The Metric.

Renzo's math 474

Given a point P of a regular surface S, we have identified the tangent plane $T_P(S)$ with $\mathbb{R}^2$ via the differential of a local parameterization near P. The tangent plane is a subspace of $\mathbb{R}^3$, and hence its vectors naturally have an inner product simply by restriction from the inner product in $\mathbb{R}^3$. We want to change our point of view, and instead think that we are assigning a non-standard inner product to $\mathbb{R}^2$. We start by reviewing the general notion of an inner product.

1 Inner Products

Given a real vector space V, an **inner product** on V is a function

$$\phi : V \times V \to \mathbb{R}$$

such that:

1. ϕ is *bilinear.* For every $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V and $\lambda, \mu \in \mathbb{R}$, it must be that

$$\phi(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = \lambda\phi(\mathbf{x}, \mathbf{z}) + \mu\phi(\mathbf{y}, \mathbf{z}).$$

$$\phi(\mathbf{x}, \lambda\mathbf{y} + \mu\mathbf{z}) = \lambda\phi(\mathbf{x}, \mathbf{y}) + \mu\phi(\mathbf{x}, \mathbf{z}).$$

2. ϕ is *symmetric.* For every $\mathbf{x}, \mathbf{y} \in V$

$$\phi(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{y}, \mathbf{x}).$$

3. ϕ is *positive definite.* For every $\mathbf{x} \neq 0 \in V$

$$\phi(\mathbf{x}, \mathbf{x}) > 0.$$

4. ϕ is *non-degenerate.* If $\mathbf{x} \in V$ is such that for every $\mathbf{y} \in V$ $\phi(\mathbf{x}, \mathbf{y}) = 0$, then $\mathbf{x} = 0$.

Problem 1. *Make sure you are absolutely comfortable with the fact that the standard inner product in $\mathbb{R}^n$ indeed satisfies all the above conditions.*

If V is a finite dimensional vector space, after the choice of a basis, an inner product (in fact any bilinear form) is represented by a matrix G.

Problem 2. *How? What is the matrix that represents the standard inner product in $\mathbb{R}^n$?*

Then conditions $2, 3, 4$ impose conditions on the matrix G.

Problem 3. *Assume V is two dimensional (which is the case we will be concerned with). Figure out what these conditions are.*

We often denote an inner product with angle brackets or with a dot. If we want to stress that an inner product is nonstandard we may put a subscript indicating that (a subscript under a dot is kind of confusing in LaTeX but bear with it):

$$\phi(\mathbf{x}, \mathbf{y}) \triangleq \langle \mathbf{x}, \mathbf{y} \rangle_G \triangleq \mathbf{x} \cdot_G \mathbf{y}.$$

2 The Metric on Tangent Spaces

Let S be a regular surface, and $\varphi : U \to S$ a local parameterization around a point $P \in S$ such that $\varphi(\underline{0}) = P$. We proved that for every $(\bar{u}, \bar{v}) \in U$, $d\varphi_{|(\bar{u},\bar{v})}$ gives an isomorphism between $\mathbb{R}^2$ and $T_q(S)$ for any point in the image of φ. We use this isomorphisms to define a family of inner products on $\mathbb{R}^2$. Denote $e_1 = (1, 0)$ and $e_2 = (0, 1)$ the standard basis vectors on the plane, then

$$\langle e_1, e_1 \rangle_{G_\varphi} := \left\langle \frac{\partial \varphi}{\partial u}_{|(u,v)}, \frac{\partial \varphi}{\partial u}_{|(u,v)} \right\rangle \triangleq \varphi_u(u, v) \cdot \varphi_u(u, v).$$

$$\langle e_1, e_2 \rangle_{G_\varphi} := \left\langle \frac{\partial \varphi}{\partial u}_{|(u,v)}, \frac{\partial \varphi}{\partial v}_{|(u,v)} \right\rangle \triangleq \varphi_u(u, v) \cdot \varphi_v(u, v).$$

$$\langle e_2, e_2 \rangle_{G_\varphi} := \left\langle \frac{\partial \varphi}{\partial v}_{|(u,v)}, \frac{\partial \varphi}{\partial v}_{|(u,v)} \right\rangle \triangleq \varphi_v(u, v) \cdot \varphi_v(u, v).$$

Problem 4. *What is the matrix $G_\varphi(u, v)$ that represents this inner products?*

Definition. *The collection of ALL non-standard inner products G_φ on ALL possible local parameterizations of S is called a* ***metric*** *on S.*

To "understand" a metric on a surface, it is sufficient to understand it on a collection of local parameterizations that cover the surface.

Problem 5. *Describe the metric on the cylinder, using either of the two parameterizations we studied.*

Problem 6. *Consider a local parameterization for the sphere given by*

$$\varphi(u, v) = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)).$$

Describe the matrix G_φ.

Problem 7. *Give a collection of parameterizations that cover the torus and compute the metric on the torus.*

Problem 8. *Compute the metric for a regular surface S which is the graph of a differentiable function $F : \mathbb{R}^2 \to \mathbb{R}$*

3 Norm of Tangent Vectors and Measuring Angles

We can use the metric to define a non-standard norm on plane vectors. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

$$|\mathbf{x}|_{G_\varphi} = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle_{G_\varphi}}.$$

Now we can use the formula:

$$\cos(\theta) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle_{G_\varphi}}{|\mathbf{x}|_{G_\varphi} |\mathbf{y}|_{G_\varphi}}.$$

In both cases, what we are really talking about is the norms or the angle between the vectors $d\varphi_{|(u,v)}(\mathbf{x})$ and $d\varphi_{|(u,v)}(\mathbf{y})$.

Problem 9. *Compute a few norms and angles of the basis vectors e_1 and e_2 for the various examples of metrics studied in the previous section.*

Recall that the **local coordinate lines** of a local parameterization φ are the images of the lines $u = const.$, $v = const.$ via φ.

Problem 10. *Under what condition on the metric do local coordinate lines (that intersect) intersect at right angles?*

A parameterization is called **orthogonal** if all coordinate lines (that intersect) intersect at right angles.

Problem 11. *Which of the parameterizations in the previous section are orthogonal?*

4 Measuring Lengths and Areas

We can use the metric to compute lengths of curves on a surface, and areas on the surface as integrals in the plane rather than in three-dimensional space.

Let $\alpha(t) = (x(t), y(t), z(t)) : [0, 1] \to \mathbb{R}^3$ a parameterized curve whose trace lives inside a regular surface S. To make our statements simpler let us assume that the image of α is contained in a local chart (U, φ). Denote by $\beta(t) = (u(t), v(t))$ the composition $b = \varphi^{-1} \circ \alpha$.

Then the length of the trace of α is:

$$L(\alpha) = \int_0^1 |\beta'(t)|_{G_\varphi} dt = \int_0^1 \sqrt{Eu'^2 + 2Fu'v' + Gv'2} dt, \tag{1}$$

where for rather irrelevant historical reasons the letters E, F, G are used to denote the entries of the matrix G_φ (up to you to figure out which is which).

Problem 12. *Understand formula* (1)*.*

Problem 13. *Play with the examples of Section 2 and compute some lengths of some natural curves on such surfaces.*

Now suppose R is a bounded region inside a local chart (U, φ) in a regular surface. Let $Q = \varphi^{-1}(R)$. We compute the area of R:

$$A(R) = \int_Q |e_1|_{G_\varphi} |e_2|_{G_\varphi} |\sin(\theta_{G_\varphi})| dudv, \tag{2}$$

where θ_{G_φ} is the angle between e_1 and e_2 computed using the metric G_φ.

Problem 14. *Understand* (2) *and then derive an expression in terms of the entries of the metric E, F, G.*

Problem 15. *Use your formula to compute some areas of some of the examples in Section 2.*