*Math 281*

Homework

Oct. 13th

These exercises are meant to explore the concept of a group.

# 1 Examples of groups and not groups

Decide if the following pairs (Set, Operation) form a group or not. When you have a group, write down the identity element and describe how to find the inverse element. When you don't have a group explain why.

1. $(\mathbb{N}, +)$: the set of natural numbers and addition.
2. $(X, +)$: $X$ is the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ (just the good old functions from calc 1), and the operation is pointwise addition $((f+g)(x) = f(x) + g(x))$.
3. $(X, \cdot)$: $X$ is the set of rotations that take a square to itself (i.e. imagine to have a square on the table and be allowed to rotate it in such a way that after you move it it looks the same as before - how many such rotations do you have?). The operation is defined as follows: $r_2 \cdot r_1 :=$ first do the rotation $r_1$, then follow it by the rotation $r_2$.
4. $(X, +)$: $X$ is the set of even integers, $+$ is addition.
5. $(X, +)$: $X$ is the set of odd integers, $+$ is addition.

# 2 Giving the same group different dresses.

Consider the following groups with 4 elements:

1. $(\mathbb{Z}/4, +)$
2. the Klein group ($X = \{1, a, b, c\}$, 1 is the identity element, $a^2 = b^2 = c^2 = 1$, the product of any two elements different from each other and different from 1 gives the third non-1 element)
3. ( 4-th roots of 1, multiplication)
4. Rotations of the square - as described above.
5. $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, +)$, where the sum is component by component.
6. the group consisting of the following permutations of a set with four elements: $\{1, (12)(34), (13)(24), (14)(23)\}$, the operation being composition of permutations.

For each of this groups write down the composition table. Now I would like you to explore this vague idea: are some of these groups just different incarnations of the same group?

A little bit more precisely. Take two of the groups above, call them $G_1$ and $G_2$. You would like to explore the possibility of a bijection between the elements of the two groups, say $f : G_1 \to G_2$, with this special property: take the composition table for $G_1$. If you substitute in every entry $g$ with $f(g)$, you obtain the composition table of $G_2$. If this is possible you say that $G_1$ and $G_2$ are isomorphic.

1. Show that being isomorphic is an equivalence relation.

2. Which of the groups above are isomorphic? Which aren't? Why?