A look into the mirror (II)

The quintic

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Topics in Algebraic Geometry Seminar
Outline

1. Numerology of the quintic
2. A-model
3. B-model
4. Number of rational curves
Our main character

\[ Q \subset \mathbb{P}^4 \]

is the zero set of a generic degree 5 homogeneous polynomial in five variables.

Facts:

- By adjunction, \( Q \) is a CY threefold.
- \( H^2(Q, \mathbb{Z}) \cong \text{Pic}(Q) = \mathbb{Z} = \langle H \rangle \).
- \( H_2(Q, \mathbb{Z}) = \mathbb{Z} = \langle \ell \rangle \).
- \( \dim(H^1(TQ)) = 101 \).
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- \( \dim(H^1(TQ)) = 101 \).
Recall:
For $D_1, D_2, D_3 \in H^2(X, \mathbb{Z})$, define:

$$
\langle D_1, D_2, D_3 \rangle := D_1 \cdot D_2 \cdot D_3 + \sum_{0 \neq \beta \in H_2(X, \mathbb{Z})} \langle D_1, D_2, D_3 \rangle^g_{\beta} q^\beta,
$$

where

$$
\langle D_1, D_2, D_3 \rangle^g_{\beta} = \int_{[\overline{M}_{0,3}(X, \beta)]^{\text{vir}}} \ev_1^*(D_1) \cdot \ev_2^*(D_2) \cdot \ev_3^*(D_3)
$$

is a three pointed Gromov-Witten invariant for $X$. 


A-model Yukawa coupling

In this case:

\[ \langle H, H, H \rangle = 5 + \sum_{d > 0} \langle H, H, H \rangle^{g=0}_{\ell} q^d. \]

Divisor equation:

\[ \langle H, H, H \rangle_{\ell} = d^3 \langle>_{\ell}. \]

Multiple covers:

\[ \langle>_{\ell} = n_d + \sum_{k|d} \frac{1}{(d/k)^3} n_k, \]

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If we regroup our generating function by collecting $n_d$’s, we obtain:

$$\langle H, H, H \rangle = 5 + \sum_{d>0} d^3 n_d \left( q^d + q^{2d} + q^{3d} + \ldots \right).$$

Adding up the geometric series:

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On the B-model side things are quite a bit more involved. We must:

1. identify a mirror family.
2. identify a large complex structure (LC) limit point in the family.
3. compute the periods near the LC point to obtain canonical coordinates.
4. compute the Yukawa coupling.

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Consider the short exact sequence of lattices:

\[ 0 \to \mathbb{Z} \xrightarrow{R} \mathbb{Z}^6 \xrightarrow{A} \mathbb{Z}^5 \to 0, \]

where

\[ R = \begin{bmatrix} -5 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]
The mirror family

Construct a family of hypersurfaces in $(\mathbb{C}^*)^5 / \mathbb{C}^*$ from the matrix $A$ using the following recipe:

- Associate a coordinate $x_i$ of $(\mathbb{C}^*)^5$ to each row.
- Associate a family parameter $u_i$ to each column.
- Think of the entries of the matrix as the exponents of the $x_i$'s.

(This will all be clear in a second with the explicit example)
The mirror family

In practice:

\[
\begin{pmatrix}
  u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
  1 & 1 & 1 & 1 & 1 & 1 \\
  0 & 1 & 0 & 0 & 0 & -1 \\
  0 & 0 & 1 & 0 & 0 & -1 \\
  0 & 0 & 0 & 1 & 0 & -1 \\
  0 & 0 & 0 & 0 & 1 & -1 \\
\end{pmatrix}
\]

\[
x_1 \left( u_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 + u_5 x_5 + \frac{u_6}{x_2 x_3 x_4 x_5} \right).
\]

Homogeneity in \( x_1 \) \( \Rightarrow \) this family can be viewed in \((\mathbb{C}^*)^4\).
Now we can compactify to a family of quintics in $\mathbb{P}^4$ by homogenizing:

$$\left( u_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 + u_5 x_5 + \frac{u_6}{x_2 x_3 x_4 x_5} \right).$$

$\downarrow$

$$P(X) = \left( u_1 x_1 x_2 x_3 x_4 x_5 + u_2 x_2^2 x_3 x_4 x_5 + u_3 x_2 x_3^2 x_4 x_5 + u_4 x_2 x_3 x_4^2 x_5 + u_5 x_2 x_3 x_4 x_5^2 + u_6 x_1^5 \right).$$

$P(X)$ “is” the mirror family to the general quintic $Q \subset \mathbb{P}^4$. 
Remarks:

1. The first presentation (in COGP) of the mirror family was different: it was the quotient of the one-parameter family

\[ X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 - 5\psi X_1 X_2 X_3 X_4 X_5 \]

by a specific action of the cyclic group \((\mathbb{Z}_5)^3\).

2. We know the mirror family must be one-dimensional. The family \(P(X)\) covers the mirror family and we will be taking a one-dimensional slice of the base around a LC point!
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We know the mirror family must be one-dimensional. The family \(P(X)\) covers the mirror family and we will be taking a one-dimensional slice of the base around a LC point!
The LC point for our family is at $u_1 = \infty$ (or, if you prefer, to the other coordinates $= 0$).

It corresponds to a singular quintic (the union of the five coordinate hyperplanes); we will discover that the periods have logarithmic monodromy going around this point.
We define a never vanishing $(3, 0)$ form on the fibers of $P(X)$ in local coordinates $x_1, \ldots, x_4$ by:

$$\Omega(x) = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} \frac{1}{\partial P/\partial x_4}.$$ 

(This is indeed regular and never vanishing on the (smooth) fibers of a small neighborhood of the LC point).
We would like, for any closed 3-cycle $\Upsilon$, to compute:

$$I(u) = \int_{\Upsilon} \Omega.$$ 

Our first step will be to find one period. **Trick:** we can reduce the computation to an integral over the 4-torus $T^4 = \{|x_i| = 1\}$:

$$I(u) = \int_{T^4} \frac{1}{P} \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \frac{dx_3}{x_3} \wedge \frac{dx_4}{x_4}.$$ 

**Note:** close to the LC point the hypersurface is “close to” the arrangement of hyperplanes and hence does not intersect $T^4$ - which makes the above formula valid.
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GKZ differential equations:

The period $I(u)$ is a solution of a GKZ system of differential equations, corresponding to the matrices $R$ and $A$ written above and to the complex vector $\beta = [-1, 0, 0, 0, 0]$.

mixed partials:

$$\frac{\partial^5}{\partial u_1^5} = \frac{\partial}{\partial u_2} \frac{\partial}{\partial u_3} \frac{\partial}{\partial u_4} \frac{\partial}{\partial u_5} \frac{\partial}{\partial u_6}$$

homogeneity 1:

$$u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} + u_3 \frac{\partial}{\partial u_3} + u_4 \frac{\partial}{\partial u_4} + u_5 \frac{\partial}{\partial u_5} + u_6 \frac{\partial}{\partial u_6} = -1$$

homogeneity 2: for $2 \leq i \leq 5$,

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homogeneity 2: for \( 2 \leq i \leq 5 \),

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u_i \frac{\partial}{\partial u_i} - u_6 \frac{\partial}{\partial u_6} = 0
\]
GKZ tell us that one formal solution for this system can be given as a power series involving $\Gamma$ functions. In this particular case the answer is:

$$l_0(u) = \frac{1}{u_1} \sum_{n \geq 0} (-1)^n \frac{(5n)!}{(n!)^5} z^n,$$

where

$$z = \frac{u_2 u_3 u_4 u_5 u_6}{u_1^5}.$$
Remarks:

• this solution has trivial monodromy.
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The other periods

GKZ hands us a method to compute all the other periods ($H_3$ is 4-dimensional), by taking a deformation of this function over a special artinian ring constructed from the GKZ combinatorial data.

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GKZ hands us a method to compute all the other periods ($H_3$ is 4-dimensional), by taking a deformation of this function over a special artinian ring constructed from the GKZ combinatorial data.

In this case,

$$\overline{R} = \frac{\mathbb{C}[\varepsilon]}{\varepsilon^4}.$$
Claim: the function
\[ I^\varepsilon(u) = \frac{1}{u_1} \sum_{n \geq 0} (-1)^n \frac{(5n + \varepsilon)!}{((n + \varepsilon)!)^5} z^{n+\varepsilon}, \]

where we define
\[ (n + \varepsilon)! := (n + \varepsilon)(n - 1 + \varepsilon) \cdots (1 + \varepsilon), \]

satisfies our GKZ system of differential equations over the ring \( \overline{\mathcal{R}} \).
Punchline: expanding in $\varepsilon$

$$l^\varepsilon(z) = l_0 + l_1\varepsilon + l_2\varepsilon^2 + l_3\varepsilon^3,$$

one gets 4 independent solutions to our GKZ system!

Remark: the logarithmic monodromy comes from expanding the term

$$z^\varepsilon := e^{\varepsilon \log(z)} = 1 + \varepsilon \log(z) + \frac{(\varepsilon \log(z))^2}{2!} + \frac{(\varepsilon \log(z))^3}{3!}$$
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Finally, we can define the canonical coordinates

\[ w := \frac{l_1}{l_0} \]

and

\[ q := e^{2\pi i w}. \]
Most of the remaining work is now simply tedious computations and a few tricks. We quickly outline how these computations go. Mark Gross’s notes are detailed and clear.

It is not too hard to see that the Yukawa coupling is:

$$\langle \frac{\partial}{\partial z'}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = \frac{c_1}{z^3 (5^5 z - 1) l_0^2},$$

for some constant $c_1$ to be determined.

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**Problem!** The Yukawa coupling is in the wrong coordinates!!
By the chain rule one can write:

$$\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right\rangle = \left( \frac{\partial z}{\partial w} \right)^3 \left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right\rangle,$$

and after some laborious substitution and series manipulation one can expand the above expression in terms of $q$ to get:

$$\left\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \right\rangle = \sum \frac{c_1}{c_2} \frac{h_j(0)}{j!} q^j,$$

where

- $c_1, c_2$ are constants to be determined;
- $h_j(z)$ is defined inductively. (in the next slide)
Sketch:

\[ h_0(z) := \frac{1}{(5^5 z - 1) \eta_0^2 (1 + z \frac{dw}{dz})^3} \]

\[ h_j(z) := \frac{1}{(1 + z \frac{dw}{dz}) e^w} \frac{dh_{j-1}}{dz} \]
At the end of the day...

Putting everything together, one can finally expand both Yukawa couplings in $q$ and match coefficients.

$H \cdot H \cdot H = 5$ and $n_1 = 2875$ are needed as initial conditions to determine $c_1$ and $c_2$. Then all other numbers are predicted:

\[
\begin{align*}
n_2 &= 609250 \\
n_3 &= 317206375 \\
n_4 &= 242467530000
\end{align*}
\]

*et cetera*