Math 366 Final Exam Preparation

This worksheet is specifically intended to help you prepare for the final exam. The questions asked here will be very relevant to the final, so you want to make sure you are comfortable with all these concepts. If not, bring it up in class or come talk to me.

1. Write down the definition of a subgroup being **normal**. In fact, can you recall two equivalent definitions?

2. Write down a couple examples of normal subgroups and a couple examples of subgroups that are not normal.

3. Is the dyhedral group

$$D_{2\cdot 4} = \{e, (1234), (13)(24), (1432), (13), (24), (12)(34), (14)(23)\}$$

a normal subgroup of S_4? Write down the left cosets and the right cosets of $D_{2\cdot 4}$ in S_4.

4. Consider $S_3 = \{e, (12), (13), (23), (123), (132)\}$ as a subgroup of S_4.Is it a normal subgroup of S_4? Write down the left cosets and the right cosets of S_3 in S_4.

5. Can you generalize the previous question to S_{n-1} being a subgroup of S_n. How many cosets are there? Can you give a characterization in words of what the left and right cosets are? (By this I mean some sentence of the form: *a left coset that we denote "blah" consists of all permutations that have the following property...*, in analogy to how we can describe the subgroup S_{n-1} in S_n as the set of all permutations that do not move the last number n.

6. Write down all subgroups of $\mathbb{Z}/12\mathbb{Z}$? Which subgroups are normal? Can you say something in general about subrgoups of $\mathbb{Z}/n\mathbb{Z}$ for arbitrary n? Why?

7. Suppose that you have a group G and a subgroup H such that every element of h commutes with every elements of G. Then is H a normal subgroup of G?

8. If G is a group of order 36 and H a normal subgroup of order 9, what is the order of the quotient group G/H?

9. Prove that the function $y = e^x$ is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R} \setminus \{0\}, \cdot)$. What is $Ker(f)$? What is $Im(f)$?

10. Prove that the function $y = e^{2\pi ix}$ is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{C} \setminus \{0\}, \cdot)$. What is $Ker(f)$? What is $Im(f)$? What is the quotient group $\mathbb{R}/Ker(f)$?

11. Prove that the function $(y_1, y_2) = (x, 4x)$ is a group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^2, +)$. What is $Ker(f)$? What is $Im(f)$? What is the quotient group $\mathbb{R}/Ker(f)$?

12. What familiar functions are group homomorphisms from $(\mathbb{R}^n, +)$ to $(\mathbb{R}^m, +)$?

13. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is not a group homomorphism (the operation is addition).

14. Given a group G, the two trivial subgroups $\{e_G\}$ and G are always normal subgroups. What are the corresponding quotient groups $G/\{e_G\}$ and G/G isomorphic to?

15. For any two elements a and b in a group G, prove that
$$(ab)^{-1} = a^{-1}b^{-1}.$$
Prove also that for any positive integer n,
$$(ab)^{-n} = (b^{-1}a^{-1})^n.$$

16. What is the order of $[5]$ in $G = \mathbb{Z}/10\mathbb{Z}$? Write down the cyclic subgroup H generated by $[5]$. How many elements are in the quotient group G/H?

17. What is the order of $[6]$ in $\mathbb{Z}/8\mathbb{Z}$? Write down the cyclic subgroup generated by $[6]$. How many elements does the quotient group G/H have?

18. Prove that if $g \in G$ is an element of order n, then the cyclic subgroup generated by g has n elements.

19. What is the relation between the index $[G : H]$ of a normal subgroup and the order of the quotient group G/H?

20. What are the orders of all the elements of S_4?

21. Use the previous question to decide if the following statement is true of false: if n divides $|G|$, then there exists an element $g \in G$ of order n.

22. Use some of the questions above (identify which ones) to decide if the following statement is true of false: if there exists an element $g \in G$ of order n, then n divides $|G|$.

23. Let $g \in G$ be a fixed element of a group G. Define the **centralizer** of g as the set of all elements in G that commute with g. In math terms:
$$C(g) := \{x \in G \quad \text{such that} \quad xg = gx\}.$$
Prove that $C(g)$ is a subgroup of G. What is $C(g)$ if G is an abelian group? Prove that the cyclic group generated by g is always contained in the centralizer $C(g)$. Give an example in which $C(g)$ is strictly larger than $< g >$.

24. Suppose that $g \in G$ is an element such that $g^n = e_G$ and $g^m = e_G$. What can you say about $g^{gcd(n,m)}$?