1. Suppose conservative vector field $G$ has potential function $g(x, y, z) = x^2 + yz$. Compute the work done when moving through this vector field along any closed, simple curve from $(0, 1, 1)$ to $(2, 0, 1)$.

Solution. We use the fundamental theorem of Calculus for line integrals and have, for $C$ a curve from $(0, 1, 1)$ to $(2, 0, 1)$, that
\[
\begin{align*}
\int_C G \cdot dr &= \\
\int_C \nabla g \cdot dr &= g(2, 0, 1) - g(0, 1, 1) = \\
((2)^2 + (0)(1)) - ((0)^2 + (1)(1)) &= 4 - 1 = 3.
\end{align*}
\]

2. Find the potential function $f(x, y, z)$ for vector field
\[
F = \langle \sin(y), x \cos(y) + z \cos(y), \sin(y) + 2z \rangle
\]
such that $f(9, 0, 1) = 2$. You may assume that $F$ is conservative.

Solution. We have that
\[
f_x = \sin(y)
\Rightarrow f(x, y, z) = x \sin(y) + g(y, z)
\]
for some function $g(y, z)$. Differentiating the above expression for $f$ with respect to $y$ gives
\[
f_y = x \cos(y) + \partial g/\partial y = x \cos(y) + z \cos(y)
\Rightarrow \partial g/\partial y = z \cos(y)
\Rightarrow g(y, z) = z \sin(y) + h(z)
\]
for some function $h(z)$, which gives

$$f(x, y, z) = x \sin(y) + z \sin(y) + h(z).$$

Now differentiating the above expression of $f$ with respect to $z$ gives

$$f_z = \sin(y) + h'(z) = \sin(y) + 2z$$

$$\Rightarrow h(z) = z^2 + C$$

for some constant $C \in \mathbb{R}$. We then have that

$$f(x, y, z) = x \sin(y) + z \sin(y) + z^2 + C,$$

and finally we use the extra condition that $f(9, 0, 1) = 2$ to have

$$f(9, 0, 1) = 9 \sin(0) + 1 \sin(0) + 1^2 + C = 2$$

$$\Rightarrow C = 1.$$

We then have that

$$f(x, y, z) = x \sin(y) + z \sin(y) + z^2 + 1.$$

We can check this answer by computing the gradient of $f$ to be

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle \sin(y), x \cos(y) + z \cos(y), \sin(y) + 2z \rangle = F.$$ 

\[ \square \]

3. Use the component test ($M_y = N_x$, etc.) to show that the vector field

$$H = \langle ze^{xz} - \sin(x + 2y), \frac{1}{y} - 2 \sin(x + 2y) + 1, xe^{xz} + \frac{1}{z} \rangle = \langle M, N, P \rangle$$

is conservative.

Solution. We note that

$$M_y = -2 \cos(x + 2y) = N_x$$
$$M_z = e^{xz} + xe^{xz} = P_x$$
$$N_z = 0 = P_y,$$

which proves that $H$ is conservative by a theorem from class. \[ \square \]