Worksheet 4: bases

The definition and what it means

Intuitively, you should think of a **basis** for a vector space V as a set of vectors that provide a reference frame so that V obtains a system of coordinates.

For example, if V is the plane with an origin, any two vectors which are non proportional to each other can be thought of as unit vectors on axes that point in their direction. Those axes are a reference frame and you can use them to give coordinates on the plane.

Problem 1 *Draw a picture, with a caption, to illustrate the previous example.*

Here is a more formal definition. Given a vector space V, the vectors $v_1, \ldots, v_n$ are a **basis** for V if any vector $v \in V$ can be expressed as a linear combination of $v_1, \ldots, v_n$ in a unique way. The coefficents of such a linear combination are the **coordinates** of v induced by the basis $v_1, \ldots, v_n$.

If

$$v = a_1 v_1 + a_2 v_2 + \ldots a_n v_n$$

then the coordinates of v are

$$[a_1, a_2, \ldots, a_n]$$

Problem 2 *Let V be the plane with an origin.*

1. *Can a single vector v be a basis for V?*
2. *Can two vectors v_1, v_2 be a basis for V?*
3. *Can three vectors v_1, v_2, v_3 be a basis for V?*

Problem 3 *Let V be the plane with an origin.*

1. *Draw a set of vectors such that every vector of V can be obtained as a linear combination of these vectors, but not in a unique way.*
2. *Draw a set of vectors such that every vector that can be obtained as a linear combination is obtained in a unique way, but not every vector in V is obtained as a linear combination.*

Problem 4 *Answer these questions, but make sure you also really understand why!*

1. *Can the zero vector ever be part of a basis?*
2. *Can the vectors v, λv simultaneously be part of a basis?*
3. *Can the vectors $v_1, v_2, v_1 + v_2$ simultaneously be part of a basis?*

Problem 5 *Show that $[1, 0, 0], [0, 1, 0]$ and $[0, 0, 1]$ are a basis for $\mathbb{R}^3$.*

(SILLY QUESTION) *What are the coordinates of the vector $[3, 4, 5]$?*

Now I am telling you that the vectors $[1, 0, 0], [0, 1, 0]$ and $[1, 1, 1]$ also are a basis for $\mathbb{R}^3$, so they define a NEW system of coordinates on $\mathbb{R}^3$.

(NOT SILLY QUESTION) *What are the coordinates induced by this new basis for the vector $[3, 4, 5]$?*

Basis and linear functions

There are two important pieces of information that make bases extremely important when dealing with linear functions.

1. A basis spans V. Therefore if you know the values of a linear function L for the vectors of a basis, you know the function completely!

Problem 6 *Suppose V has a basis given by v_1, v_2, v_3, and you have a linear function $L : V \to \mathbb{R}^2$ such that*

$$L(v_1) = [1, 2] \qquad L(v_2) = [4, 1] \qquad L(v_3) = [\pi, \sqrt{2}]$$

What is

$$L(\pi v_1 + \sqrt{2} v_2 - v_3) = ?$$

2. Vectors of V can be written in a unique way as linear combination of the basis vectors. Therefore you can assign at random values in a vector space W for the vectors of a basis, and there always exists a linear function $L : V \to W$ that extends your assignment.

Problem 7 *Let $[1, 0, 0], [0, 1, 0]$ and $[0, 0, 1]$ be a basis for $\mathbb{R}^3$. W is another vector space, of which you know nothing other than the fact it is a vector space. Let w_1, w_2, w_3 be vectors in W. Write down a linear function $L : \mathbb{R}^3 \to W$ such that $L([1, 0, 0]) = w_1, L([0, 1, 0]) = w_2, L([0, 0, 1]) = w_3$:*

$$L([x, y, z]) = ?$$

Problem 8 *Consider the vector space $\mathbb{R}^2$ with the standard basis $[1, 0]$ and $[0, 1]$. Write the matrix representing the linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L([1, 0]) = [2, -1]$ and $L([0, 1]) = [-1, 2]$.*

Now consider the vectors $v_1 = [1, 1]$ and $v_2 = [1, -1]$. This is also a basis (I am telling you, you don't have to check it), so they induce a new system of coordinates on $\mathbb{R}^2$. Write the matrix for the same function L in the new system of coordinates.