HW 5 Solutions Math 261 F18

August 21, 2018

1. Find the derivative of \( h(x, y, z) = x + 2y^2 + 3z^3 \) at the point \((2, 0, \sqrt{2})\) in the direction of the vector \( \mathbf{v} = \langle 1, 1, 0 \rangle \).

Solution. We use the formula
\[
(D_u h)|_P = \nabla h|_P \cdot u
\]
where \( u \) is a unit vector in the direction of \( \mathbf{v} \), \( P = (2, 0, \sqrt{2}) \) is the point we are considering, and \( \nabla f|_P \) is the gradient of \( h \) evaluated at \( P \). We compute
\[
\nabla h = \langle h_x, h_y, h_z \rangle = \langle 1, 4y, 9z^2 \rangle
\]
\[
\Rightarrow \nabla h|_P = \nabla h|_{(2,0,\sqrt{2})} = \langle 1, 0, 9 \cdot 2 \rangle = (1, 0, 18).
\]
and \( u \), being a unit vector in the direction of \( \mathbf{v} \) can be computed as
\[
u = \frac{\mathbf{v}}{||\mathbf{v}||} = \frac{\langle 1, 1, 0 \rangle}{||(1, 1, 0)||} = \frac{\langle 1, 1, 0 \rangle}{\sqrt{2}} = \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle.
\]
Hence we can compute
\[
(D_u h)|_P = \langle 1, 0, 18 \rangle \cdot \langle 1/\sqrt{2}, 1/\sqrt{2}, 0 \rangle
\]
\[
= 1(1/\sqrt{2}) + 0(1/\sqrt{2}) + 18(0) = 1/\sqrt{2}.
\]

2. Find the equation for the tangent plane to the surface \( x^2 - xy - y^2 - z = 0 \) at the point \( (1, 1, -1) \). Please give your answer in the form \( Ax + By + Cz = D \).

Solution. We let \( f(x, y, z) = x^2 - xy - y^2 - z \) and compute \( \nabla f \) at \( (1, 1, -1) \), which gives a normal vector to the tangent plane to the surface at \((1, 1, -1)\):
\[
\nabla f = \langle f_x, f_y, f_z \rangle = \langle 2x - y, -x - 2y, -1 \rangle
\]
\[
\Rightarrow \nabla f|_{(1,1,-1)} = \langle 2(1) - (1), -(1) - 2(1), -1 \rangle = \langle 1, -3, -1 \rangle.
\]
Now, noting that \((1, 1, −1)\) is a point on the tangent plane, we can give
the equation of the tangent plane as
\[
1(x - 1) + (-3)(y - 1) + (-1)(z + 1) = 0 \\
⇔ x - 1 - 3y + 3 - z - 1 = 0 \\
⇔ x - 3y - z = -1.
\]

3. Give the best possible upper bound (using the technique from class) for
the error in approximating \(f(x, y) = x^2 + 3xy - 2y^2\) at the point \((1, 1)\),
over the rectangle \(|x - 1| \leq 0.1, |y - 1| \leq 0.3\). It is OK to leave your
answer as a numerical expression (i.e., not simplified down to a number).

Solution. We have the formula for an upper bound of error given by
\[
E \leq \frac{M(|x - x_0| + |y - y_0|)^2}{2}
\]
where \(M\) has the property that
\[
|f_{xx}| \leq M \quad |f_{xy}| \leq M \quad |f_{yy}| \leq M
\]
for all points in the specified interval. We note first that in this problem,
\(x_0 = y_0 = 1\), and we have
\[
|x - x_0| = |x - 1| \leq 0.1 \\
|y - y_0| = |y - 1| \leq 0.3.
\]
We then compute
\[
f_x = 2x + 3y \quad f_y = 3x - 4y
\]
\[
⇒ f_{xx} = 2 \\
f_{xy} = -4 \\
f_{yy} = 3
\]
\[
⇒ |f_{xx}| = 2 \\
|f_{xy}| = 4 \\
|f_{yy}| = 3 \Rightarrow |f_{xx}|, |f_{yy}|, |f_{xy}| \leq 4
\]
meaning the best value for \(M\) with this error bound is 4. Finally, we plug
our results into our formula and have
\[
E \leq \frac{4(0.1 + 0.3)^2}{2},
\]
and we prefer the answer be left in this form on homework and exams.

4. Let \(f\) be some function of the plane, such that \((1, 1)\) and \((1, −1)\) are critical
points. Suppose \(f_{xx} = x + 2, f_{xy} = x + y - 2,\) and \(f_{yy} = y + 1\). Classify
(min/max/SP) the critical points \((1, 1)\) and \((1, −1)\), clearly indicating any
computed values you used to make your decision.
Solution. For $P = (1,1)$ we have

\[ f_{xx}|_P = 3 > 0 \] and
\[ f_{xx}|_P f_{yy}|_P - (f_{xy}|_P)^2 = (3)(2) - (1 + 1 - 2)^2 = 6 > 0, \]
which means that $(1,1)$ is a local minimum of $f(x,y)$.

For $Q = (1,-1)$ we have

\[ f_{xx}|_Q = 3 > 0 \] and
\[ f_{xx}|_Q f_{yy}|_Q - (f_{xy}|_Q)^2 = (3)(0) - (1 + (-1) - 2) = -(-2)^2 = -4 < 0, \]
which means that $(1,-1)$ is a saddle point for $f(x,y)$.

Be sure to study up on Lagrange multipliers, too, since we won’t be able to give you homework on that before the exam!