II Homomorphisms

This is just another name for "LINEAR FUNCTIONS"

The definition of isomorphism has two conditions. In this section we will consider the second one. We will study maps that are required only to preserve structure, maps that are not also required to be correspondences.

Experience shows that these maps are tremendously useful. For one thing we shall see in the second subsection below that while isomorphisms describe how spaces are the same, we can think of these maps as describing how spaces are alike.

II.1 Definition

1.1 Definition A function between vector spaces $h\colon V \to W$ that preserves addition

$$\text{if } \vec{v}_1, \vec{v}_2 \in V \text{ then } h(\vec{v}_1 + \vec{v}_2) = h(\vec{v}_1) + h(\vec{v}_2)$$

and scalar multiplication

$$\text{if } \vec{v} \in V \text{ and } r \in \mathbb{R} \text{ then } h(r \cdot \vec{v}) = r \cdot h(\vec{v})$$

is a *homomorphism* or *linear map*.

1.2 Example The projection map $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

This is a good example

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{\pi} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a homomorphism. It preserves addition

$$\pi(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}) = \pi(\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}) = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \pi(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) + \pi(\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix})$$

and scalar multiplication.

$$\pi(r \cdot \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}) = \pi(\begin{pmatrix} rx_1 \\ ry_1 \\ rz_1 \end{pmatrix}) = \begin{pmatrix} rx_1 \\ ry_1 \end{pmatrix} = r \cdot \pi(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix})$$

This is not an isomorphism since it is not one-to-one. For instance, both $\vec{0}$ and $\vec{e}_3$ in $\mathbb{R}^3$ map to the zero vector in $\mathbb{R}^2$.

1.3 Example The domain and codomain can be other than spaces of column vectors. Both of these are homomorphisms; the verifications are straightforward.

(1) $f_1\colon \mathcal{P}_2 \to \mathcal{P}_3$ given by

$$a_0 + a_1x + a_2x^2 \;\mapsto\; a_0x + (a_1/2)x^2 + (a_2/3)x^3$$

(2) $f_2\colon M_{2\times 2} \to \mathbb{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

Silly but important

1.4 Example Between any two spaces there is a *zero homomorphism*, mapping every vector in the domain to the zero vector in the codomain.

1.5 Example These two suggest why we use the term 'linear map'.

(1) The map $g\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{g} 3x + 2y - 4.5z$$

is linear, that is, is a homomorphism. The check is easy. In contrast, the map $\hat{g}\colon \mathbb{R}^3 \to \mathbb{R}$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{\hat{g}} 3x + 2y - 4.5z + 1$$

is not linear. To show this we need only produce a single linear combination that the map does not preserve. Here is one.

$$\hat{g}(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = 4 \qquad \hat{g}(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}) + \hat{g}(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}) = 5$$

(2) The first of these two maps $t_1, t_2\colon \mathbb{R}^3 \to \mathbb{R}^2$ is linear while the second is not.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{t_1} \begin{pmatrix} 5x - 2y \\ x + y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{t_2} \begin{pmatrix} 5x - 2y \\ xy \end{pmatrix}$$

Finding a linear combination that the second map does not preserve is easy.

So one way to think of 'homomorphism' is that we are generalizing 'isomorphism' (by dropping the condition that the map is a correspondence), motivated by the observation that many of the properties of isomorphisms have only to do with the map's structure-preservation property. The next two results are examples of this motivation. In the prior section we saw a proof for each that only uses preservation of addition and preservation of scalar multiplication, and therefore applies to homomorphisms.

We already saw this!

1.6 Lemma A homomorphism sends the zero vector to the zero vector.

1.7 Lemma The following are equivalent for any map $f\colon V \to W$ between vector spaces.

And this!

(1) f is a homomorphism
(2) $f(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2) = c_1 \cdot f(\vec{v}_1) + c_2 \cdot f(\vec{v}_2)$ for any $c_1, c_2 \in \mathbb{R}$ and $\vec{v}_1, \vec{v}_2 \in V$
(3) $f(c_1 \cdot \vec{v}_1 + \cdots + c_n \cdot \vec{v}_n) = c_1 \cdot f(\vec{v}_1) + \cdots + c_n \cdot f(\vec{v}_n)$ for any $c_1, \ldots, c_n \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_n \in V$

1.8 Example The function $f\colon \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{f} \begin{pmatrix} x/2 \\ 0 \\ x+y \\ 3y \end{pmatrix}$$

is linear since it satisfies item (2).

$$\begin{pmatrix} r_1(x_1/2) + r_2(x_2/2) \\ 0 \\ r_1(x_1+y_1) + r_2(x_2+y_2) \\ r_1(3y_1) + r_2(3y_2) \end{pmatrix} = r_1 \begin{pmatrix} x_1/2 \\ 0 \\ x_1+y_1 \\ 3y_1 \end{pmatrix} + r_2 \begin{pmatrix} x_2/2 \\ 0 \\ x_2+y_2 \\ 3y_2 \end{pmatrix}$$

However, some things that hold for isomorphisms fail to hold for homomorphisms. One example is in the proof of Lemma I.2.4, which shows that an isomorphism between spaces gives a correspondence between their bases. Homomorphisms do not give any such correspondence; Example 1.2 shows this and another example is the zero map between two nontrivial spaces. Instead, for homomorphisms we have a weaker but still very useful result.

And this!

1.9 Theorem A homomorphism is determined by its action on a basis: if V is a vector space with basis $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$, if W is a vector space, and if $\vec{w}_1, \ldots, \vec{w}_n \in W$ (these codomain elements need not be distinct) then there exists a homomorphism from V to W sending each $\vec{\beta}_i$ to $\vec{w}_i$, and that homomorphism is unique.

Proof For any input $\vec{v} \in V$ let its expression with respect to the basis be $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$. Define the associated output by using the same coordinates $h(\vec{v}) = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n$. This is well defined because, with respect to the basis, the representation of each domain vector $\vec{v}$ is unique.

This map is a homomorphism because it preserves linear combinations: where $\vec{v_1} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ and $\vec{v_2} = d_1\vec{\beta}_1 + \cdots + d_n\vec{\beta}_n$, here is the calculation.

$$\begin{aligned} h(r_1\vec{v}_1 + r_2\vec{v}_2) &= h(\,(r_1c_1 + r_2d_1)\vec{\beta}_1 + \cdots + (r_1c_n + r_2d_n)\vec{\beta}_n\,) \\ &= (r_1c_1 + r_2d_1)\vec{w}_1 + \cdots + (r_1c_n + r_2d_n)\vec{w}_n \\ &= r_1h(\vec{v}_1) + r_2h(\vec{v}_2) \end{aligned}$$

This map is unique because if $\hat{h}\colon V \to W$ is another homomorphism satisfying that $\hat{h}(\vec{\beta}_i) = \vec{w}_i$ for each i then h and $\hat{h}$ have the same effect on all of the vectors in the domain.

$$\begin{aligned} \hat{h}(\vec{v}) = \hat{h}(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) &= c_1\hat{h}(\vec{\beta}_1) + \cdots + c_n\hat{h}(\vec{\beta}_n) \\ &= c_1\vec{w}_1 + \cdots + c_n\vec{w}_n = h(\vec{v}) \end{aligned}$$

They have the same action so they are the same function. QED

And this! This chapter is essentially talking about our Worksheet 3

1.10 Definition Let V and W be vector spaces and let $B = \langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ be a basis for V. A function defined on that basis $f\colon B \to W$ is *extended linearly* to a function $\hat{f}\colon V \to W$ if for all $\vec{v} \in V$ such that $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$, the action of the map is $\hat{f}(\vec{v}) = c_1 \cdot f(\vec{\beta}_1) + \cdots + c_n \cdot f(\vec{\beta}_n)$.

1.11 Example If we specify a map $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ that acts on the standard basis $\mathcal{E}_2$ in this way

$$h(\begin{pmatrix}1\\0\end{pmatrix}) = \begin{pmatrix}-1\\1\end{pmatrix} \qquad h(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}-4\\4\end{pmatrix}$$

then we have also specified the action of h on any other member of the domain. For instance, the value of h on this argument

$$h(\begin{pmatrix}3\\-2\end{pmatrix}) = h(3 \cdot \begin{pmatrix}1\\0\end{pmatrix} - 2 \cdot \begin{pmatrix}0\\1\end{pmatrix}) = 3 \cdot h(\begin{pmatrix}1\\0\end{pmatrix}) - 2 \cdot h(\begin{pmatrix}0\\1\end{pmatrix}) = \begin{pmatrix}5\\-5\end{pmatrix}$$

is a direct consequence of the value of h on the basis vectors.

Later in this chapter we shall develop a convenient scheme for computations like this one, using matrices.

1.12 Definition A linear map from a space into itself $t\colon V \to V$ is a *linear transformation*.

1.13 Remark In this book we use 'linear transformation' only in the case where the codomain equals the domain. However, be aware that other sources may instead use it as a synonym for 'homomorphism'.

1.14 Example The map on $\mathbb{R}^2$ that projects all vectors down to the x-axis is a linear transformation.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

1.15 Example The derivative map $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$

$$a_0 + a_1x + \cdots + a_nx^n \overset{d/dx}{\longmapsto} a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

is a linear transformation as this result from calculus shows: $d(c_1f + c_2g)/dx = c_1\,(df/dx) + c_2\,(dg/dx)$.

1.16 Example The matrix transpose operation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is a linear transformation of $\mathcal{M}_{2\times 2}$. (Transpose is one-to-one and onto and so is in fact an automorphism.)

We finish this subsection about maps by recalling that we can linearly combine maps. For instance, for these maps from $\mathbb{R}^2$ to itself

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix} 2x \\ 3x - 2y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \overset{g}{\longmapsto} \begin{pmatrix} 0 \\ 5x \end{pmatrix}$$

the linear combination $5f - 2g$ is also a transformation of R^2.

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{5f-2g}{\longmapsto} \begin{pmatrix} 10x \\ 5x - 10y \end{pmatrix}$$

This is cool, but ignore it if it is confusing. It is a concept that belongs more to M469 than to this class.

1.17 Lemma For vector spaces V and W, the set of linear functions from V to W is itself a vector space, a subspace of the space of all functions from V to W.

We denote the space of linear maps from V to W by $\mathcal{L}(V, W)$.

Proof This set is non-empty because it contains the zero homomorphism. So to show that it is a subspace we need only check that it is closed under the

operations. Let $f, g\colon V \to W$ be linear. Then the operation of function addition is preserved

$$\begin{aligned}(f+g)(c_1\vec{v}_1 + c_2\vec{v}_2) &= f(c_1\vec{v}_1 + c_2\vec{v}_2) + g(c_1\vec{v}_1 + c_2\vec{v}_2)\\ &= c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2) + c_1 g(\vec{v}_1) + c_2 g(\vec{v}_2)\\ &= c_1\big(f+g\big)(\vec{v}_1) + c_2\big(f+g\big)(\vec{v}_2)\end{aligned}$$

as is the operation of scalar multiplication of a function.

$$\begin{aligned}(r\cdot f)(c_1\vec{v}_1 + c_2\vec{v}_2) &= r(c_1 f(\vec{v}_1) + c_2 f(\vec{v}_2))\\ &= c_1(r\cdot f)(\vec{v}_1) + c_2(r\cdot f)(\vec{v}_2)\end{aligned}$$

Hence $\mathcal{L}(V, W)$ is a subspace. QED

We started this section by defining 'homomorphism' as a generalization of 'isomorphism', by isolating the structure preservation property. Some of the points about isomorphisms carried over unchanged, while we adapted others.

Note, however, that the idea of 'homomorphism' is in no way somehow secondary to that of 'isomorphism'. In the rest of this chapter we shall work mostly with homomorphisms. This is partly because any statement made about homomorphisms is automatically true about isomorphisms but more because, while the isomorphism concept is more natural, our experience will show that the homomorphism concept is more fruitful and more central to progress.

Exercises

✓ 1.18 Decide if each $h\colon \mathbb{R}^3 \to \mathbb{R}^2$ is linear.

(a) $h(\begin{pmatrix}x\\y\\z\end{pmatrix}) = \begin{pmatrix}x\\x+y+z\end{pmatrix}$ **(b)** $h(\begin{pmatrix}x\\y\\z\end{pmatrix}) = \begin{pmatrix}0\\0\end{pmatrix}$ **(c)** $h(\begin{pmatrix}x\\y\\z\end{pmatrix}) = \begin{pmatrix}1\\1\end{pmatrix}$

(d) $h(\begin{pmatrix}x\\y\\z\end{pmatrix}) = \begin{pmatrix}2x+y\\3y-4z\end{pmatrix}$

✓ 1.19 Decide if each map $h\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ is linear.

(a) $h(\begin{pmatrix}a&b\\c&d\end{pmatrix}) = a+d$

(b) $h(\begin{pmatrix}a&b\\c&d\end{pmatrix}) = ad-bc$

(c) $h(\begin{pmatrix}a&b\\c&d\end{pmatrix}) = 2a+3b+c-d$

(d) $h(\begin{pmatrix}a&b\\c&d\end{pmatrix}) = a^2+b^2$

✓ 1.20 Show that these are homomorphisms. Are they inverse to each other?

(a) $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_2$ given by $a_0 + a_1x + a_2x^2 + a_3x^3$ maps to $a_1 + 2a_2x + 3a_3x^2$

(b) $\int\colon \mathcal{P}_2 \to \mathcal{P}_3$ given by $b_0 + b_1x + b_2x^2$ maps to $b_0x + (b_1/2)x^2 + (b_2/3)x^3$

1.21 Is (perpendicular) projection from $\mathbb{R}^3$ to the xz-plane a homomorphism? Projection to the yz-plane? To the x-axis? The y-axis? The z-axis? Projection to the origin?

1.22 Verify that each map is a homomorphism.

(a) $h\colon \mathcal{P}_3 \to \mathbb{R}^2$ given by

$$ax^2 + bx + c \mapsto \begin{pmatrix} a+b \\ a+c \end{pmatrix}$$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ x-y \\ 3y \end{pmatrix}$$

1.23 Show that, while the maps from Example 1.3 preserve linear operations, they are not isomorphisms.

1.24 Is an identity map a linear transformation?

✓ 1.25 Stating that a function is 'linear' is different than stating that its graph is a line.

(a) The function $f_1\colon \mathbb{R} \to \mathbb{R}$ given by $f_1(x) = 2x - 1$ has a graph that is a line. Show that it is not a linear function.

(b) The function $f_2\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + 2y$$

does not have a graph that is a line. Show that it is a linear function.

✓ 1.26 Part of the definition of a linear function is that it respects addition. Does a linear function respect subtraction?

1.27 Assume that h is a linear transformation of V and that $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis of V. Prove each statement.

(a) If $h(\vec{\beta}_i) = \vec{0}$ for each basis vector then h is the zero map.

(b) If $h(\vec{\beta}_i) = \vec{\beta}_i$ for each basis vector then h is the identity map.

(c) If there is a scalar r such that $h(\vec{\beta}_i) = r \cdot \vec{\beta}_i$ for each basis vector then $h(\vec{v}) = r \cdot \vec{v}$ for all vectors in V.

1.28 Consider the vector space $\mathbb{R}^+$ where vector addition and scalar multiplication are not the ones inherited from $\mathbb{R}$ but rather are these: $a + b$ is the product of a and b, and $r \cdot a$ is the r-th power of a. (This was shown to be a vector space in an earlier exercise.) Verify that the natural logarithm map $\ln\colon \mathbb{R}^+ \to \mathbb{R}$ is a homomorphism between these two spaces. Is it an isomorphism?

1.29 Consider this transformation of the plane $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x/2 \\ y/3 \end{pmatrix}$$

Find the image under this map of this ellipse.

$$\{\begin{pmatrix} x \\ y \end{pmatrix} \mid (x^2/4) + (y^2/9) = 1\}$$

✓ 1.30 Imagine a rope wound around the earth's equator so that it fits snugly (suppose that the earth is a sphere). How much extra rope must we add to raise the circle to a constant six feet off the ground?

✓ 1.31 Verify that this map $h\colon \mathbb{R}^3 \to \mathbb{R}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 3x - y - z$$

is linear. Generalize.

1.32 Show that every homomorphism from $\mathbb{R}^1$ to $\mathbb{R}^1$ acts via multiplication by a scalar. Conclude that every nontrivial linear transformation of $\mathbb{R}^1$ is an isomorphism. Is that true for transformations of $\mathbb{R}^2$? $\mathbb{R}^n$?

1.33 **(a)** Show that for any scalars $a_{1,1}, \ldots, a_{m,n}$ this map $h\colon \mathbb{R}^n \to \mathbb{R}^m$ is a homomorphism.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n \\ \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n \end{pmatrix}$$

(b) Show that for each i, the i-th derivative operator d^i/dx^i is a linear transformation of $\mathcal{P}_n$. Conclude that for any scalars $c_k, \ldots, c_0$ this map is a linear transformation of that space.

$$f \mapsto c_k \frac{d^k}{dx^k} f + c_{k-1} \frac{d^{k-1}}{dx^{k-1}} f + \cdots + c_1 \frac{d}{dx} f + c_0 f$$

1.34 Lemma 1.17 shows that a sum of linear functions is linear and that a scalar multiple of a linear function is linear. Show also that a composition of linear functions is linear.

1.35 Where $f\colon V \to W$ is linear, suppose that $f(\vec{v}_1) = \vec{w}_1, \ldots, f(\vec{v}_n) = \vec{w}_n$ for some vectors $\vec{w}_1, \ldots, \vec{w}_n$ from W.

(a) If the set of $\vec{w}$'s is independent, must the set of $\vec{v}$'s also be independent?

(b) If the set of $\vec{v}$'s is independent, must the set of $\vec{w}$'s also be independent?

(c) If the set of $\vec{w}$'s spans W, must the set of $\vec{v}$'s span V?

(d) If the set of $\vec{v}$'s spans V, must the set of $\vec{w}$'s span W?

1.36 Generalize Example 1.16 by proving that for every appropriate domain and codomain the matrix transpose map is linear. What are the appropriate domains and codomains?

1.37 **(a)** Where $\vec{u}, \vec{v} \in \mathbb{R}^n$, by definition the line segment connecting them is the set $\ell = \{t \cdot \vec{u} + (1-t) \cdot \vec{v} \mid t \in [0..1]\}$. Show that the image, under a homomorphism h, of the segment between $\vec{u}$ and $\vec{v}$ is the segment between $h(\vec{u})$ and $h(\vec{v})$.

(b) A subset of $\mathbb{R}^n$ is *convex* if, for any two points in that set, the line segment joining them lies entirely in that set. (The inside of a sphere is convex while the skin of a sphere is not.) Prove that linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ preserve the property of set convexity.

✓ 1.38 Let $h\colon \mathbb{R}^n \to \mathbb{R}^m$ be a homomorphism.

(a) Show that the image under h of a line in $\mathbb{R}^n$ is a (possibly degenerate) line in $\mathbb{R}^m$.

(b) What happens to a k-dimensional linear surface?

1.39 Prove that the restriction of a homomorphism to a subspace of its domain is another homomorphism.

1.40 Assume that $h\colon V \to W$ is linear.

(a) Show that the *range space* of this map $\{h(\vec{v}) \mid \vec{v} \in V\}$ is a subspace of the codomain W.

(b) Show that the *null space* of this map $\{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$ is a subspace of the domain V.

(c) Show that if U is a subspace of the domain V then its image $\{h(\vec{u}) \mid \vec{u} \in U\}$ is a subspace of the codomain W. This generalizes the first item.

(d) Generalize the second item.

1.41 Consider the set of isomorphisms from a vector space to itself. Is this a subspace of the space $\mathcal{L}(V, V)$ of homomorphisms from the space to itself?

1.42 Does Theorem 1.9 need that $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis? That is, can we still get a well-defined and unique homomorphism if we drop either the condition that the set of $\vec{\beta}$'s be linearly independent, or the condition that it span the domain?

1.43 Let V be a vector space and assume that the maps $f_1, f_2\colon V \to \mathbb{R}^1$ are linear.

(a) Define a map $F\colon V \to \mathbb{R}^2$ whose component functions are the given linear ones.

$$\vec{v} \mapsto \begin{pmatrix} f_1(\vec{v}) \\ f_2(\vec{v}) \end{pmatrix}$$

Show that F is linear.

(b) Does the converse hold — is any linear map from V to $\mathbb{R}^2$ made up of two linear component maps to $\mathbb{R}^1$?

(c) Generalize.

II.2 Range space and Null space

Isomorphisms and homomorphisms both preserve structure. The difference is that homomorphisms have fewer restrictions, since they needn't be onto and needn't be one-to-one. We will examine what can happen with homomorphisms that cannot happen with isomorphisms.

First consider the fact that homomorphisms need not be onto. Of course, each function is onto some set, namely its range. For example, the injection map $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

is a homomorphism, and is not onto $\mathbb{R}^3$. But it is onto the xy-plane.

This is important!

2.1 Lemma Under a homomorphism, the image of any subspace of the domain is a subspace of the codomain. In particular, the image of the entire space, the range of the homomorphism, is a subspace of the codomain.

PROOF Let $h\colon V \to W$ be linear and let S be a subspace of the domain V. The image $h(S)$ is a subset of the codomain W, which is nonempty because S is nonempty. Thus, to show that $h(S)$ is a subspace of W we need only show that it is closed under linear combinations of two vectors. If $h(\vec{s}_1)$ and $h(\vec{s}_2)$ are members of $h(S)$ then $c_1 \cdot h(\vec{s}_1) + c_2 \cdot h(\vec{s}_2) = h(c_1 \cdot \vec{s}_1) + h(c_2 \cdot \vec{s}_2) = h(c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2)$ is also a member of $h(S)$ because it is the image of $c_1 \cdot \vec{s}_1 + c_2 \cdot \vec{s}_2$ from S. QED

2.2 Definition The *range space* of a homomorphism $h\colon V \to W$ is

$$\mathscr{R}(h) = \{h(\vec{v}) \mid \vec{v} \in V\}$$

sometimes denoted $h(V)$. The dimension of the range space is the map's *rank*.

We shall soon see the connection between the rank of a map and the rank of a matrix.

2.3 Example For the derivative map $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_1 + 2a_2x + 3a_3x^2$ the range space $\mathscr{R}(d/dx)$ is the set of quadratic polynomials $\{r + sx + tx^2 \mid r, s, t \in \mathbb{R}\}$. Thus, this map's rank is 3.

2.4 Example With this homomorphism $h\colon M_{2\times 2} \to \mathcal{P}_3$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a + b + 2d) + cx^2 + cx^3$$

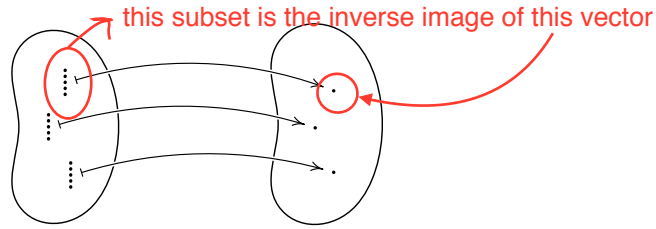
an image vector in the range can have any constant term, must have an x coefficient of zero, and must have the same coefficient of x^2 as of x^3. That is, the range space is $\mathscr{R}(h) = \{r + sx^2 + sx^3 \mid r, s \in \mathbb{R}\}$ and so the rank is 2.

The prior result shows that, in passing from the definition of isomorphism to the more general definition of homomorphism, omitting the onto requirement doesn't make an essential difference. Any homomorphism is onto some space, namely its range.

However, omitting the one-to-one condition does make a difference. A homomorphism may have many elements of the domain that map to one element of the codomain. Below is a bean sketch of a many-to-one map between sets.* It shows three elements of the codomain that are each the image of many members of the domain. (Rather than picture lots of individual $\mapsto$ arrows, each association of many inputs with one output shows only one such arrow.)

* More information on many-to-one maps is in the appendix.

The inverse image of a vector in W is a subset of V: it can be empty, it can have one element, or it can have infinitely many elements.

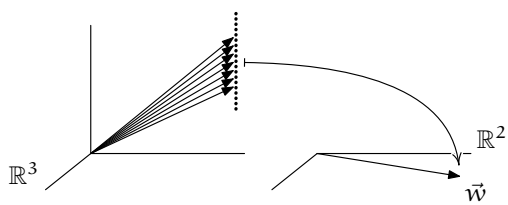


Recall that for any function $h\colon V \to W$, the set of elements of V that map to $\vec{w} \in W$ is the *inverse image* $h^{-1}(\vec{w}) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{w}\}$. Above, the left side shows three inverse image sets.

2.5 Example Consider the projection $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{\pi} \begin{pmatrix} x \\ y \end{pmatrix}$$

which is a homomorphism that is many-to-one. An inverse image set is a vertical line of vectors in the domain.



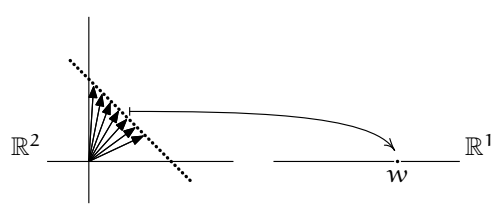
One example is this.

$$\pi^{-1}(\begin{pmatrix} 1 \\ 3 \end{pmatrix}) = \{\begin{pmatrix} 1 \\ 3 \\ z \end{pmatrix} \mid z \in \mathbb{R}\}$$

2.6 Example This homomorphism $h\colon \mathbb{R}^2 \to \mathbb{R}^1$

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{h} x + y$$

is also many-to-one. For a fixed $w \in \mathbb{R}^1$ the inverse image $h^{-1}(w)$

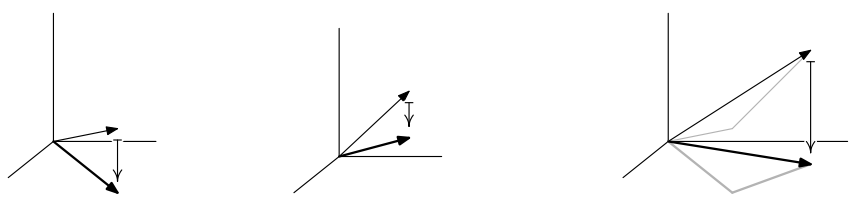


is the set of plane vectors whose components add to w.

In generalizing from isomorphisms to homomorphisms by dropping the one-to-one condition we lose the property that, intuitively, the domain is "the same" as the range. We lose, that is, that the domain corresponds perfectly to the range. The examples below illustrate that what we retain is that a homomorphism describes how the domain is "analogous to" or "like" the range.

2.7 Example We think of $\mathbb{R}^3$ as like $\mathbb{R}^2$ except that vectors have an extra component. That is, we think of the vector with components x, y, and z as like the vector with components x and y. Defining the projection map π makes precise which members of the domain we are thinking of as related to which members of the codomain.

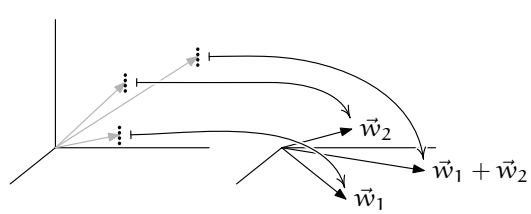
To understanding how the preservation conditions in the definition of homomorphism show that the domain elements are like the codomain elements, start by picturing $\mathbb{R}^2$ as the xy-plane inside of $\mathbb{R}^3$ (the xy plane inside of $\mathbb{R}^3$ is a set of three-tall vectors with a third component of zero and so does not precisely equal the set of two-tall vectors $\mathbb{R}^2$, but this embedding makes the picture much clearer). The preservation of addition property says that vectors in $\mathbb{R}^3$ act like their shadows in the plane.



$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix} \text{ above } \begin{pmatrix}x_1\\y_1\end{pmatrix} \text{ plus } \begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix} \text{ above } \begin{pmatrix}x_2\\y_2\end{pmatrix} \text{ equals } \begin{pmatrix}x_1+y_1\\y_1+y_2\\z_1+z_2\end{pmatrix} \text{ above } \begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}$$

Thinking of $\pi(\vec{v})$ as the "shadow" of $\vec{v}$ in the plane gives this restatement: the sum of the shadows $\pi(\vec{v}_1)+\pi(\vec{v}_2)$ equals the shadow of the sum $\pi(\vec{v}_1+\vec{v}_2)$. Preservation of scalar multiplication is similar.

Drawing the codomain $\mathbb{R}^2$ on the right gives a picture that is uglier but is more faithful to the bean sketch above.



Again, the domain vectors that map to $\vec{w}_1$ lie in a vertical line; one is drawn, in gray. Call any member of this inverse image $\pi^{-1}(\vec{w}_1)$ a "$\vec{w}_1$ vector." Similarly, there is a vertical line of "$\vec{w}_2$ vectors" and a vertical line of "$\vec{w}_1+\vec{w}_2$ vectors."

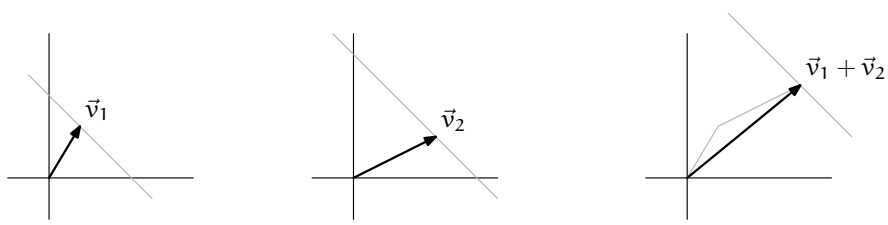
Now, saying that π is a homomorphism is recognizing that if $\pi(\vec{v}_1) = \vec{w}_1$ and $\pi(\vec{v}_2) = \vec{w}_2$ then $\pi(\vec{v}_1 + \vec{v}_2) = \pi(\vec{v}_1) + \pi(\vec{v}_2) = \vec{w}_1 + \vec{w}_2$. That is, the classes add: any $\vec{w}_1$ vector plus any $\vec{w}_2$ vector equals a $\vec{w}_1 + \vec{w}_2$ vector. Scalar multiplication is similar.

So although $\mathbb{R}^3$ and $\mathbb{R}^2$ are not isomorphic π describes a way in which they are alike: vectors in $\mathbb{R}^3$ add as do the associated vectors in $\mathbb{R}^2$ — vectors add as their shadows add.

2.8 Example A homomorphism can express an analogy between spaces that is more subtle than the prior one. For the map from Example 2.6

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{h} x + y$$

fix two numbers in the range $w_1, w_2 \in \mathbb{R}$. A $\vec{v}_1$ that maps to w_1 has components that add to w_1, so the inverse image $h^{-1}(w_1)$ is the set of vectors with endpoint on the diagonal line $x + y = w_1$. Think of these as "w_1 vectors." Similarly we have "w_2 vectors" and "$w_1 + w_2$ vectors." The addition preservation property says this.



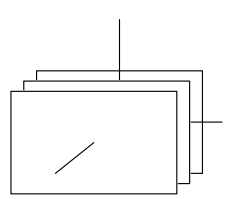
a "w_1 vector" plus a "w_2 vector" equals a "$w_1 + w_2$ vector"

Restated, if we add a w_1 vector to a w_2 vector then h maps the result to a $w_1 + w_2$ vector. Briefly, the sum of the images is the image of the sum. Even more briefly, $h(\vec{v}_1) + h(\vec{v}_2) = h(\vec{v}_1 + \vec{v}_2)$.

2.9 Example The inverse images can be structures other than lines. For the linear map $h\colon \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ x \end{pmatrix}$$

the inverse image sets are planes $x = 0$, $x = 1$, etc., perpendicular to the x-axis.



We won't describe how every homomorphism that we will use is an analogy because the formal sense that we make of "alike in that ..." is 'a homomorphism exists such that ...'. Nonetheless, the idea that a homomorphism between two spaces expresses how the domain's vectors fall into classes that act like the range's vectors is a good way to view homomorphisms.

Another reason that we won't treat all of the homomorphisms that we see as above is that many vector spaces are hard to draw, e.g., a space of polynomials. But there is nothing wrong with leveraging spaces that we can draw: from the three examples 2.7, 2.8, and 2.9 we draw two insights.

The first insight is that in all three examples the inverse image of the range's zero vector is a line or plane through the origin. It is therefore a subspace of the domain.

This is a key concept, which leads to the notion of Kernel.

2.10 Lemma For any homomorphism the inverse image of a subspace of the range is a subspace of the domain. In particular, the inverse image of the trivial subspace of the range is a subspace of the domain.

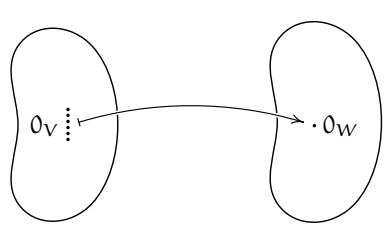
(The examples above consider inverse images of single vectors but this result is about inverse images of sets $h^{-1}(S) = \{\vec{v} \in V \mid h(\vec{v}) \in S\}$. We use the same term for both by taking the inverse image of a single element $h^{-1}(\vec{w})$ to be the inverse image of the one-element set $h^{-1}(\{\vec{w}\})$.)

Proof Let $h\colon V \to W$ be a homomorphism and let S be a subspace of the range space of h. Consider the inverse image of S. It is nonempty because it contains $\vec{0}_V$, since $h(\vec{0}_V) = \vec{0}_W$ and $\vec{0}_W$ is an element of S as S is a subspace. To finish we show that $h^{-1}(S)$ is closed under linear combinations. Let $\vec{v}_1$ and $\vec{v}_2$ be two of its elements, so that $h(\vec{v}_1)$ and $h(\vec{v}_2)$ are elements of S. Then $c_1\vec{v}_1 + c_2\vec{v}_2$ is an element of the inverse image $h^{-1}(S)$ because $h(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)$ is a member of S. QED

2.11 Definition The *null space* or *kernel* of a linear map $h\colon V \to W$ is the inverse image of $\vec{0}_W$.

$$\mathscr{N}(h) = h^{-1}(\vec{0}_W) = \{\vec{v} \in V \mid h(\vec{v}) = \vec{0}_W\}$$

The dimension of the null space is the map's *nullity*.



2.12 Example The map from Example 2.3 has this null space $\mathscr{N}(d/dx) = \{a_0 + 0x + 0x^2 + 0x^3 \mid a_0 \in \mathbb{R}\}$ so its nullity is 1.

2.13 Example The map from Example 2.4 has this null space, and nullity 2.

$$\mathscr{N}(h) = \{\begin{pmatrix} a & b \\ 0 & -(a+b)/2 \end{pmatrix} \mid a, b \in \mathbb{R}\}$$

Now for the second insight from the above examples. In Example 2.7 each of the vertical lines squashes down to a single point — in passing from the domain to the range, π takes all of these one-dimensional vertical lines and maps them to a point, leaving the range smaller than the domain by one dimension. Similarly, in Example 2.8 the two-dimensional domain compresses to a one-dimensional range by breaking the domain into the diagonal lines and maps each of those to a single member of the range. Finally, in Example 2.9 the domain breaks into planes which get squashed to a point and so the map starts with a three-dimensional domain but ends two smaller, with a one-dimensional range. (The codomain is two-dimensional but the range is one-dimensional and the dimension of the range is what matters.)

And this is the COOLEST result in elementary linear algebra!

2.14 Theorem A linear map's rank plus its nullity equals the dimension of its domain.

Proof Let $h\colon V \to W$ be linear and let $B_N = \langle \vec{\beta}_1, \ldots, \vec{\beta}_k \rangle$ be a basis for the null space. Expand that to a basis $B_V = \langle \vec{\beta}_1, \ldots, \vec{\beta}_k, \vec{\beta}_{k+1}, \ldots, \vec{\beta}_n \rangle$ for the entire domain, using Corollary Two.III.2.13. We shall show that $B_R = \langle h(\vec{\beta}_{k+1}), \ldots, h(\vec{\beta}_n) \rangle$ is a basis for the range space. Then counting the size of the bases gives the result.

To see that B_R is linearly independent, consider $\vec{0}_W = c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$. We have $\vec{0}_W = h(c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n)$ and so $c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$ is in the null space of h. As B_N is a basis for the null space there are scalars $c_1, \ldots, c_k$ satisfying this relationship.

$$c_1\vec{\beta}_1 + \cdots + c_k\vec{\beta}_k = c_{k+1}\vec{\beta}_{k+1} + \cdots + c_n\vec{\beta}_n$$

But this is an equation among members of B_V, which is a basis for V, so each c_i equals 0. Therefore B_R is linearly independent.

To show that B_R spans the range space consider a member of the range space $h(\vec{v})$. Express $\vec{v}$ as a linear combination $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ of members of B_V. This gives $h(\vec{v}) = h(c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) = c_1 h(\vec{\beta}_1) + \cdots + c_k h(\vec{\beta}_k) + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$ and since $\vec{\beta}_1, \ldots, \vec{\beta}_k$ are in the null space, we have that $h(\vec{v}) = \vec{0} + \cdots + \vec{0} + c_{k+1}h(\vec{\beta}_{k+1}) + \cdots + c_n h(\vec{\beta}_n)$. Thus, $h(\vec{v})$ is a linear combination of members of B_R, and so B_R spans the range space. QED

2.15 Example Where $h\colon \mathbb{R}^3 \to \mathbb{R}^4$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{h} \begin{pmatrix} x \\ 0 \\ y \\ 0 \end{pmatrix}$$

the range space and null space are

$$\mathscr{R}(h) = \{ \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix} \mid a, b \in \mathbb{R} \} \quad \text{and} \quad \mathscr{N}(h) = \{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \mid z \in \mathbb{R} \}$$

and so the rank of h is 2 while the nullity is 1.

2.16 Example If $t\colon \mathbb{R} \to \mathbb{R}$ is the linear transformation $x \mapsto -4x$, then the range is $\mathscr{R}(t) = \mathbb{R}$. The rank is 1 and the nullity is 0.

2.17 Corollary The rank of a linear map is less than or equal to the dimension of the domain. Equality holds if and only if the nullity of the map is 0.

We know that an isomorphism exists between two spaces if and only if the dimension of the range equals the dimension of the domain. We have now seen that for a homomorphism to exist a necessary condition is that the dimension of the range must be less than or equal to the dimension of the domain. For instance, there is no homomorphism from $\mathbb{R}^2$ onto $\mathbb{R}^3$. There are many homomorphisms from $\mathbb{R}^2$ into $\mathbb{R}^3$, but none onto.

The range space of a linear map can be of dimension strictly less than the dimension of the domain and so linearly independent sets in the domain may map to linearly dependent sets in the range. (Example 2.3's derivative transformation on $\mathcal{P}_3$ has a domain of dimension 4 but a range of dimension 3 and the derivative sends $\{1, x, x^2, x^3\}$ to $\{0, 1, 2x, 3x^2\}$). That is, under a homomorphism independence may be lost. In contrast, dependence stays.

2.18 Lemma Under a linear map, the image of a linearly dependent set is linearly dependent.

Proof Suppose that $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_V$ with some c_i nonzero. Apply h to both sides: $h(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n)$ and $h(\vec{0}_V) = \vec{0}_W$. Thus we have $c_1 h(\vec{v}_1) + \cdots + c_n h(\vec{v}_n) = \vec{0}_W$ with some c_i nonzero. QED

When is independence not lost? The obvious sufficient condition is when the homomorphism is an isomorphism. This condition is also necessary; see

Exercise 37. We will finish this subsection comparing homomorphisms with isomorphisms by observing that a one-to-one homomorphism is an isomorphism from its domain onto its range.

2.19 Example This one-to-one homomorphism $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{\iota} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

gives a correspondence between $\mathbb{R}^2$ and the xy-plane subset of $\mathbb{R}^3$.

2.20 Theorem Where V is an n-dimensional vector space, these are equivalent statements about a linear map $h\colon V \to W$.

(1) h is one-to-one
(2) h has an inverse from its range to its domain that is a linear map
(3) $\mathscr{N}(h) = \{\vec{0}\,\}$, that is, $\text{nullity}(h) = 0$
(4) $\text{rank}(h) = n$
(5) if $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for V then $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n) \rangle$ is a basis for $\mathscr{R}(h)$

Proof We will first show that (1) $\iff$ (2). We will then show that (1) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (2).

For (1) $\implies$ (2), suppose that the linear map h is one-to-one, and therefore has an inverse $h^{-1}\colon \mathscr{R}(h) \to V$. The domain of that inverse is the range of h and thus a linear combination of two members of it has the form $c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)$. On that combination, the inverse h^{-1} gives this.

$$\begin{aligned} h^{-1}(c_1 h(\vec{v}_1) + c_2 h(\vec{v}_2)) &= h^{-1}(h(c_1\vec{v}_1 + c_2\vec{v}_2)) \\ &= h^{-1} \circ h\ (c_1\vec{v}_1 + c_2\vec{v}_2) \\ &= c_1\vec{v}_1 + c_2\vec{v}_2 \\ &= c_1 \cdot h^{-1}(h(\vec{v}_1)) + c_2 \cdot h^{-1}(h(\vec{v}_2)) \end{aligned}$$

Thus if a linear map has an inverse then the inverse must be linear. But this also gives the (2) $\implies$ (1) implication, because the inverse itself must be one-to-one.

Of the remaining implications, (1) $\implies$ (3) holds because any homomorphism maps $\vec{0}_V$ to $\vec{0}_W$, but a one-to-one map sends at most one member of V to $\vec{0}_W$.

Next, (3) $\implies$ (4) is true since rank plus nullity equals the dimension of the domain.

For (4) $\implies$ (5), to show that $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n) \rangle$ is a basis for the range space we need only show that it is a spanning set, because by assumption the range has dimension n. Consider $h(\vec{v}) \in \mathscr{R}(h)$. Expressing $\vec{v}$ as a linear

combination of basis elements produces $h(\vec{v}) = h(c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n)$, which gives that $h(\vec{v}) = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n)$, as desired.

Finally, for the (5) $\implies$ (2) implication, assume that $\langle\vec{\beta}_1, \ldots, \vec{\beta}_n\rangle$ is a basis for V so that $\langle h(\vec{\beta}_1), \ldots, h(\vec{\beta}_n)\rangle$ is a basis for $\mathscr{R}(h)$. Then every $\vec{w} \in \mathscr{R}(h)$ has the unique representation $\vec{w} = c_1 h(\vec{\beta}_1) + \cdots + c_n h(\vec{\beta}_n)$. Define a map from $\mathscr{R}(h)$ to V by

$$\vec{w} \mapsto c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$$

(uniqueness of the representation makes this well-defined). Checking that it is linear and that it is the inverse of h are easy. QED

We have seen that a linear map expresses how the structure of the domain is like that of the range. We can think of such a map as organizing the domain space into inverse images of points in the range. In the special case that the map is one-to-one, each inverse image is a single point and the map is an isomorphism between the domain and the range.

Exercises

✓ 2.21 Let $h\colon \mathcal{P}_3 \to \mathcal{P}_4$ be given by $p(x) \mapsto x \cdot p(x)$. Which of these are in the null space? Which are in the range space?

(a) x^3 **(b)** 0 **(c)** 7 **(d)** $12x - 0.5x^3$ **(e)** $1 + 3x^2 - x^3$

2.22 Find the range space and the rank of each homomorphism.

(a) $h\colon \mathcal{P}_3 \to \mathbb{R}^2$ given by

$$ax^2 + bx + c \mapsto \begin{pmatrix} a+b \\ a+c \end{pmatrix}$$

(b) $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ x-y \\ 3y \end{pmatrix}$$

✓ 2.23 Find the range space and rank of each map.

(a) $h\colon \mathbb{R}^2 \to \mathcal{P}_3$ given by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto a + ax + ax^2$$

(b) $h\colon \mathcal{M}_{2\times 2} \to \mathbb{R}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$$

(c) $h\colon \mathcal{M}_{2\times 2} \to \mathcal{P}_2$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + b + c + dx^2$$

(d) the zero map $Z\colon \mathbb{R}^3 \to \mathbb{R}^4$

✓ 2.24 For each linear map in the prior exercise, find the null space and nullity.

✓ 2.25 Find the nullity of each map below.

(a) $h\colon \mathbb{R}^5 \to \mathbb{R}^8$ of rank five **(b)** $h\colon \mathcal{P}_3 \to \mathcal{P}_3$ of rank one
(c) $h\colon \mathbb{R}^6 \to \mathbb{R}^3$, an onto map **(d)** $h\colon \mathcal{M}_{3\times 3} \to \mathcal{M}_{3\times 3}$, onto

✓ 2.26 What is the null space of the differentiation transformation $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$? What is the null space of the second derivative, as a transformation of $\mathcal{P}_n$? The k-th derivative?

2.27 For the map $h\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x+z \end{pmatrix}$$

find the range space, rank, null space, and nullity.

2.28 Example 2.7 restates the first condition in the definition of homomorphism as 'the shadow of a sum is the sum of the shadows'. Restate the second condition in the same style.

2.29 For the homomorphism $h\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by $h(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + (a_0 + a_1)x + (a_2 + a_3)x^3$ find these.

(a) $\mathcal{N}(h)$ **(b)** $h^{-1}(2 - x^3)$ **(c)** $h^{-1}(1 + x^2)$

✓ 2.30 For the map $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = 2x + y$$

sketch these inverse image sets: $f^{-1}(-3)$, $f^{-1}(0)$, and $f^{-1}(1)$.

✓ 2.31 Each of these transformations of $\mathcal{P}_3$ is one-to-one. For each, find the inverse.

(a) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_1x + 2a_2x^2 + 3a_3x^3$
(b) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_0 + a_2x + a_1x^2 + a_3x^3$
(c) $a_0 + a_1x + a_2x^2 + a_3x^3 \mapsto a_1 + a_2x + a_3x^2 + a_0x^3$
(d) $a_0+a_1x+a_2x^2+a_3x^3 \mapsto a_0+(a_0+a_1)x+(a_0+a_1+a_2)x^2+(a_0+a_1+a_2+a_3)x^3$

2.32 Describe the null space and range space of a transformation given by $\vec{v} \mapsto 2\vec{v}$.

2.33 List all pairs $(\mathrm{rank}(h), \mathrm{nullity}(h))$ that are possible for linear maps from $\mathbb{R}^5$ to $\mathbb{R}^3$.

2.34 Does the differentiation map $d/dx\colon \mathcal{P}_n \to \mathcal{P}_n$ have an inverse?

✓ 2.35 Find the nullity of this map $h\colon \mathcal{P}_n \to \mathbb{R}$.

$$a_0 + a_1x + \cdots + a_nx^n \mapsto \int_{x=0}^{x=1} a_0 + a_1x + \cdots + a_nx^n \, dx$$

2.36 **(a)** Prove that a homomorphism is onto if and only if its rank equals the dimension of its codomain.

(b) Conclude that a homomorphism between vector spaces with the same dimension is one-to-one if and only if it is onto.

2.37 Show that a linear map is one-to-one if and only if it preserves linear independence.

2.38 Corollary 2.17 says that for there to be an onto homomorphism from a vector space V to a vector space W, it is necessary that the dimension of W be less than or equal to the dimension of V. Prove that this condition is also sufficient; use Theorem 1.9 to show that if the dimension of W is less than or equal to the dimension of V, then there is a homomorphism from V to W that is onto.

✓ 2.39 Recall that the null space is a subset of the domain and the range space is a subset of the codomain. Are they necessarily distinct? Is there a homomorphism that has a nontrivial intersection of its null space and its range space?

2.40 Prove that the image of a span equals the span of the images. That is, where $h\colon V \to W$ is linear, prove that if S is a subset of V then $h([S])$ equals $[h(S)]$. This generalizes Lemma 2.1 since it shows that if U is any subspace of V then its image $\{h(\vec{u}) \mid \vec{u} \in U\}$ is a subspace of W, because the span of the set U is U.

2.41 **(a)** Prove that for any linear map $h\colon V \to W$ and any $\vec{w} \in W$, the set $h^{-1}(\vec{w})$ has the form

$$h^{-1}(\vec{w}) = \{\vec{v} + \vec{n} \mid \vec{v}, \vec{n} \in V \text{ and } \vec{n} \in \mathscr{N}(h) \text{ and } h(\vec{v}) = \vec{w}\}$$

(if h is not onto and $\vec{w}$ is not in the range of h then this set is empty since its third condition cannot be satisfied). Such a set is a *coset* of $\mathscr{N}(h)$ and we denote it as $\vec{v} + \mathscr{N}(h)$.

(b) Consider the map $t\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{t} \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

for some scalars a, b, c, and d. Prove that t is linear.

(c) Conclude from the prior two items that for any linear system of the form

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

we can write the solution set (the vectors are members of $\mathbb{R}^2$)

$$\{\vec{p} + \vec{h} \mid \vec{h} \text{ satisfies the associated homogeneous system}\}$$

where $\vec{p}$ is a particular solution of that linear system (if there is no particular solution then the above set is empty).

(d) Show that this map $h\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n \\ \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n \end{pmatrix}$$

for any scalars $a_{1,1}, \ldots, a_{m,n}$. Extend the conclusion made in the prior item.

(e) Show that the k-th derivative map is a linear transformation of $\mathcal{P}_n$ for each k. Prove that this map is a linear transformation of the space

$$f \mapsto \frac{d^k}{dx^k}f + c_{k-1}\frac{d^{k-1}}{dx^{k-1}}f + \cdots + c_1\frac{d}{dx}f + c_0 f$$

for any scalars $c_k, \ldots, c_0$. Draw a conclusion as above.

2.42 Prove that for any transformation $t\colon V \to V$ that is rank one, the map given by composing the operator with itself $t \circ t\colon V \to V$ satisfies $t \circ t = r \cdot t$ for some real number r.

2.43 Let $h\colon V \to \mathbb{R}$ be a homomorphism, but not the zero homomorphism. Prove that if $\langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for the null space and if $\vec{v} \in V$ is not in the null space then $\langle \vec{v}, \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for the entire domain V.

2.44 Show that for any space V of dimension n, the *dual space*

$$\mathcal{L}(V, \mathbb{R}) = \{h\colon V \to \mathbb{R} \mid h \text{ is linear}\}$$

is isomorphic to $\mathbb{R}^n$. It is often denoted V^*. Conclude that $V^* \cong V$.

2.45 Show that any linear map is the sum of maps of rank one.

2.46 Is 'is homomorphic to' an equivalence relation? (*Hint:* the difficulty is to decide on an appropriate meaning for the quoted phrase.)

2.47 Show that the range spaces and null spaces of powers of linear maps $t\colon V \to V$ form descending

$$V \supseteq \mathscr{R}(t) \supseteq \mathscr{R}(t^2) \supseteq \ldots$$

and ascending

$$\{\vec{0}\} \subseteq \mathscr{N}(t) \subseteq \mathscr{N}(t^2) \subseteq \ldots$$

chains. Also show that if k is such that $\mathscr{R}(t^k) = \mathscr{R}(t^{k+1})$ then all following range spaces are equal: $\mathscr{R}(t^k) = \mathscr{R}(t^{k+1}) = \mathscr{R}(t^{k+2}) \ldots$. Similarly, if $\mathscr{N}(t^k) = \mathscr{N}(t^{k+1})$ then $\mathscr{N}(t^k) = \mathscr{N}(t^{k+1}) = \mathscr{N}(t^{k+2}) = \ldots$.