1. Using cylindrical coordinates, set up the integral to find the volume of the region enclosed by the vertical cylinder \( x^2 + y^2 = 4 \) and the planes \( z = 0 \) and \( y + z = 4 \). Do NOT evaluate the integral; just set it up.

**Solution.** To solve this integral, we first find the double integral over the area of the projection to the \( xy \)-plane in terms of polar coordinates. Note that projecting to the \( xy \)-plane gives the area bounded by the circle of radius 2 centered at the origin in \( \mathbb{R}^2 \). We can integrate over this area using the bounds \( 0 \leq r \leq 2 \) and \( 0 \leq \theta \leq 2\pi \). Hence we just need to put the bounds of \( z \) in terms of \( r \) and \( \theta \). To do this, we substitute in the equation \( y + z = 4 \) with \( y = r \sin \theta \) and have \( 0 \leq z \leq 4 - r \sin \theta \). We then have that the volume is

\[
\int \int_R \int_0^{2\pi} \int_0^{2 - r \sin \theta} rdzdrd\theta.
\]

2. Using spherical coordinates, set up the integral to find the volume of the region enclosed by the vertical cylinder \( x^2 + y^2 = 4 \) and the planes \( z = 0 \) and \( z = 2 \). Do NOT evaluate the integral; just set it up.

**Solution.** Note that we can use the \( \theta \) bounds of \( 0 \leq \theta \leq 2\pi \), since we want to go all the way around the cylinder in the clockwise direction in the \( xy \)-plane. We will need to use 2 triple integrals here, since the upper bound of \( \rho \) will change depending on whether we are in the region of the cylinder with \( \phi \leq \pi/4 \) (i.e. the portion of the cylinder above the cone \( \phi = \pi/4 \)) or in the region \( \phi > \pi/4 \). The \( \pi/4 \) is determined by noting that \( \rho = \sqrt{x^2 + y^2} = 2\sqrt{2} \) for any point in the boundary of the top circle of the cylinder. We then have that \( \cos \phi = 2/2\sqrt{2} = \sqrt{2}/2 \), giving \( \phi = \pi/4 \) as the cone we want. For \( \phi \leq \pi/4 \), we are integrating over the volume of the cylinder lying above the cone \( \phi = \pi/4 \). For a given value of \( \phi \) and \( \theta \) in this region, we make a slice which is a right triangle making a right angle with the plane \( z = 2 \), and we have that \( \cos \phi = 2/\rho \), so \( \rho = 2/\cos \phi = 2 \sec \phi \).
Then below the cone $\phi = \pi/4$, we note that a given value of $\theta$ and $\phi$ makes a slice which is a right triangle making a right angle with the plane $z = 0$, which gives $\cos(\pi/2 - \phi) = \sin(\phi) = 2/\rho$ so that $\rho = 2/\sin \phi = 2 \csc \phi$.

Finally we can write our integral as

$$
\int_{\phantom{0}}^{\pi/4} \int_{\phantom{0}}^{2 \sec \theta} \int_{\phantom{0}}^{\rho^2 \sin \phi} d\rho d\phi d\theta + \int_{\phantom{\pi/4}}^{\pi/2} \int_{\phantom{\pi/4}}^{2 \csc \theta} \int_{\phantom{0}}^{\rho^2 \sin \phi} d\rho d\phi d\theta.
$$

3. Consider using the substitution

$$
\begin{aligned}
x &= u - v, \\
y &= 2u + v
\end{aligned}
$$

for the integral of $x + y^2 - 2$. What is the integrand in terms of $u$ and $v$?

(Don’t bother with the integral signs, the bounds, or the $dudv$.)

**Solution.** We immediately have

$$
x + y^2 - 2 = (u - v) + (2u + v)^2 - 2,
$$

which we do not simplify for this problem. Then, to compute the Jacobian, we take partial derivatives

$$
\begin{aligned}
\frac{\partial x}{\partial u} &= 1, & \frac{\partial x}{\partial v} &= -1, \\
\frac{\partial y}{\partial u} &= 2, & \frac{\partial y}{\partial v} &= 1
\end{aligned}
$$

and compute

$$
\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = (1)(1) - (-1)(2) = 1 + 2 = 3,
$$

so our integrand is then

$$
((u - v) + (2u + v)^2 - 2)3
$$

4. Using the same substitution as in the previous problem, suppose the $(x, y)$ region over which we wish to integrate includes the boundary line $2x - y = 3$. Convert this line into a $(u, v)$ boundary line.

**Solution.** We convert the line given in terms of $x$ and $y$ by substituting for $x$ and $y$ in terms of $u$ and $v$ and have

$$
2(u - v) - (2u + v) = 2u - 2v - 2u - v = -3v = 3,
$$

so $v = -1$ is the bound in $(u, v)$ coordinates.