A look into the mirror (I)
an overview of Mirror Symmetry

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Topics in Algebraic Geometry Seminar
Outline

1. Physics: the big black box

2. Math
   - Calabi-Yau threefolds
   - The A-model
   - The B-model
   - The Mirror Map

3. Mirror conjecture
Mirror Symmetry is a correspondence between pairs of (families of) Calabi-Yau threefolds

\[ X \leftrightarrow \check{X} \]

that interchanges complex and symplectic geometry.

Mirror Symmetry is motivated by physics.
A physical theory should satisfy some natural axioms that give it the structure of a SCFT.

SUSY is a required feature of a SCFT. It eliminates in a very natural way a lot of the difficulties arising in constructing a string theory.

A mathematical realization of a SCFT is given by a sigma model, a construction depending upon the choice of:

- a Calabi-Yau threefold $X$;
- a complexified Kahler class $\omega$. 
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Mirror Symmetry
(X, ω)

Moduli space of SCFT

M_{compl}
X varies
ω constant

M_{kah}
X constant
ω varies
SUSY suggests the existence of an involution on the moduli space of SCFT such that:

\[ H^q(X, \Lambda^p T_X) \cong H^q(\check{X}, \Lambda^p \Omega_{\check{X}}) \]
\[ H^q(X, \Lambda^p \Omega_X) \cong H^q(\check{X}, \Lambda^p T_{\check{X}}) \]
In particular, looking at $p = q = 1$

\[ T_{M_{\text{compl}}} = H^1(X, T_X) \cong H^1(\tilde{X}, \Omega_{\tilde{X}}) = T_{M_{\text{kah}}} \]

\[ T_{M_{\text{kah}}} = H^1(X, \Omega_X) \cong H^1(\tilde{X}, T_{\tilde{X}}) = T_{M_{\text{compl}}} \]

we obtain an identification of tangent spaces, and hence local isomorphisms between complex and kahler moduli spaces of the mirror pair. Such isomorphisms are called the **Mirror Maps**.
Physics hands us two trilinear forms called Yukawa couplings:

- **A-model YC**: \((T_{M_{\text{kah}}}^3 \rightarrow \mathbb{C})\);
- **B-model YC**: \((T_{M_{\text{compl}}}^3 \rightarrow \mathbb{C})\).

Mirror symmetry postulates that such functions should get identified via the mirror maps!

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Physics hands us two trilinear forms called **Yukawa couplings**:

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This is how mirror symmetry makes **enumerative predictions** about rational curves in CY threefolds.
A CY threefold $X$ is a projective threefold (possibly with mild singularities) such that:

- $K_X \cong \mathcal{O}_X$.
- $H^i(X, \mathcal{O}_X) = 0$, for $i = 1, 2$. 

Definition
Combining the above definition with Serre duality and $h^{p,q} = h^{q,p}$ we obtain that the Hodge diamond of a CY threefold is:

\[
\begin{array}{ccc}
  b_6 & : & 1 \\
  b_5 & : & 0 \\
  b_4 & : & 0 \quad h^{1,1} \quad h^{2,1} \quad 0 \\
  b_3 & : & 1 \quad h^{2,1} \quad h^{2,1} \quad 0 \\
  b_2 & : & 0 \quad h^{1,1} \quad 0 \\
  b_1 & : & 0 \\
  b_0 & : & 1
\end{array}
\]
A kahler form $\omega$ is a closed $(1, 1)$ (real) form such that $\omega^3$ is non-degenerate. The kahler cone $\mathcal{K}(X)$ is the space of all possible kahler forms. It is an open subset of $H^{1,1}(X, \mathbb{R})$. 
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The **kahler cone** $\mathcal{K}(X)$ is the space of all possible kahler forms. It is an open subset of $H^{1,1}(X, \mathbb{R})$. 
The complexified kahler moduli space of $X$ is

$$M_{kah} := H^2(X, \mathbb{R})/H^2(X, \mathbb{Z}) + iK(X).$$

A basis $\{C_\beta\}$ of $H_2(X, \mathbb{Z})$ gives coordinates (called kahler parameters) on $M_{kah}$,

$$z_i = \int_{C_\beta} B + i\omega$$

only defined up to periods.
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only defined up to periods.
If $\text{Pic}(X) = \mathbb{Z} = \langle H \rangle$, then

$$M_{\text{kah}} = \mathbb{R}/\mathbb{Z} + i\mathbb{R}_{>0}$$

is equivalent to the punctured disk $\Delta^*$ via the exponential coordinates

$$q = e^{2\pi iz}$$
For higher Picard number, a **framing** is a choice of a basis for $H^2(X, \mathbb{Z})$, that identifies a **simplicial cone** in $\overline{\mathcal{K}}(X)$.

An exponential transformation from the kahler parameters identifies the corresponding portion in $M_{kah}$ with a punctured polydisc.
The Yukawa coupling

For $D_1, D_2, D_3 \in H^2(X, \mathbb{Z})$, define:

$$\langle D_1, D_2, D_3 \rangle := D_1 \cdot D_2 \cdot D_3 + \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} \langle D_1, D_2, D_3 \rangle^g_{\beta} \cdot q^{\beta},$$

where

$$\langle D_1, D_2, D_3 \rangle^g_{\beta} = \int_{[\overline{M}_{0,3}(X,\beta)]^{vir}} \text{ev}_1^*(D_1) \cdot \text{ev}_2^*(D_2) \cdot \text{ev}_3^*(D_3)$$

is a three pointed Gromov-Witten invariant for $X$.

**Note:** from the above formula we can extract, after correcting for multiple cover contributions, the (virtual) number of rational curves on the threefold in any given homology class.
Deformation spaces

Idea: the moduli space of complex structures is too complicated, so we study it locally.

A deformation space for $X$ is the data illustrated in the following universal property diagram:

$$
\begin{array}{cccc}
\mathcal{X} & \longrightarrow & U \\
\downarrow & & \downarrow \\
S & \longrightarrow & Def(X) \\
\downarrow & & \\
X_0 & & \\
\end{array}
$$
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\downarrow \\
U
\end{array}$$
Facts and observations

1. The tangent space to $\text{Def}(X)$ at $x_0$ is classically identified with $H^1(X, T_X)$.

2. For a CY threefold, the choice of a global non-vanishing holomorphic 3-form gives an isomorphism

$$H^1(X, T_X) \cong H^1(X, \Lambda^2 \Omega_X) = H^{2,1}(X)$$

(⇒ symmetry in the Hodge diamond of a mirror pair)

3. **Bogomolov-Tian-Todorov theorem:** for a CY threefold, the deformation problem is unobstructed. (i.e. any infinitesimal deformation can be integrated).

4. A family $\mathcal{X} \to S$ induces a map $T_{S,s_0} \to T_{\text{Def}(X)}$ called the Kodaira-Spencer morphism. If we assume it to be an isomorphism, we can work on the tangent space of a concrete family rather than on $T_{\text{Def}(X)}$. 

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Given a family of CY threefolds $\pi : \mathcal{X} \to S$ we can define the Hodge bundle to be

$$E := R^3\pi_* (\mathbb{C}) \otimes O_S.$$ 

What is going on:

$$H^3(\mathcal{X}_s, \mathbb{C}) \to E \quad \downarrow \quad s \to S$$
Given a family of CY threefolds $\pi : X \to S$ we can define the Hodge bundle to be

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A basis $\{\sigma_i\}$ for $H^3(X, \mathbb{Z})$ gives a local frame for $E$: any local section is

$$\sigma = \sum f_i(s) \sigma_i(s).$$

Gauss-Manin connection:

$$\nabla_{\frac{\partial}{\partial s_j}} \sigma = \sum \frac{\partial f_i}{\partial s_j} \sigma_i.$$
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The Yukawa Coupling

We can now define a cubic form on $T_{\text{Def}}(X)^{KS} \cong T_{S,s}$. Choose a family of Calabi-Yau forms $\Omega(s)$ (non-vanishing $(3,0)$ forms).

$$\left\langle \frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3} \right\rangle := \int_X \Omega \wedge \nabla \frac{\partial}{\partial s_1} \nabla \frac{\partial}{\partial s_2} \nabla \frac{\partial}{\partial s_3} \Omega$$
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third derivatives are necessary to obtain something non-trivial, by Griffiths transversality.

the coupling depends on the choice of $\Omega(s)$. Any two Calabi-Yau families differ by a non-vanishing holomorphic function $f(s)$, and the coupling transforms by multiplication by $f^2(s)$. 
Remarks

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2. The coupling depends on the choice of $\Omega(s)$. Any two Calabi-Yau families differ by a non-vanishing holomorphic function $f(s)$, and the coupling transforms by multiplication by $f^2(s)$. 
The Mirror Map

Mirror Map “=” a set of canonical coordinates $q$ on $Def(X)$ that we can identify with the $q$’s on (part of) $M_{kah}$ coming from the choice of a framing.

Observation: on the kahler side $q = 0$ corresponded to a degenerate kahler metric. This suggests that we should try and “center” our canonical coordinates somewhere on the “boundary” of the complex moduli space.

Simplification: from now on, let us restrict our attention to the situation of $dim(Def(X)) = 1$ and look very locally around some point. I.e., we consider families $X \rightarrow \Delta^*$. 

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$$\mathcal{X} \to \Delta^*.$$
For a fixed pair \((X, \Omega)\), the period map is

\[
P_{X,\Omega} : H_3(X, \mathbb{C}) \quad \beta \quad \mapsto \quad \mathbb{C} \\
\quad \beta \quad \mapsto \quad \int_{\beta} \Omega.
\]

Local torelli tells us the period map is a local coordinate for the complex moduli space.

Problems:

1. for a family \(X \to \Delta^*\) we can define a period map only on the universal cover \(\mathcal{H}\) of the punctured disc.

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P(z) := P_{X_z,\Omega(z)}
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2. This definition still depends upon the choice of a family of Calabi-Yau forms.
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\[ P(z + 1) = P(z) \circ T, \]

where \( T : H_3(X, \mathbb{C}) \to H_3(X, \mathbb{C}) \) is a linear map called monodromy transformation.

If we were lucky enough to have a basis for \( H_3(X, \mathbb{C}) \) such that

\[
T = \begin{bmatrix}
1 & n & * & * \\
0 & 1 & * & * \\
0 & 0 & 1 & * \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

we could simultaneously solve problems (1) and (2) by setting...
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Mirror Symmetry
Canonical coordinates

\[ w(z) := \frac{\int_{A_1} \Omega(z)}{\int_{A_0} \Omega(z)} \]

and the *canonical coordinate* (recall \( s = e^{2\pi i z} \)):

\[ q(s) := e^{2\pi i w} \]

Such luck happens only around special points in the boundary of the complex moduli space, called *large complex structure limit points*.
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Such luck happens only around special points in the boundary of the complex moduli space, called large complex structure limit points.
The periods of a family of CY threefolds are the solutions of a **GKZ** system of differential equations, called **Picard-Fuchs** equations.

The technology we have developed this semester allows us to systematically:

1. find the solutions to the Picard-Fuchs equations.
2. identify a family centered around a large complex structure limit point.
3. extract the basis vectors necessary to define canonical coordinates.

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Mirror Symmetry
The Mirror conjecture

It is possible to correspond:

<table>
<thead>
<tr>
<th>$\mathcal{X} \to (\Delta^*)^s$</th>
<th>$\leftrightarrow$</th>
<th>$\check{X}$</th>
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<tbody>
<tr>
<td>0 a large CS limit point</td>
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<tr>
<th>canonical coordinates $q$</th>
<th>$\leftrightarrow$</th>
<th>a framing on $\mathcal{K}(\check{X})$ giving coordinates $q$ for $M_{kah}$</th>
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<th>(2, 1)-YC (Quantum Cohomology)</th>
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Stay tuned!