Affine Geometry
and the Discrete Legendre Transform

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Outline

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Hypersurfaces in toric varieties coming from reflexive polytopes.
Duality: $\Delta \leftrightarrow \Delta^\circ$.
These polygons are polar (and we will see them again):
Complete intersections in toric varieties coming from NEF-partitions of reflexive polytopes.

Duality: \( \{ \Delta_1, \ldots, \Delta_r \} \leftrightarrow \{ \nabla_1, \ldots, \nabla_r \} \), such that

\[
\Delta = \text{Conv} \{ \Delta_1, \ldots, \Delta_r \}, \quad \Delta^\circ = \nabla_1 + \cdots + \nabla_r,
\]

\[
\nabla = \text{Conv} \{ \nabla_1, \ldots, \nabla_r \}, \quad \nabla^\circ = \Delta_1 + \cdots + \Delta_r.
\]
Calabi-Yau manifolds in Grassmannians and (partial) flags. Duality is obtained using degeneration to toric varieties.

Gross and Siebert propose another degeneration approach. Duality: discrete Legendre transform of a polarized positive integral tropical manifold $(B, P, \varphi) \leftrightarrow (\tilde{B}, \tilde{P}, \tilde{\varphi})$.

Let's make sense of the last sentence!
An integral (or lattice) rational convex polyhedron $\sigma$ is the (possibly unbounded) intersection of finitely many closed affine half-spaces in $\mathbb{R}^n$ with at least one vertex, s.t. functions defining these half-spaces can be taken with rational coefficients and all vertices are integral. Let $\text{LPoly}$ be the category of integral convex polyhedra with integral affine isomorphisms onto faces as morphisms.

An integral polyhedral complex is a category $\mathcal{P}$ and a functor $F : \mathcal{P} \rightarrow \text{LPoly}$ s.t. if $\sigma \in F(\mathcal{P})$ and $\tau$ is a face of $\sigma$, then $\tau \in F(\mathcal{P})$.

To avoid self-intersections: there is at most one morphism between any two objects of $\mathcal{P}$. (This requirement is not essential.)
Topological Manifold and Tangent Spaces

**Definition**

Let $B$ be the topological space associated to $\mathcal{P}$: the quotient of $\bigcup_{\sigma \in \mathcal{P}} F(\sigma)$ by the equivalence relation of face inclusion.

From now on we will denote $F(\sigma)$ just by $\sigma$ and call them cells.

**Definition**

$\Lambda_\sigma \cong \mathbb{Z}^{\dim \sigma}$ is the free abelian group of integral vector fields along $\sigma$.

For any $y \in \text{Int} \sigma$ there is a canonical injection $\Lambda_\sigma \to T_{\sigma, y}$, inducing the isomorphism $T_{\sigma, y} \cong \Lambda_{\sigma, \mathbb{R}} = \Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$.

Using the exponential map, we can identify $\sigma$ with a polytope $\tilde{\sigma} \subset T_{\sigma, y}$. (Different choices of $y$ will correspond to translations of $\tilde{\sigma}$.)
**Definition**

Let $N_{\tau}$ be a lattice of rank $k$ and $N_{\tau}^{\mathbb{R}} = N_{\tau} \otimes_{\mathbb{Z}} \mathbb{R}$. A *fan structure* along $\tau \in \mathcal{P}$ is a continuous map $S_{\tau} : U_{\tau} \to N_{\tau}^{\mathbb{R}}$, where $U_{\tau}$ is the open star of $\tau$, s.t.

- $S_{\tau}^{-1}(0) = \text{Int } \tau$
- for each $e : \tau \to \sigma$ the restriction $S_{\tau}|_{\text{Int } \sigma}$ is induced by an epimorphism $\Lambda_{\sigma} \to W \cap N_{\tau}$ for some vector subspace $W \subset N_{\tau}^{\mathbb{R}}$
- cones $K_e = \mathbb{R}_{\geq 0} S_{\tau}(\sigma \cap U_{\tau})$, $e : \tau \to \sigma$, form a finite fan $\Sigma_{\tau}$ in $N_{\tau}^{\mathbb{R}}$
Tropical Manifold

Definition

If \( \tau \subset \sigma \), the \textit{fan structure along} \( \sigma \) \textit{induced by} \( S_\tau \) is the composition

\[
U_\sigma \rightarrow U_\tau \xrightarrow{S_\tau} N_\tau,\mathbb{R} \rightarrow N_\tau,\mathbb{R}/L_\sigma = N_\sigma,\mathbb{R},
\]

where \( L_\sigma \subset N_\mathbb{R} \) is the linear span of \( S_\tau(\text{Int} \sigma) \).

Definition

An \textit{integral tropical manifold} of dimension \( n \) is a (countable) integral polyhedral complex \( \mathcal{P} \) with a fan structure \( S_v : U_v \rightarrow N_v,\mathbb{R} \simeq \mathbb{R}^n \) at each vertex \( v \in \mathcal{P} \) s.t.

- for any vertex \( v \) the support \( |\Sigma_v| = \bigcup_{C \in \Sigma_v} C \) is (non-strictly) convex with nonempty interior;
- if \( v \) and \( w \) are vertices of \( \tau \), then the fan structures along \( \tau \) induced from \( S_v \) and \( S_w \) are equivalent.
Let $\Delta$ be a reflexive polytope.
Let $\mathcal{P} = \partial \Delta$.
The fan structures are given by projections along the vertices.
Fan structures at vertices define an affine structure on $B$ away from the closed discriminant locus $\Delta$ of codimension two.

For each bounded $\tau \in \mathcal{P}$, s.t. $\dim \tau \neq 0$, $n$, choose $a_\tau \in \text{Int} \, \tau$:
Constructing Discriminant Locus

- For each unbounded $\tau$, s.t. $\dim \tau \neq n$, choose a $0 \neq a_\tau \in \Lambda_{\tau,\mathbb{R}}$, s.t. $a_\tau + \tau \subset \tau$.

- For each chain $\tau_1 \subset \cdots \subset \tau_{n-1}$ with $\dim \tau_i = i$ and $\tau_i$ bounded for $i \leq r$, where $r \geq 1$, let
  \[
  \Delta_{\tau_1 \cdots \tau_{n-1}} = \text{Conv} \{ a_{\tau_i} : 1 \leq i \leq r \} + \sum_{i>r} \mathbb{R}_{\geq 0} \cdot a_{\tau_i} \subset \tau_{n-1}.
  \]

- Let $\Delta$ be the union of such polyhedra.
Constructing Discriminant Locus

(Only “visible” half of $\Delta$ is shown for the bounded case.)
Remarks on Discriminant Locus

- If $\varrho \in \mathcal{P}$ and $\dim \varrho = n - 1$, the connected components of $\varrho \setminus \Delta$ are in bijection with the vertices of $\varrho$.
- Interiors of top dimensional cells and fan structures at vertices define an affine structure on $B \setminus \Delta$ (fan structures give charts via the exponential maps).
- This defines a flat connection on $T_{B\setminus\Delta}$ which we may use for parallel transport.
- There is some flexibility in constructing $\Delta$.
- “Generic” discriminant locus: coordinates of $a_\tau$ are “as algebraically independent as possible.” Then $\Delta$ contains no rational points.
Choose the following data:

- $\omega \in \mathcal{P}$ — a bounded edge (i.e. a one-dimensional cell)
- $v^+$ and $v^-$ — vertices of $\omega$
- $\varrho \in \mathcal{P}$, $\dim \varrho = n - 1$, $\varrho \not\subset \partial B$, $\omega \subset \varrho$
- $\sigma^+$ and $\sigma^-$ — top dimensional cells containing $\varrho$

Follow the change of affine charts given by

- the fan structure at $v^+$
- the polyhedral structure of $\sigma^+$
- the fan structure at $v^-$
- the polyhedral structure of $\sigma^-$
- again the fan structure at $v^+$
Local Monodromy

- We obtain a transformation $T_{\omega \varrho} \in \text{SL}(\Lambda_{v^+})$, where $\Lambda_{v^+}$ is the lattice of integral tangent vectors to $B$ at the point $v^+$. (Not the integral vector fields along the zero-dimensional face $v^+$!)
- Let $d_\omega \in \Lambda_\omega \subset \Lambda_{v^+}$ be the primitive vector from $v^+$ to $v^-$. 
- Let $\tilde{d}_{\varrho} \in \Lambda_{\varrho}^\perp \subset \Lambda_{v^+}^*$ be the primitive vector s.t. $\langle \tilde{d}_{\varrho}, \sigma^+ \rangle \geq 0$.
- Then $T_{\omega \varrho}(m) = m + \chi_{\omega \varrho} \langle m, \tilde{d}_{\varrho} \rangle d_\omega$ for some constant $\chi_{\omega \varrho}$.
- $\chi_{\omega \varrho}$ is independent on the ordering of $v^\pm$ and $\sigma^\pm$.
- If $m \in \Lambda_{\varrho}$, then $T_{\omega \varrho}(m) = m$.
- For any $m$ we have $T_{\omega \varrho}(m) - m \in \Lambda_{\varrho}$.
- $\chi_{\omega \varrho} \geq 0$ for “geometrically meaningful manifolds”.
- If all $\chi_{\omega \varrho} \geq 0$, $B$ is called positive.
- The affine structure extends to a neighborhood of $\tau \in \mathcal{P}$ iff $\chi_{\omega \varrho} = 0$ for all $\omega$ and $\varrho$ s.t. $\omega \subset \tau \subset \varrho$. 
Functions on Tropical Manifolds

**Definition**
An *affine function* on an open set $U \subset B$ is a continuous map $U \to \mathbb{R}$ that is affine on $U \setminus \Delta$.

**Definition**
A *piecewise-linear (PL) function* on $U$ is a continuous map $\varphi : U \to \mathbb{R}$ s.t. for all fan structures $S_\tau : U_\tau \to N_{\tau,\mathbb{R}}$ along $\tau \in \mathcal{P}$ we have $\varphi|_{U \cap U_\tau} = \lambda + S_\tau^\ast(\varphi_\tau)$ for some affine function $\lambda : U_\tau \to \mathbb{R}$ and a function $\varphi_\tau : N_{\tau,\mathbb{R}} \to \mathbb{R}$ which is PL w.r.t. the fan $\Sigma_\tau$.

(This definition ensures that $\varphi$ is “good enough” near $\Delta$.)
Multivalued PL Functions and Polarizations

Definition

A multivalued piecewise-linear (MPL) function $\varphi$ on $U$ is a collection of PL functions $\{\varphi_i\}$ on an open cover $\{U_i\}$ of $U$ s.t. $\varphi_i$’s differ by affine functions on overlaps.

MPL functions can be given by specifying maps $\varphi_\tau : N_{\tau, \mathbb{R}} \to \mathbb{R}$. All of the above definitions can be restricted to integral functions in the obvious way.

Definition

If all local PL representatives of an integral MPL function $\varphi$ are strictly convex, $\varphi$ is a polarization of $(B, \mathcal{P})$ and $(B, \mathcal{P}, \varphi)$ is a polarized integral tropical manifold.

(If $\partial B \neq \emptyset$, we also require that $|\Sigma_\tau|$ is convex for every $\tau \in \mathcal{P}$.)
Example of a Polarization

- Suppose \((B, \mathcal{P})\) is constructed from a reflexive polytope \(\Delta\).
- Let \(\psi\) be a PL function on the fan generated by \(\Delta\), s.t. \(\psi|_{\partial \Delta} \equiv 1\).
- For each vertex \(v\) choose an integral affine function \(\psi_v\) s.t. \(\psi_v(v) = 1 = \psi(v)\) and let \(\varphi_v = \psi - \psi_v\) on \(U_v\).
The discrete Legendre transform is a duality transformation of the set of polarized (positive) integral tropical manifolds $(B, P, \varphi) \leftrightarrow (\tilde{B}, \tilde{P}, \tilde{\varphi})$.

As a category, $\tilde{P}$ is the opposite of $P$.

The functor $\tilde{F} : \tilde{P} \to \textbf{LPoly}$ is given by $\tilde{F}(\tilde{\tau}) = \text{Newton}(\varphi_\tau)$, where $\tilde{\tau} = \tau$ as objects and $\text{Newton}(\varphi_\tau)$ is the Newton polyhedron of $\varphi_\tau$:

$$\text{Newton}(\varphi_\tau) = \left\{ x \in \mathbb{N}_{\tau, \mathbb{R}}^* : \varphi_\tau + x \geq 0 \right\}.$$

For the fans from our example we get:

\[ \left\langle 2, > \right\rangle \quad \left\langle 1, > \right\rangle \quad \text{-1} \quad 2 \]
Discrete Legendre Transform

- These data are enough to construct $\bar{B}$ as a topological manifold.
- It also can be obtained as the dual cell complex of $(B, P)$ using the barycentric subdivision, which gives a homeomorphism between $B$ and $\bar{B}$, but is not a polyhedral complex.
- In our example we get the boundary of the polar reflexive polygon:
We still need to give fan structures and a polarization.

Let $\sigma \in \mathcal{P}$, $\dim \sigma = n$. Then $\tilde{\sigma}$ is a vertex.

Let $\Sigma_{\tilde{\sigma}}$ be the normal fan of $\sigma$ in $\Lambda^*_\sigma,\mathbb{R}$.

Using the parallel transport, we can identify $\Lambda^*_\sigma,\mathbb{R}$ with $\Lambda^*_v,\mathbb{R} = T^*_B,\mathbb{R}$ for a vertex $v$ of $\sigma$.

Let $\tilde{\sigma} \subset T_{B,v}$ be the inverse image of $\sigma$ under the exponential map.

Let $\tilde{\varphi}_{\tilde{\sigma}} : |\Sigma_{\tilde{\sigma}}| \to \mathbb{R}$ be given by $\tilde{\varphi}_{\tilde{\sigma}}(m) = -\inf(m(\tilde{\sigma}))$.

If $\Sigma_{\tilde{\sigma}}$ is incomplete, extend $\tilde{\varphi}_{\tilde{\sigma}}$ to $\Lambda^*_\tau,\mathbb{R}$ by infinity.

A different choice of $v$ corresponds to the translation of $\tilde{\sigma}$ by an integral vector and the change of $\tilde{\varphi}_{\tilde{\sigma}}$ by an integral affine function.

In our example we get the following (same as if we repeated construction with vectors!):

![Diagram](image-url)
Consider the cones

\[ \{ m \in \Lambda^*_{v,R} : m(\tilde{\sigma}) \geq 0 \} \in \Sigma_{\tilde{\sigma}}, \quad \{ m \in N^*_{v,R} : m(dS_v(\tilde{\sigma})) \geq 0 \} \in \Sigma^*_v. \]

\( dS_v \) is an integral affine identification of cones corresponding to cells containing \( v \) ("tangent wedges") in \( T_{B,v} \approx \Lambda_{v,R} \) and \( N_{v,R} \).

Cones above can be identified via \( dS^*_v : N^*_{v,R} \to \Lambda^*_{v,R} \).

These maps induce the fan structure at \( \tilde{\sigma} \).

This gives the polarized integral tropical manifold \((\tilde{B}, \tilde{P}, \tilde{\varphi})\).

Can take \( \tilde{\Delta} = \Delta \) (using homeomorphism \( B \to \tilde{B} \)).

Discrete Legendre transform of a positive manifolds is positive.
Thank you!