H^*(S^{[n]})$ in case S is a projective complex surface with numerically trivial canonical bundle.

Notation 3.1. The notation introduced in Section 1 remains valid.

For a finite set I, denote by S^I the manifold whose set of points is the set of maps from I to S; it is isomorphic to S^r, where $r = |I|$, the cardinality of Y. In particular we identify S^n with $S^{\{1,\dots,n\}}$.

A set map $\varphi : I \to J$ induces a morphism $\tilde{\varphi} : S^J \to S^I$ which is injective if φ is surjective and conversely: in the first case it is the inclusion of a multidiagonal, in the second it is a projection on some of the factors. Denote by $\tilde{\varphi}_*$ and $\tilde{\varphi}^*$ the induced maps on cohomology.

For a subgroup H of $\mathfrak{S}_n$, let $O(H)$ be the set of orbits of H in $\{1, \dots, n\}$; write $O(g)$ for $O(\langle g \rangle)$ and $O(g,h)$ for $O(\langle g,h \rangle)$. For $g \in \mathfrak{S}_n$, let $l(g)$ be the minimal number of traspositions whose product is g. Note that $n - l(g)$ is the cardinality of $O(g)$.

For a graded vector space V^*, denote by $V^*[a]$ the graded vector space defined by $V^i[a] = V^{i+a}$.

Remark 3.2. Let H be a subgroup of $\mathfrak{S}_n$, and $\varphi : \{1, \dots, n\} \to O(H)$ the natural surjection. Then the image of $\tilde{\varphi} : S^{O(H)} \to S^n$ is the fixed locus of H.

Lemma 3.3. *Let $g \in \mathfrak{S}_n$, $y \in Y^g$. The age of g at y is equal to $a(g) = d \cdot l(g)/2$.*

In particular the age is always a half-integer, and is an integer if $d = \dim S$ is even (the case which will interest us most being $\dim S = 2$). The ring $H^*(Y, \mathfrak{S}_n)$ is therefore integrally graded.

Corollary 3.4. *Let $g, h \in \mathfrak{S}_n$, then the rank of $F(g,h)$ is*

$$r = \frac{d}{2}(n + 2|O(g,h)| - |O(g)| - |O(h)| - |O(gh)|).$$

Proof. This follows immediately from Lemma 1.12 and Lemma 3.3. □

For a graded Frobenius algebra A (as defined in [LS2]), let $A\{\mathfrak{S}_n\}$ be the Frobenius algebra defined in [LS2]. Its definition can be extended to the case where $A = H^*(S)[d]$ and S is a noncompact manifold of dimension d although in this case no duality is defined on A; it is enough to replace e by $c_d(T_S)$ and to use the natural pushforward map on cohomology whenever needed.

Proposition 3.5. *There is a canonical isomorphism of graded vector spaces with $\mathfrak{S}_n$ action*

$$\lambda : H^*(Y, \mathfrak{S}_n)[nd] \to H^*(S)[d]\{\mathfrak{S}_n\}.$$

If S is compact, then the duality structures also agree.

Proof. Both vector spaces are defined as direct sums over the elements of $\mathfrak{S}_n$, so it is enough to define λ componentwise. As already remarked, the fixed locus of g on Y is naturally isomorphic to the submanifold $S^{O(g)}$ of S^n induced by the natural surjection $\{1, \dots, n\} \to O(g)$. This isomorphism determines λ. The grading is preserved, since in both cases it is chosen

so as to have the graded pieces distributed symmetrical around zero: in $H^*(Y, \mathfrak{S}_n)[nd]$ because $a(g)$ is equal to half the real codimension of Y^g in Y, and in $H^*(S)[d]\{\mathfrak{S}_n\}$ because in [LS2] the grading of each summand of $A\{\mathfrak{S}_n\}$ is centered around zero. The morphism λ is $\mathfrak{S}_n$-equivariant by comparing Definition 1.2 with paragraph 2.8 in [LS2].

We see that λ also preserves the duality, by comparing Definition 1.26 with Proposition 2.16 in [LS2]. □

In order to compare the product structure one has to compute the bundle $F(g, h)$ defined in Construction 1.9. We begin by doing so in a special case.

Lemma 3.6. *Assume that g, h are two elements in $\mathfrak{S}_n$ such that $\langle g, h\rangle$ acts transitively on $\{1, \dots, n\}$; in other words, $Y^{g,h}$ is the small diagonal Δ, canonically isomorphic to S. Then the bundle $F(g, h)$ is isomorphic to a direct sum of copies of T_Δ.*

Proof. We use the notation of Construction 1.9. Let V be the representation of H on $\mathbb{C}^n$ induced by the natural action of $\mathfrak{S}_n$. As an H-equivariant vector bundle, $T_Y|_\Delta$ is isomorphic to the tensor product of T_Δ (with the trivial H-action) and of V. This implies that $F(g, h) = R^1\pi_*^H(E)$ is isomorphic to $T_\Delta \otimes W$, where W is $H^1(C, \mathcal{O}_C \otimes V)^H$. □

Corollary 3.7. *In the assumptions of the Lemma, $c(g, h)$ only depends on the rank r of $F(g, h)$; it has value 1 if $r = 0$, $c_d(T_\Delta)$ if $r = d$, and is zero otherwise.*

Theorem 3.8. *The linear isomorphism $\lambda : H^*(Y, \mathfrak{S}_n)[nd] \to H^*(S)[d]\{\mathfrak{S}_n\}$ defined in Proposition 3.5 is a ring isomorphism.*

Proof. Let $h, l \in \mathfrak{S}_n$, and write $H := \langle h, l\rangle$. We have to prove that for every $\alpha \in H^*(Y^h)$ and $\beta \in H^*(Y^l)$

$$\lambda(\alpha_h \cdot \beta_l) = \lambda(\alpha_h) \cdot \lambda(\beta_h).$$

The fixed locus of H is naturally isomorphic to $S^{O(H)}$: for every $o \in O(H)$, let $p_o : S^{O(H)} \to S$ be the natural projection. Comparing with the definition of the product in Proposition 2.13 of [LS2], we see that we have to prove that

$$c(h, l) = \prod_{o \in O(H)} p_o^* \left(c_d(T_S)^{g(h,l)(o)}\right) \tag{4}$$

where $g(h, l)(o)$ is the graph defect defined in 2.6 in [LS2]. Note that $c_d(T_S)$ coincides with e as defined in [LS2]. Because of the splitting Lemma 1.13,

$$F(h, l) = \bigoplus_{o \in O(h)} p_o^* F_o(h, l)$$

where $F_o(h, l) = F(S^o, h, l)$ viewed as a bundle on S (canonically isomorphic to $(S^o)^H$).

Hence is it is enough to prove (4) in the case where H acts transitively on $\{1, \dots, n\}$; write $g(h, l)$ for $g(h, l)(\{1, \dots, n\})$ (the unique orbit of H).

In this case the rank r of $F(h, l)$ is equal, by comparing Corollary 3.4 with the definition of the graph defect, to $d \cdot g(h, l)$. This completes the proof, by Corollary 3.7, since $c_d(T_S)^i = 0$ for every $i > 1$. □

Now let S be a projective surface over $\mathbb{C}$ with torsion canonical class (i.e., of Kodaira dimension zero). In [LS2] Lehn and Sorger show that after introducing some additional signs $H^*(S)[2]^{[n]} := (H^*(S)[2]\{\mathfrak{S}_n\})^{\mathfrak{S}_n}$ is naturally isomorphic to $H^*(S^{[n]})[2n]$ as a ring.

Definition 3.9. For $g, h \in \mathfrak{S}_n$ let $\epsilon(g,h) := (l(g) + l(h) - l(gh))/2$. Note that this always is an integer. We define a modified ring structure on $H^*(S^n, \mathfrak{S}_n)$ by.

$$\alpha_g * \beta_h = (-1)^{\epsilon(g,h)} \alpha_g \cdot \beta_h.$$

By the obvious identity $\epsilon(g,h) + \epsilon(gh,k) = (l(g) + l(h) + l(k) - l(ghk))/2$ this defines an associative product.

This defines an induced ring structure on $H^*_o([S^n/\mathfrak{S}_n])$, which we denote by $H^*_{so}([S^n/\mathfrak{S}_n])$, and we define the pairing on $H^*_{so}([S^n/\mathfrak{S}_n])$ by

$$\langle \alpha, \beta \rangle := \int_{[S^n/\mathfrak{S}_n]} \alpha * \beta.$$

Let $g \in \mathfrak{S}_n$, and let $N = |O(g)|$. We identify as above $(S^n)^g$ with $S^{O(g)}$. Choosing a numbering $\varphi : \{1, \dots, N\} \simeq O(g)$ gives an isomorphism $\tilde{\varphi}_* : H^*(S^N) \to H^*((S^n)^g)$. So any class in $H^*((S^n)^g)$ can be written as $\tilde{\varphi}_*(\alpha_1 \otimes \dots \otimes \alpha_N)$. We write

$$\overline{\varphi}(\alpha_1 \otimes \dots \otimes \alpha_N) := \sum_{h \in \mathfrak{S}_n} h(\tilde{\varphi}_*(\alpha_1 \otimes \dots \otimes \alpha_N)_g) \in H^*_o([S^n/\mathfrak{S}_n]).$$

We denote by $p_k : H^*(S^{[*]}) \to H^*(S^{[*+k]})$ the generators of the Heisenberg algebra action on the cohomology of the Hilbert schemes (as defined e.g. in [N1]).

Then, using the results of [LS2], Theorem 3.8 can be reformulated as follows.

Theorem 3.10. *Let S be a complex projective surface with $K_S = 0$ and let $\mathbf{1} \in H^0(S^0)$ be the identity. There is a canonical ring isomorphism $\Psi : H^*_{so}([S^n, \mathfrak{S}_n]) \to H^*(S^{[n]})$, given by*

$$\overline{\varphi}(\alpha_1 \otimes \dots \otimes \alpha_N) \mapsto p_{|\varphi(1)|}(\alpha_1) \dots p_{|\varphi(N)|}(\alpha_N)\mathbf{1},$$

which is compatible with the duality pairing.

Proof. Let A be the Frobenius algebra $H^*(S)[2]$ with the degree map given by $-\int_S$. Replace also the degree map on $S^{[n]}$ by $-\int_{S^{[n]}}$. Then in [LS2] a ring isomorphism $A^{[n]} \to H^*(S^{[n]})[2n]$ compatible with the duality pairing is constructed.

In the definition of the product $\lambda(\alpha_g) \cdot \lambda(\beta_h)$ on $A\{\mathfrak{S}_n\}$ the degree is used in two places. First, for the definition of the pushforward $i_*(\alpha|_{(S^n)^{g,h}} \cdot \beta|_{(S^n)^{g,h}})$ where $i : (S^n)^{g,h} \to (S^n)^{gh}$ is the inclusion. With the new definition of the degree the pushforward changes by a sign $(-1)^{|O(g,h)| - |O(gh)|}$.

Second, in the definition of the class e in Section 2.2 of [LS2]. In follows from the definition that e is changed from $c_2(T_S)$ to $-c_2(T_S)$. Let $h, l \in \mathfrak{S}_n$. Then, for the products to be the same, $c_{top}(F(h,l))$ in (4) has to be replaced by $(-1)^{rk(F(h,l))/2} c_{top}(F(h,l))$.

Putting this together, we have to change $\alpha_g \cdot \beta_h$ by a factor of $(-1)^{b(g,h)}$, where by Lemma 3.4

$$\begin{aligned} b(g,h) &= |O(gh)| - |O(g,h)| + \frac{1}{2}(n + 2|O(g,h)| - |O(g)| - |O(h)| - |O(gh)|) \\ &= \frac{1}{2}(n - |O(g)| - |O(h)| + |O(gh)|) = \frac{1}{2}(l(g) + l(h) - l(gh)), \end{aligned}$$

i.e. $\cdot$ is replaced by $*$.

The explicit formula in terms of the Heisenberg operators follows from the definition of Φ in [LS2] directly before Prop. 2.11. □

Note that, by deformation invariance of the cohomology ring, the assumption S projective in Theorem 3.10 can be replaced by S compact, since every compact complex surface with torsion canonical class can be deformed to a projective one. In fact, Theorem 3.10 is also true for $S = \mathbb{A}^2$ by [LS1].

If S is a not necessarily compact complex surface which is also an abelian group (e.g. an abelian variety), then the structure of the orbifold cohomology $H^*([S^n/\mathfrak{S}_n])$ is particularly simple.

Corollary 3.11. *Let S be a smooth complex surface which is also an abelian group. Then the ring structure on $H^*(S^n, \mathfrak{S}_n)$ is given by $\alpha_g \cdot \beta_h = \gamma_{gh}$, where*

(5)
$$\gamma = \begin{cases} i_*(\alpha|_{(S^n)^{g,h}} \cdot \beta|_{(S^n)^{g,h}}) & \text{if } |O(g)| + |O(h)| + |O(gh)| = 2|O(g,h)| + n, \\ 0 & \text{otherwise.} \end{cases}$$

Here i is the embedding of $(S^n)^{g,h}$ into $(S^n)^{gh}$.

Proof. As all the Chern classes of T_S vanish, we get $c(g,h) \neq 0$ if an only if the rank of $F(g,h)$ is zero, in which case $c(g,h) = 1$. By Lemma 3.4 the rank of $F(g,h)$ is $n + 2|O(g,h)| - |O(g)| - |O(h)| - |O(gh)|$. □

We want to generalize the definition of $H^*_{so}([Y/G])$ from the case $Y = S^n$ for S a surface with $K_S = 0$ and $G = \mathfrak{S}_n$ to arbitrary complex symplectic actions of a finite group G on a complex symplectic manifold Y. This is based on the fact that in the above case $l(g) = a(g)$.

Definition 3.12. Let Y be a complex manifold with an action of a finite group G. Assume that for any pair of elements $g, h \in G$ with $Y^{g,h}$ nonempty, $\epsilon(g,h) := (a(g) + a(h) - a(gh))/2$ is an integer.

Then we can define a new associate ring structure on $H^*(Y, G)$ by $\alpha_g * \beta_h = (-1)^{\epsilon(g,h)} \alpha_g \cdot \beta_h$. This is associative because obviously $\epsilon(g,h) + \epsilon(gh,k) = (a(g) + a(h) + a(k) - a(ghk))/2$. This defines a new ring structure on $H^*_o([Y/G])$, which we denote by $H^*_{so}([Y/G])$.

4. Generalized Kummer varieties

Now we want to compute the orbifold cohomology for the orbifold quotients $[S_0^n/\mathfrak{S}_n]$ whose resolutions are the higher order Kummer varieties $K(S)_{n-1}$ of Beauville [Be]. In analogy with Theorem 3.10, we expect that there is a canonical isomorphism from $H^*_{so}([S_0^n/\mathfrak{S}_n])$ to $H^*(K(S)_{n-1})$.

Assume that S is a complex surface which is an abelian group, whose identity element we denote by 0. Let $\sigma := \sigma_n : S^{(n)} \to S$ be the morphism

that associates to a 0-cycle its sum in S. Let $\omega : S^{[n]} \to S^{(n)}$ be the Hilbert-Chow morphism. Then we define $K(S)_{n-1} := \omega^{-1}\sigma^{-1}(0)$. The most important case is if S is compact, i.e. a 2-dimensional torus, when one gets the higher order Kummer varieties introduced and studied by Beauville in [Be]. The varieties $K(S)_{n-1}$ are smooth and complex symplectic: The proof in [Be] works for any S, not necessarily compact (of course in that case $K(S)_{n-1}$ will also be not necessarily compact).

Write $S_0^n := \{(a_1, \dots, a_n) \in S^n \mid \sum a_i = 0\}$. Then S_0^n is isomorphic to S^{n-1} and the symmetric group $\mathfrak{S}_n$ acts on S_0^n by permuting the factors. $K(S)_{n-1}$ is a crepant resolution of the quotient $S_0^n/\mathfrak{S}_n$. We now compute the orbifold cohomology ring $H_o^*([S_0^n/\mathfrak{S}_n])$. We describe $H^*(S) \times H^*(S_0^{n-1}, \mathfrak{S}_n)$. We denote by $S[k]$ the set of k-division points of S. For any subgroup H of $\mathfrak{S}_n$ let $m(H) := gcd\{|o| \mid o \in O(H)\}$ be the greatest common divisor of the numbers of elements of the orbits of H and $m(g_1, \dots, g_r) := m(\langle g_1, \dots, g_r\rangle)$.

Proposition 4.1. 1. *There is a canonical $\mathfrak{S}_n$-equivariant isomorphism*

$$H^*(S) \times H^*(S_0^{n-1}, \mathfrak{S}_n) \simeq \bigoplus_{g \in \mathfrak{S}_n} \bigoplus_{y \in S[m(g)]} H^*((S^n)^g).$$

We denote by $\alpha_{g,y}$ a class $\alpha \in H^*((S^n)^g)$ in the summand corresponding to (g, y). Then the action of $\mathfrak{S}_n$ on the right hand side is given by $h(\alpha_{g,y}) = (h_*(\alpha))_{hgh^{-1},y}$.

2. *The ring structure on $H^*(S) \times H^*(S_0^{n-1}, \mathfrak{S}_n)$ induces via the above isomorphism the following ring structure*

$$\alpha_{g,x} \cdot \beta_{h,y} = \sum_{z \in S[m(g,h)]} n_{g,h}(x, y, z)\gamma_{gh,z}. \tag{6}$$

Here $\gamma \in H^*((S^n)^{gh})$ is given by (5) and

$$n_{g,h}(x, y, z) = \left|\left\{w \in S[m(g, h)] \mid \tfrac{m(g,h)}{m(g)}w = x, \tfrac{m(g,h)}{m(h)}w = y, \tfrac{m(g,h)}{m(gh)}w = z\right\}\right|.$$

Proof. The proof will occupy the rest of this section. We use some of the ideas of [GS] p. 243.

Lemma 4.2. *Let H be a subgroup of $\mathfrak{S}_n$ and assume $m(H) = 1$. Then $(S_0^n)^H$ is connected and there is a canonical isomorphism $H^*(S \times (S_0^n)^H) \simeq H^*((S^n)^H)$.*

Proof. There is an $\mathfrak{S}_n$-equivariant morphism $q : S \times S_0^n \to S^n$ given on points by $(a, (b_i)_i) \mapsto (a + b_i)_i$; for any subgroup H of $\mathfrak{S}_n$ its restriction is a morphism $q : S \times (S_0^n)^H \to (S^n)^H$. The action of $S[n]$ on $S \times S_0^n$ by $c(a, (b_i)_i) = (a - c, (b_i + c)_i)$ commutes with the $\mathfrak{S}_n$-action and the map q is just the quotient map for this action.

Let $H \subset \mathfrak{S}_n$ with $m(H) = 1$. As in [GS] p. 243 one shows that $S \times (S_0^n)^H$ is isomorphic to $(S^n)^H$ (in particular $(S_0^n)^H$ is connected) and that the action of $S[n]$ on $H^*(S \times (S_0^n)^H)$ is trivial. Therefore q^* is a natural isomorphism $H^*((S^n)^H) \to H^*(S \times (S_0^n)^H)$. □

Let H be a subgroup of $\mathfrak{S}_n$, and let $m = m(H)$. We identify $(S^n)^H$ with $S^{O(H)}$. Note that the restriction of σ to $(S^n)^H$ is given by sending $(b_o)_{o \in O(H)}$

to $\sum_{o\in O(H)}|o|b_o$. As all $|o|$ are divisible by m, we can define

$$\sigma/m : (S^n)^H \to S, (b_o)_{o\in O(H)} \mapsto \sum_{o\in O(H)} \frac{|o|}{m} b_o,$$

and for $y \in S[m]$ we define $(S^n)^H_y := (\sigma/m)^{-1}y$. By definition we get

Remark 4.3. $(S^n_0)^H = \coprod_{y\in S[m]}(S^n)^H_y$.

The proof of the next lemma shows that this is the decomposition of $(S^n_0)^H$ into connected components.

Lemma 4.4. *There is a canonical isomorphism* $H^*(S\times(S^n)^H_y) \simeq H^*((S^n)^H)$.

Proof. $(S^n)^H_y$ is identified with

$$S^{O(H)}_y := \big\{(b_o)_{o\in O(H)} \in S^{O(H)} \mid \sum_{o\in O(H)} \frac{|o|}{m} = y\big\}.$$

For any $z \in S$ with $\frac{n}{m}z = y$, we get an isomorphism $S^{O(H)}_y \to S^{O(H)}_0, (b_o)_o \mapsto (b_o - z)_o$. Grouping the elements of $\{1,\dots,n\}$ in sets of m elements, each of which contained in an orbit of H, defines a surjection $\{1,\dots,n\} \to \{1,\dots,n/m\}$, giving isomorphisms $S^{O(H)} \simeq S^{O(\overline{H})}$ and $S^{O(H)}_0 \simeq S^{O(\overline{H})}_0 = (S^{n/m}_0)^{\overline{H}}$ for a subgroup $\overline{H}$ of $\mathfrak{S}_{n/m}$ with $m(\overline{H}) = 1$.

Furthermore with the same proof as in [GS] p. 243 the induced isomorphism $H^*((S^{n/m}_0)^{\overline{H}}) \to H^*((S^n)^H_y)$ is independent of the choice of z. As $m(\overline{H}) = 1$, there is a canonical isomorphism

$$H^*(S \times (S^{n/m}_0)^{\overline{H}}) \simeq H^*((S^{n/m})^{\overline{H}}) \simeq H^*((S^n)^H).$$

□

Putting everything together we get a canonical isomorphism

$$H^*(S) \times H^*(S^n_0, \mathfrak{S}_n) \simeq \bigoplus_{g\in\mathfrak{S}_n} \bigoplus_{y\in S[m(g)]} H^*((S^n)^g),$$

which by definition commutes with the $\mathfrak{S}_n$-action. This shows part 1.

We note that the action of $\mathfrak{S}_n$ on S^n_0 is just the restriction of the action on S^n. Furthermore for any subgroup $H \subset \mathfrak{S}_n$, the normal bundle of $(S^n_0)^H$ in (S^n_0) is just the restriction of the the normal bundle of $(S^n)^H$ in (S^n). In particular the age $a(g)$ of an element $g \in \mathfrak{S}_n$ is the same for both actions, and the bundle $F(g,h)$ on $(S^n_0)^{g,h}$ is the restriction of the corresponding bundle on $(S^n)^{g,h}$(in view of Lemma 1.11). Therefore the ring structure on $H^*(S^n_0, \mathfrak{S}_n)$ is determined in the same way as for $H^*(S^n, \mathfrak{S}_n)$: $H^*(S^n_0, \mathfrak{S}_n) = \bigoplus_{g\in\mathfrak{S}_n} H^*((S^n_0)^g)$ and if we write α_g for a class $\alpha \in H^*((S^n_0)^g)$ in the summand corresponding to g, then $\alpha_g \cdot \beta_h = \gamma_h$ with γ given by the formula (5) with the restriction to $(S^n)^{g,h}$ replaced by that over $(S^n_0)^{g,h}$. Here i is the embedding of $(S^n_0)^{g,h}$ into $(S^n_0)^{gh}$.

We need to determine how the product is distributed over the connected components $(S^n)^H_y$.

We write $\alpha_{(g,y)}$ for a class $\alpha \in H^*((S^n)^H_y)$ and β_g for a class $\beta \in H^*((S^n_0)^H)$. Let $g, h \in \mathfrak{S}_n$, $H := \langle g, h\rangle$, let $x \in S[m(g)]$, $y \in S[m(h)]$.

Then $\alpha_{(g,x)} \cdot \beta_{(h,y)} = 0$ if $|O(g)| + |O(h)| + |O(gh)| \neq 2|O(g,h)| + n$. Otherwise

$$\begin{aligned}\alpha_{(g,x)} \cdot \beta_{(h,y)} &= i_*(\alpha|_{(S_0^n)^H}\beta|_{(S_0^n)^H})_{gh} \\ &= \sum_w (i_w)_*(\alpha|_{(S^n)^H_w}\beta|_{(S^n)^H_w})_{gh} \\ &= \sum_{z\in S[m(gh)]}\sum_w (i_{w,z})_*(\alpha|_{(S^n)^H_w}\beta|_{(S^n)^H_w})_{(gh,z)}. \end{aligned} \tag{7}$$

Here in the second line the inner sum is over all $w \in S[m(H)]$ such that $(S^n)^H_w \subset (S^n)^g_x$ and $(S^n)^H_w \subset (S^n)^h_y$. In the last row we require in addition that $(S^n)^H_w \subset (S^n)^{gh}_z$. Note that this is equivalent to $\frac{m(g,h)}{m(g)}w = x$, $\frac{m(g,h)}{m(h)}w = y$, $\frac{m(g,h)}{m(gh)}w = z$. Finally $i : (S_0^n)^H \to (S_0^n)^{gh}$, $i_w : (S^n)^H_w \to (S_0^n)^{gh}$, $i_{w,z} : (S^n)^H_w \to (S^n)^{gh}_z$ are the inclusions.

From the definitions it is also obvious that, if $(S^n)^H_y \subset (S^n)^G_z$ and thus also $(S^n)^H \subset (S^n)^G$, then the isomorphisms $H^*(S \times (S^n)^H_y) \simeq H^*((S^n)^H)$ and $H^*(S \times (S^n)^G_z) \simeq H^*((S^n)^G)$ commute with the pullbacks induced by the inclusions. Therefore (6) follows from (7), thus completing the proof of Proposition 4.1. □

Appendix

We collect here for the reader's convenience a few facts about Galois coverings of rational smooth and nodal curves which are of an elementary nature and probably well known but for which we couldn't find a suitable reference.

Let D be a smooth complex curve and U a contractible neighborhood of a point p. Then the fundamental group of $U \setminus p$ is infinite cyclic and has a canonical generator γ_p, defined by the condition that the integral over γ_p of dz/z be $2\pi i$, where z is a local coordinate at p. Note that we don't need to specify a basepoint since the fundamental group is abelian. Moreover, γ_p doesn't depend on the neighborhood U chosen, in an obvious sense.

If D is complete and rational, given n points $p_1, \dots, p_n$ and a basepoint p_0 we can choose $\gamma_i \in F = \pi_1(D\setminus\{p_1, \dots, p_n\}, p_0)$ such that $\gamma_1 \cdot \ldots \cdot \gamma_n$ is the identity and γ_i is the pushforward of γ_{p_i} from any contractible neighborhood of p_i that doesn't contain any other p_j. In fact F is the free group generated by any $n-1$ of the γ_i. So given any group G and any set of n elements g_i such that $g_1 \cdot \ldots \cdot g_n$ is the identity, there is an induced homomorphism $F \to G$ sending γ_i to g_i; this defines a Galois covering $C^o(D, g_i, G)$ of $D\setminus\{p_1, \dots, p_n\}$ with Galois group G.

If in the above construction G is finite, then one can uniquely complete $C^o(D, g_i, G)$ to a proper smooth Galois G cover $C = C(D, g_i, G)$ of D branched over $p_1, \dots, p_n$. Note that the representation of G defined by $H^a(C, \mathcal{O}_C)$ (where a is 0 or 1) does not depend on the choice of $p_0, \dots, p_n$ or the γ_i's, because any two curves obtained by different choices can be deformed to each other.

Note that C is connected if the g_i's generate G. Otherwise, if H is the subgroup generated by the g_i, one can find a connected component

of $C(D, g_i, G)$ which is H invariant and isomorphic to $C(D, g_i, H)$ as a curve with H action; the connected components of $C(D, g_i, G)$ are then in bijection with the cosets of H in G. This implies that the cohomology group $H^a(C(D, g_i, G), \mathcal{O})$ is isomorphic, as a representation of G, to $Ind_H^G H^a(C(D, g_i, H), \mathcal{O})$ (the induced representation, see page 32 of [FH]).

Assume now that D is the union of two smooth, proper rational curves D' and D'' meeting transversely at one point p. Choose $p_1, \dots, p_k$ on D' and $p_{k+1}, \dots, p_n$ on D'', and choose again $g_i \in G$ a finite group such that $g_1 \cdot \dots \cdot g_n$ is the identity. Then the previous construction defines a Galois cover $C' = C(D', g_1, \dots, g_k, g, G)$ (resp. $C'' = C(D'', g^{-1}, g_{k+1}, \dots, g_n, G)$) of D' (resp. D'') with Galois group G, branched over $p_1, \dots, p_k, p$ (respectively $p, p_{k+1}, \dots, p_n$); here $g = g_{k+1} \cdot \dots \cdot g_n = (g_1 \cdot \dots \cdot g_k)^{-1}$.

Let Z' (resp. Z'') be the inverse image of p in C' (resp. C''); then there are points q' in Z' and q'' in Z'' such that their stabilizer is generated by g and such that g acts on $T_{C',q'}$ and $T_{C'',q''}$ with two roots of unity with product one. One can therefore naturally identify Z' with Z'' by identifying gq' with gq'' for every $g \in G$, thus obtaining a nodal curve $C = C(D, g_i, G)$ which is a Galois cover of D with Galois group G, branched over the marked points and the node. It is easy to see that C is connected if and only if the g_i generate G.

Again the definition of C depends on a number of choices but the representation $H^a(C, \mathcal{O}_C)$ of G only depends on the elements $g_1, \dots, g_n$.

The construction of the cover of the nodal curve is closely related to the notion of admissible cover introduced in [HM] and coincides with (a special case of) that of twisted stable n-pointed map into BG in [AV].

References

[AV] D. Abramovich, A. Vistoli, *Compactifying the space of stable maps*, preprint math.AG/9908167 v2.

[BB] V. Batyrev, L. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math. 126 (1996), no. 1, 183–203.

[Be] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom. 18 (1983), no. 4, 755–782.

[CR1] W. Chen, Y. Ruan, *A new cohomology theory for orbifold*, math.AG/0004129.

[CR2] W. Chen, Y. Ruan, *Stringy geometry and topology of orbifolds*, math.AG/0011149.

[D*1] L. Dixon, J. A. Harvey, C. Vafa, E. Witten, *Strings on orbifolds*, Nuclear Phys. B 261 (1985), no. 4, 678–686.

[D*2] L. Dixon, J. A. Harvey, C. Vafa, E. Witten, *Strings on orbifolds. II*, Nuclear Phys. B 274 (1986), no. 2, 285–314.

[EG] P. Etingof, V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, preprint math.AG/0011114.

[F] B. Fantechi, *The orbifold Chow ring*, in preparation.

[FH] W. Fulton, J. Harris, *Representation Theory: a first course*, GTM **129**, Springer-Verlag, New York 1991.

[G] L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. 286 (1990), no. 1-3, 193–207.

[GS] L. Göttsche, W. Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*. Math. Ann. 296 (1993), no. 2, 235–245.

[Gr] I. Grojnowski, *Instantons and affine algebras. I. The Hilbert scheme and vertex operators*, Math. Res. Lett. 3 (1996), no. 2, 275–291.

[Gro] A. Grothendieck, *Sur le mémoire de Weil: généralisation des fonctions abéliennes*. Séminaire Bourbaki, Vol. 4, Exp. No. 141, 57–71, Soc. Math. France, Paris, 1995.

[HM] J. Harris, D. Mumford, *On the Kodaira dimension of the moduli space of curves*. With an appendix by William Fulton. Invent. Math. 67 (1982), no. 1, 23–88.

[HH] F. Hirzebruch, T. Höfer, *On the Euler number of an orbifold*, Math. Ann. 286 (1990), no. 1-3, 255–260.

[L] M. Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. 136 (1999), no. 1, 157–207.

[LS1] M. Lehn, C. Sorger, *Symmetric groups and the cup product on the cohomology of Hilbert schemes*, math.AG/0009131.

[LS2] M. Lehn, C. Sorger, *The cup product of the Hilbert scheme for $K3$ surfaces*, math.AG/0012166.

[N1] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) 145 (1997), no. 2, 379–388.

[N2] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999. xii+132 pp.

[LQW] W. Li, Z. Qin, W. Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces*, math.AG/0009132.

[PB] *Susanna, prima bisiaca del millennio*. Il Piccolo, edizione di Monfalcone, 2 gennaio 2000.

[Q] D. Quillen, *Elementary proofs of some results of cobordism using Steenrod Operations*, Adv. Math. **7** (1971), 29–56.

[R] M. Reid, *La correspondance de McKay*, Séminaire Bourbaki, exp. 867, vol. 1999/2000.

[U] B. Uribe, Ph.D. Thesis, University of Wisconsin, in preparation.

[V] E. Vasserot, *Sur l'anneau de cohomologie du schéma de Hilbert de $\mathbb{C}^2$*, math.AG/0009127.

[Z] E. Zaslow *Topological orbifold models and quantum cohomology rings*. Comm. Math. Phys. 156 (1993), no. 2, 301–331.

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