# Homework

M474 Fall 2013

Name:______________________________ Score: ____________

1. Consider the 2 dimensional vector space $V$, with a chosen basis $e_1$ and $e_2$. $V$ has an inner product which in this basis is represented by the matrix:

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

Consider the linear function $A : V \to V$ defined, in the same basis, by the matrix (notice it is not symmetric):

$$A = \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}$$

Define a bilinear form $B : V \times V \to \mathbb{R}$ in the standard way by:

$$B(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, A\mathbf{y} \rangle_G$$

   (a) Prove that $B$ is a self adjoint bilinear form;
   (b) Compute $GA$. Observe that it is a symmetric matrix;
   (c) Compute the eigenvalues and eigenvectors of $A$;
   (d) Normalize the eigenvectors of $A$ to be of $G$-norm 1. Call these vectors $\mathbf{v}$ and $\mathbf{w}$.
   (e) Check that they are $G$-orthogonal.
   (f) Compute $B(\mathbf{v}, \mathbf{v})$, $B(\mathbf{v}, \mathbf{w})$, $B(\mathbf{w}, \mathbf{w})$. What is the matrix that represents the bilinear form $B$ in the basis $\mathbf{v}, \mathbf{w}$?
   (g) Compute eigenvalues and eigenvectors for $GA$. Notice that the eigenvectors are not orthogonal, and that the matrix of the bilinear form in the eigenvectors basis is not diagonal.

2. Consider a curve $\alpha$ parameterized by arclength in the $x, y$ plane. Define the cylinder over $\alpha$ ($:= C_\alpha$) to be the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $(x, y, 0) \in tr(\alpha)$.

   (a) describe the lines of curvature of $C_\alpha$
   (b) prove that all points of $C_\alpha$ are either parabolic or planar.
   (c) what are the points that are planar?

3. Consider a curve $\alpha$ parameterized by arclength in the $\{z = 1\}$ plane. Define the cone over $\alpha$ ($:= K_\alpha$ ) to be the set of all points $(x, y, z) \in \{z > 0\}$ that belong to a line from the origin through a point of the trace of $\alpha$.

   (a) give a parametrization for $K_\alpha$.

   (b) compute the principal directions and principal curvatures for all points of $K_\alpha$.

   (c) prove that all points of $C_\alpha$ are either parabolic or planar.

   (d) what are the points that are planar?

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and $S = \Gamma_f$ the regular surface corresponding to its graph. Let $(\bar{x}, \bar{y})$ be a critical point of $f$ and $P = (\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$ be the corresponding point on the surface.

   (a) Compute $dN_{|P}$. Observe that it is just the Hessian of $f$ at $(\bar{x}, \bar{y})$, and therefore the classification of the critical point from Calc $III$ agrees with ours now.

   (b) For each of the following functions, determine whether $P = (0, 0, f(0, 0))$ is a hyperbolic, parabolic, elliptic or planar point:

      i. $z = xy$;
      ii. $z = x^2 y$;
      iii. $z = x^2 \cos(y)$;
      iv. $z = x^2 + \sin^2(y)$.