Worksheet 1: lines

Lines in the plane

Recall $\mathbb{R}^2$ is the set of (all) ordered pairs (a_1, a_2) of real numbers, with addition and scalar multiplication as defined in class. We want to think of $\mathbb{R}^2$ as an algebraic model for the geometric object **plane together with a choice of coordinate axes**.

The goal of this section is to understand that a line in the plane can be described algebraically in two different ways:

parametrically: this amounts to thinking of a line as the trajectory of a motion. We have an additional variable (which we can think of as "time"), and for every instant t we specify the position of the particle by giving its coordinates.

via equations: in these case the line is described as the subset of points in the plane whose coordinates satisfy certain equations.

Parametrized line

A parameterization for a line in the plane is a function

$$\ell : \mathbb{R} \to \mathbb{R}^2$$

of the form

$$\ell(t) = (x(t), y(t)) = (a_1 + b_1 t, a_2 + b_2 t)$$

(often times we will omit the $\ell(t)$ and just write $(x(t), y(t)) = (a_1 + b_1 t, a_2 + b_2 t)$). The **line** ℓ is the image of the function ℓ.

An alternative and completely equivalent notation that is used is:

$$\begin{cases} x(t) = a_1 + b_1 t \\ y(t) = a_2 + b_2 t. \end{cases}$$

Two vectors in $\mathbb{R}^2$ play a role in the parameterization of a line. The vector (a_1, a_2) is the **initial position vector**: it is the point where your particle is at time $t = 0$. The vector (b_1, b_2) is the **velocity vector**: it tells you in which direction the point is moving and, if we had a notion of how to measure how large vectors are, it would tell us also how fast the particle is moving. However, remember we are **intentionally ignoring** the fact that vectors can be "measured" in $\mathbb{R}^2$.

Problem 1 *Consider the line:*

$$\ell(t) = (x(t), y(t)) = (1 + t, 2t)$$

1. What is the initial position vector for $\ell(t)$?
2. What is the velocity vector for $\ell(t)$?
3. Does the point $(6, 10)$ belong to the line ℓ?
4. Does the point $(8, 8)$ belong to the line ℓ?
5. If I tell you that the point $(7, b)$ belongs to the line ℓ, what is b?
6. If
$$\tilde{\ell}(t) = (x(t), y(t)) = (t, t),$$
what are the points of intersections in $\ell \cap \tilde{\ell}$?
7. If
$$\tilde{\ell}(t) = (x(t), y(t)) = (t, 2t),$$
what are the points of intersections in $\ell \cap \tilde{\ell}$?
8. If
$$\tilde{\ell}(t) = (x(t), y(t)) = (6 - 2t, 10 - 4t),$$
what are the points of intersections in $\ell \cap \tilde{\ell}$?

Problem 2
1. *Write down a parameterization for the line ℓ that passes through the points $(1, 2)$ and $(5, 6)$.*
2. *Write down a parameterization for the line ℓ with y-intercept 5 and slope 3.*

Discussion 1 *Let us now look back at Problems 1 and 2 and see what observations we can generalize:*

1. *Does a line ℓ have a unique parameterization? If not, how are two different parameterizations related one to another?*
2. *How can you detect when two lines are parallel?*
3. *If you are given two parameterized lines, how do you look for their point of intersection? If you think of the two parameterized lines as the trajectories of motions of particles, does the fact that two lines intersect imply that there is a collision between the two particles?*
4. *How do you find a possible velocity vector for a parameterization of a line ℓ if you are given the information that ℓ contains two points $P = (p_1, p_2), Q = (q_1, q_2)$?*

Equations for line in the plane

A line in the plane is a subset of points of the plane whose coordinates satisfy a **polynomial equation** in x and y of degree 1. This is an equation of the form

$$p(x, y) = q(x, y),$$

where p and q are polynomials of degree one in x and y, i.e. expressions of the form $ax + by + c$.

We often give the equation of a line in some "nice" form, that allows us to read some geometric information about the line immediately.

The equation for a line is in **y-intercept/slope form** if it has the shape

$$y = mx + b,$$

where m is called the **slope** of the line and b the **y-intercept**.

Problem 3 *Consider the line ℓ defined by the equation:*

$$y = 5x + 8$$

1. *Show that it contains the point $(0, 8)$.*
2. *Does it contain the point $(2, 12)$?*
3. *Write a parameterization of the line ℓ.*

Another typical way to present a line is by just brining all the terms to one side, to obtain an equation of the form $ax + by + c = 0$.

Problem 4 *Consider the line ℓ given by the equation:*

$$2x + 5y + 30 = 0$$

1. *What is the point of intersection of ℓ with the x-axis?*
2. *What is the point of intersection of ℓ with the y-axis?*
3. *What is the point of intersection of ℓ with the line $2x + 5y = 0$?*
4. *What is the point of intersection of ℓ with the line $4x + 10y + 60 = 0$?*

Discussion 2

1. *How do you detect if two lines given by their equations are the same line? How do you detect when they are parallel?*
2. *In general, when you intersect two lines, you obtain a point. This point is identified as the unique point that satisfies simultaneously the equations of the two lines. Ponder that requiring multiple equations to be satisfied simultaneously means intersecting the solution sets of the equations.*

Lines in 3-space

Recall $\mathbb{R}^3$ is the set of (all) ordered pairs (a_1, a_2, a_3) of real numbers. We want to think of $\mathbb{R}^3$ as an algebraic model for the geometric object $3d$**-space together with a choice of coordinate axes**. As we did in the previous section in the two-dimensional case, we want to understand how a line in $3d$-space can be described algebraically in two different ways, parameterized and by equations.

Parametrized line

A parameterization for a line in the plane is a function

$$\ell : \mathbb{R} \to \mathbb{R}^3$$

of the form

$$\ell(t) = (x(t), y(t), z(t)) = (a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t)$$

The **line** ℓ is the image of the function ℓ.

An alternative and completely equivalent notation that is used is:

$$\begin{cases} x(t) = a_1 + b_1 t \\ y(t) = a_2 + b_2 t \\ z(t) = a_3 + b_3 t. \end{cases}$$

As before, two vectors, now in $\mathbb{R}^3$, play a role in the parameterization of a line. The vector (a_1, a_2, a_3) is the **initial position vector**: it is the point where your particle is at time $t = 0$. The vector (b_1, b_2, b_3) is the **velocity vector**.

Problem 5 *Consider the line:*

$$\ell(t) = (x(t), y(t), z(t)) = (1 + t, 2t, 3 + 3t)$$

1. If I tell you that the point $(7, b, c)$ belongs to the line ℓ, what are b, c?
2. If I tell you that the point $(1, b, 8)$ belongs to the line ℓ, am I lying?
3. If
$$\tilde{\ell}(t) = (x(t), y(t), z(t)) = (t, t, t),$$
what are the points of intersections in $\ell \cap \tilde{\ell}$?

Discussion 3 *1. How likely is for two lines in three dimensional space to intersect?*

2. If you are given two parameterized lines, how do you look for their point of intersection? What algebraic phenomenon tells you whether the lines intersect or do not intersect?

Equations for lines in $3d$-space

A line in $3d$-space may be described as a subset of points whose coordinates satisfy two **polynomial equations** in x and y of degree 1. In mathy notation:

$$\begin{cases} p_1(x,y,z) = q_1(x,y,z) \\ p_2(x,y,z) = q_2(x,y,z), \end{cases}$$

where p and q are polynomials of degree one in x, y and z, i.e. expressions of the form $ax + by + cz + d$.

For example, if we give the very simple equations $x = 0$ and $y = 0$, the points that simultaneously satisfy such equations give the z-axis.

Problem 6 *Consider the line ℓ:*

$$\begin{cases} x + y + z = 0 \\ 2x + y + 4 = 0 \end{cases}$$

1. *Does the point $(-2, 0, 2)$ belong to ℓ?*
2. *Find another point on ℓ.*
3. *If the point $(0, a, b)$ belongs to ℓ what are a and b?*
4. *If $\tilde{\ell}$ is the line given by*

$$\begin{cases} x + y + z = 0 \\ x - z + 4 = 0, \end{cases}$$

is $\tilde{\ell}$ the same line as ℓ or a different one?

Problem 7 *Consider the pair of equations:*

$$\begin{cases} x + y + z = 0 \\ x + y + z - 1 = 0. \end{cases}$$

What is the set of points whose coordinates satisfy both equations simultaneously?

Now consider this pair of equations:

$$\begin{cases} x + y + z = 0 \\ 2x + 2y + 2z = 0. \end{cases}$$

What is the set of points whose coordinates satisfy both equations simultaneously?

Discussion 4 *It may have become apparent that working with a line from the equation point of view is maybe a little more laborious... it is not all that easy even to find points on a line, and it is not immediate to see when two pairs of equations give the same line, parallel lines, etc... we will be developing techniques to understand how these computations are made. For now, here are a couple philosophical questions for discussion:*

1. *When do two polynomial equations of degree one fail to determine a line?*

2. *How can one determine when two distinct pairs of equations define the same line?*

3. *How can one determine when two lines intersect?*

Lines through the origin

Consider the lines:

$$\ell_1 : \{x + 5y = 0\} \subseteq \mathbb{R}^2$$

$$\ell_2 : \begin{cases} x(t) = 5t \\ y(t) = 2t \\ z(t) = 4t \end{cases} \subseteq \mathbb{R}^3$$

Problem 8 1. *Check that both lines contain the origin.*

2. *Pick a point P at random on ℓ_1 and regard it as a vector in $\mathbb{R}^2$. What can you tell me about the vector $5P$?*

3. *Pick two points P, Q at random on ℓ_2 and regard them as a vectors in $\mathbb{R}^3$. What can you tell me about the vector $5P + 2Q$?*

Discussion 5 *It is a meaningful and important question to as what subsets of vector spaces are vector spaces in their own sake. What have we learned about this question in the Exercise above? Why? Can a line not through the origin be a vector space (this is actually a tricky question...)?*