HW 7 Solutions MATH 261 F18

October 19, 2018

1. Set up but do NOT evaluate a double integral to compute the integral of $f(x, y) = \cos(xy)$ over the part of the unit disk (the region inside the circle of radius 1 centered at the origin) in the first quadrant (where $x > 0$, $y > 0$).

Proof. The first step in this proof is to draw the unit disc and shade the portion which is in the first quadrant. (The first step of the vast majority of these integration problems–especially in two dimensions–is to draw a picture first.) From there we see that we can integrate $y$ from 0 to 1, which then gives the left curve of the area as the region $x = f(y) = 0$; the right side of the area is then given by the curve $x = g(y) = \sqrt{1 - y^2}$, which we can find using just the equation of the unit circle in $\mathbb{R}^2$. Hence we can set up our integral as

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(xy) \, dx \, dy.$$ 

If you instead chose to work with the order $dy \, dx$, after an analogous reasoning you will have found:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \cos(xy) \, dy \, dx.$$ 

2. Convert the following double integral to an equivalent polar form but do NOT evaluate:

$$\int_0^1 \int_y^{\sqrt{4-y^2}} x^2 + y^2 \, dx \, dy.$$ 

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Proof. We first change the bounds of integration and then address the integrand. For the bounds of integration, we must draw the area in $\mathbb{R}^2$ and notice that it is given by bounding with the curves $y = 0$, $y = 1$, $x = y$, and $x = \sqrt{4 - y^2}$. This area is a piece of the unit disc for $\theta = 0$ to some value $\theta = \theta_0$, but then we are integrating a more rectangular piece from $\theta = \theta_0$ to $\theta = \pi/4$. To find this value $\theta_0$, we notice that it is the $\theta$ value at which the line $y = 1$ intersects the circle $x = \sqrt{4 - y^2}$. We know that, at this point, $r = 2$, and we can use the equation

$$y = r \sin(\theta)$$

to solve for

$$\theta_0 = \sin^{-1}(y/r) = \sin^{-1}(1/2) = \pi/6.$$  

Hence for the first part of our region, we have the bounds $0 \leq \theta \leq \pi/6$ and $0 \leq r \leq 2$. Next we find the bounds for the second part of our region. We have $\pi/6 \leq \theta \leq \pi/4$, and then we notice that the line $y = 1$ is the same as the equation $r \sin \theta = y = 1$, which gives $r = 1/\sin \theta = \csc \theta$ in terms of $\theta$; hence our $r$ bounds for the second part of the region are $0 \leq r \leq \csc \theta$.

Next we deal with the integrand. We convert $x^2 + y^2$ to an expression in terms of $r$ and $\theta$ by using

$$x^2 + y^2 = r^2,$$

and we note that $dA = dx dy = r dr d\theta$, so our answer is finally

$$\int_0^{\pi/6} \int_0^2 r^2 dr d\theta + \int_{\pi/6}^{\pi/4} \int_0^2 r^2 dr d\theta =$$

$$\int_0^{\pi/6} \int_0^2 r^3 dr d\theta + \int_{\pi/6}^{\pi/4} \int_0^2 r^3 dr d\theta.$$

$\square$

3. Set up but do NOT evaluate a triple integral to compute the volume of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,1)$. The top plane of the tetrahedron is given by $2x + y + 2z = 2$. USE the order $dzdydx$.

Proof. Note that the integration bounds on $x$ will be just $0 \leq x \leq 1$. To determine the bounds of $y$ in terms of $x$, we project the shape to the $xy$-plane, which in this case just corresponds to taking the triangle in $\mathbb{R}^2$ with vertices $(0,0), (1,0), (0,2),$ and $(0,0)$. (We just omit the third coordinate from each of the vertices in three dimensions in the problem.) If we draw
the picture, we can see that the bounds of $y$, the bottom and top curves of the area of the triangle in $\mathbb{R}^2$, are $0 \leq y \leq 2 - 2x$.

To determine the bounds of $z$ in terms of $x$ and $y$, we think about integrating from the bottom surface of the tetrahedron to the top surface of the tetrahedron in $\mathbb{R}^3$. The bottom surface, with the vertices given in the problem, is simply given by $z = 0$, the $xy$-plane, and the top surface is given by solving the equation of the plane given for $z$:

$$z = 1 - x - y/2.$$  

Thus we compute the volume $\int \int \int_{R} dV$ of the tetrahedron as the integral

$$\int_{0}^{1} \int_{y=2-2x}^{1} \int_{z=1-x-y/2}^{0} dz dy dx.$$

4. Consider the tetrahedron $T$ with vertices $(1, 0, 0), (1, -1, 1), (1, 1, 1)$, and $(0, 0, 1)$. How many regions must $T$ be split into in order to integrate some function over $T$ with the following variable orders (each worth 1 point)? (Each answer is just 1 number!)

(a) $dxdydz$
(b) $dxdzdy$
(c) $dydzdx$

Solution. (a) 2
(b) 2
(c) 1