Math 281

Homework

Oct. 17th

1 Symmetric Groups

1. Consider the following permutations in the symmetric group S_5:

$$\sigma = (1235), \tau = (24)$$

   Compute the following:

   - $\sigma\tau$;
   - $\tau\sigma$;
   - $\tau\sigma\tau$;
   - τ^2;
   - σ^2;
   - σ^3;
   - σ^4.

2. Describe (in cycle notation) the inverse of a permutation $\sigma \in S_d$. Motivate your answer.

3. Are the following statements true or false? If true, explain why. If false, give a counter-example.

   (a) Let $\sigma_1, \sigma_2 \in S_d$ be such that, when written in cycle notation, the numbers that appear in σ_1 don't appear in σ_2. (For example: σ_1 could be $(12)(46)$ and σ_2 could be (357).) Then the two permutations commute, in the sense that $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

   (b) If two permutations commute, then when written in cycle notation the numbers appearing in the cycles of the first permutation must not appear in the cycles of the second.

2 Order of an element

Consider an element g in a group G, and all its integer powers g^k. If for all $k \neq 0$, $g^k \neq e$, then we say g is an element of **infinite order**. If on the other hand there exist $k \neq 0$ with $g^k = e$, then the smallest positive such k is called the **order of** g.

1. Prove that all non-zero elements of $(\mathbb{Z}, +)$ have infinite order. Remember here that you are using additive notation, and hence g^k really means $g + g + \ldots + g$, k times.

2. Prove that if $g \in G$ is an element of order k, then for any integer multiple of k, $g^{nk} = e$.

3. Consider a permutation $\sigma \in S_d$. Can you describe the order of σ in terms of its cycle notation?

4. Prove that the set $\langle g \rangle := \{g^k : k \in \mathbb{Z}\}$ is a subgroup of G. This is called the **cyclic subgroup generated by** g

5. What is $\langle (1234) \rangle$? And $\langle (124)(56) \rangle$?