Affine Geometry and the Discrete Legendre Transfrom

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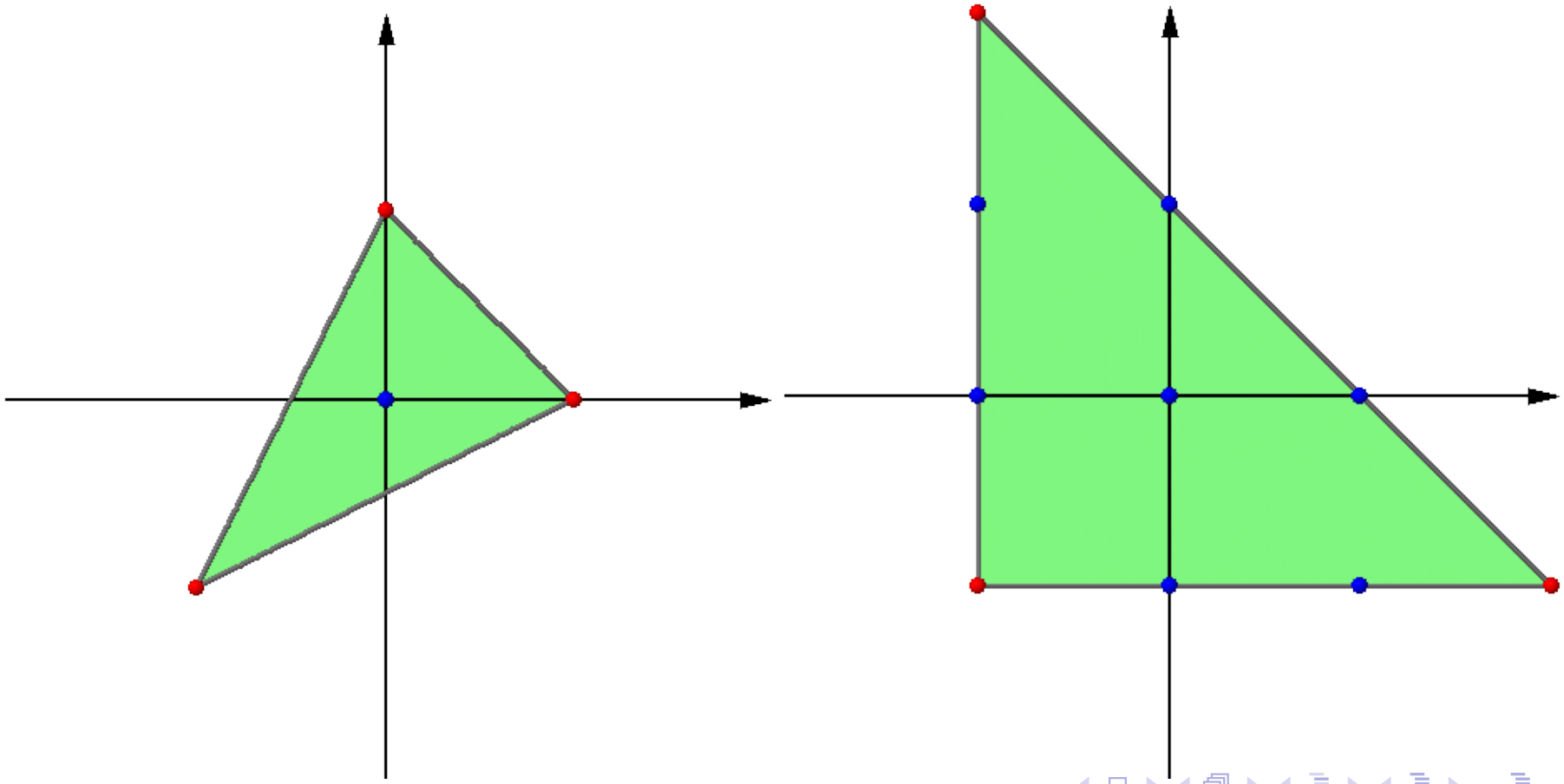
Outline

Batyrev

- Hypersurfaces in toric varieties coming from reflexive polytopes.
- Duality: $\Delta \leftrightarrow \Delta^\circ$.
- These polygons are polar (and we will see them again):



Batyrev–Borisov

- Complete intersections in toric varieties coming from NEF-partitions of reflexive polytopes.
- Duality: $\{\Delta_1, \ldots, \Delta_r\} \leftrightarrow \{\nabla_1, \ldots, \nabla_r\}$, such that

$$\Delta = \mathrm{Conv}\{\Delta_1, \ldots, \Delta_r\}, \qquad \Delta^\circ = \nabla_1 + \cdots + \nabla_r,$$
$$\nabla = \mathrm{Conv}\{\nabla_1, \ldots, \nabla_r\}, \qquad \nabla^\circ = \Delta_1 + \cdots + \Delta_r.$$

Degenerations

- Calabi-Yau manifolds in Grassmannians and (partial) flags. Duality is obtained using degeneration to toric varieties.
- Gross and Siebert propose another degeneration approach. Duality: discrete Legendre transform of a polarized positive integral tropical manifold $(B, \mathcal{P}, \varphi) \leftrightarrow (\check{B}, \check{\mathcal{P}}, \check{\varphi})$.
- Let's make sense of the last sentence!

Polyhedral Complex

Definition

An *integral (or lattice) rational convex polyhedron* σ is the (possibly unbounded) intersection of finitely many closed affine half-spaces in $\mathbb{R}^n$ with at least one vertex, s.t. functions defining these half-spaces can be taken with rational coefficients and all vertices are integral. Let **LPoly** be the category of integral convex polyhedra with integral affine isomorphisms onto faces as morphisms.

Definition

An *integral polyhedral complex* is a category $\mathcal{P}$ and a functor $F : \mathcal{P} \to$ **LPoly** s.t. if $\sigma \in F(\mathcal{P})$ and τ is a face of σ, then $\tau \in F(\mathcal{P})$.

To avoid self-intersections: there is at most one morphism between any two objects of $\mathcal{P}$. (This requirement is not essential.)

Topological Manifold and Tangent Spaces

Definition

Let B be the topological space associated to $\mathcal{P}$: the quotient of $\coprod_{\sigma\in\mathcal{P}} F(\sigma)$ by the equivalence relation of face inclusion.

From now on we will denote $F(\sigma)$ just by σ and call them cells.

Definition

$\Lambda_\sigma \simeq \mathbb{Z}^{\dim\sigma}$ is the free abelian group of integral vector fields along σ.

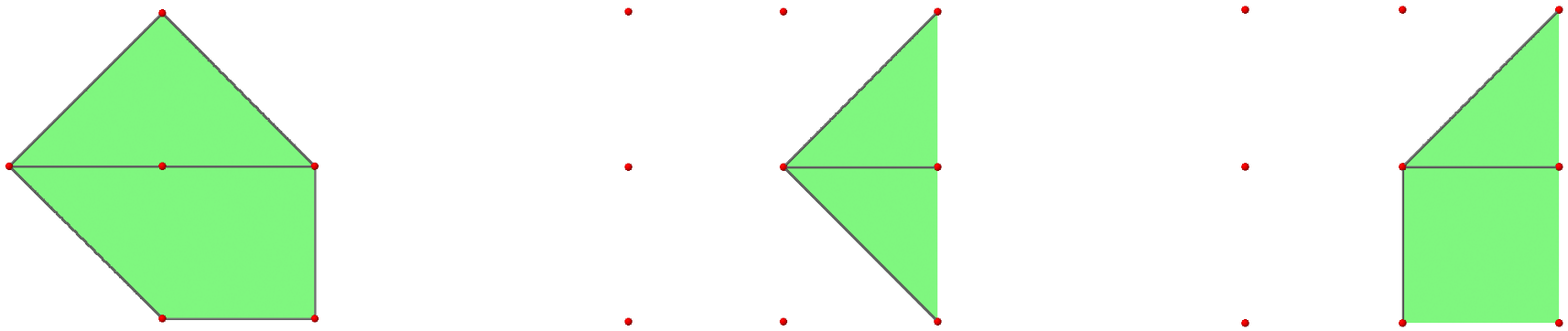
For any $y \in \operatorname{Int}\sigma$ there is a canonical injection $\Lambda_\sigma \to T_{\sigma,y}$, inducing the isomorphism $T_{\sigma,y} \simeq \Lambda_{\sigma,\mathbb{R}} = \Lambda_\sigma \otimes_{\mathbb{Z}} \mathbb{R}$.
Using the exponential map, we can identify σ with a polytope $\widetilde{\sigma} \subset T_{\sigma,y}$.
(Different choices of y will correspond to translations of $\widetilde{\sigma}$.)

Fan Structure

Definition

Let N_τ be a lattice of rank k and $N_{\tau,\mathbb{R}} = N_\tau \otimes_{\mathbb{Z}} \mathbb{R}$. A *fan structure* along $\tau \in \mathcal{P}$ is a continuous map $S_\tau : U_\tau \to N_{\tau,\mathbb{R}}$, where U_τ is the open star of τ, s.t.

- $S_\tau^{-1}(0) = \operatorname{Int} \tau$
- for each $e : \tau \to \sigma$ the restriction $S_\tau|_{\operatorname{Int}\sigma}$ is induced by an epimorphism $\Lambda_\sigma \to W \cap N_\tau$ for some vector subspace $W \subset N_{\tau,\mathbb{R}}$
- cones $K_e = \mathbb{R}_{\geqslant 0} S_\tau(\sigma \cap U_\tau)$, $e : \tau \to \sigma$, form a finite fan Σ_τ in $N_{\tau,\mathbb{R}}$



Tropical Manifold

Definition

If $\tau \subset \sigma$, the *fan structure along* σ *induced by* S_τ is the composition

$$U_\sigma \to U_\tau \xrightarrow{S_\tau} N_{\tau,\mathbb{R}} \to N_{\tau,\mathbb{R}}/L_\sigma = N_{\sigma,\mathbb{R}},$$

where $L_\sigma \subset N_\mathbb{R}$ is the linear span of $S_\tau(\operatorname{Int}\sigma)$.

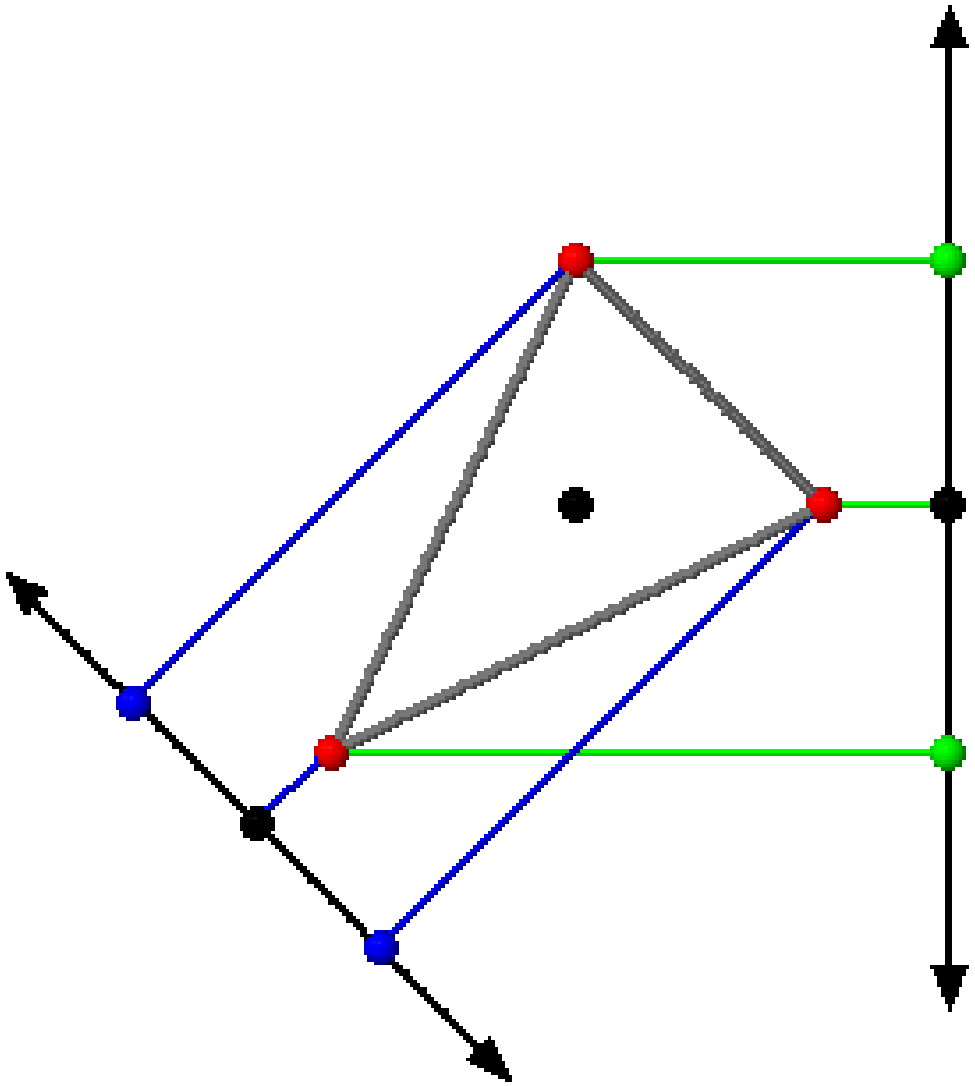
Definition

An *integral tropical manifold* of dimension n is a (countable) integral polyhedral complex $\mathcal{P}$ with a fan structure $S_v : U_v \to N_{v,\mathbb{R}} \simeq \mathbb{R}^n$ at each vertex $v \in \mathcal{P}$ s.t.

- for any vertex v the support $|\Sigma_v| = \bigcup_{C \in \Sigma_v} C$ is (non-strictly) convex with nonempty interior;
- if v and w are vertices of τ, then the fan structures along τ induced from S_v and S_w are equivalent.

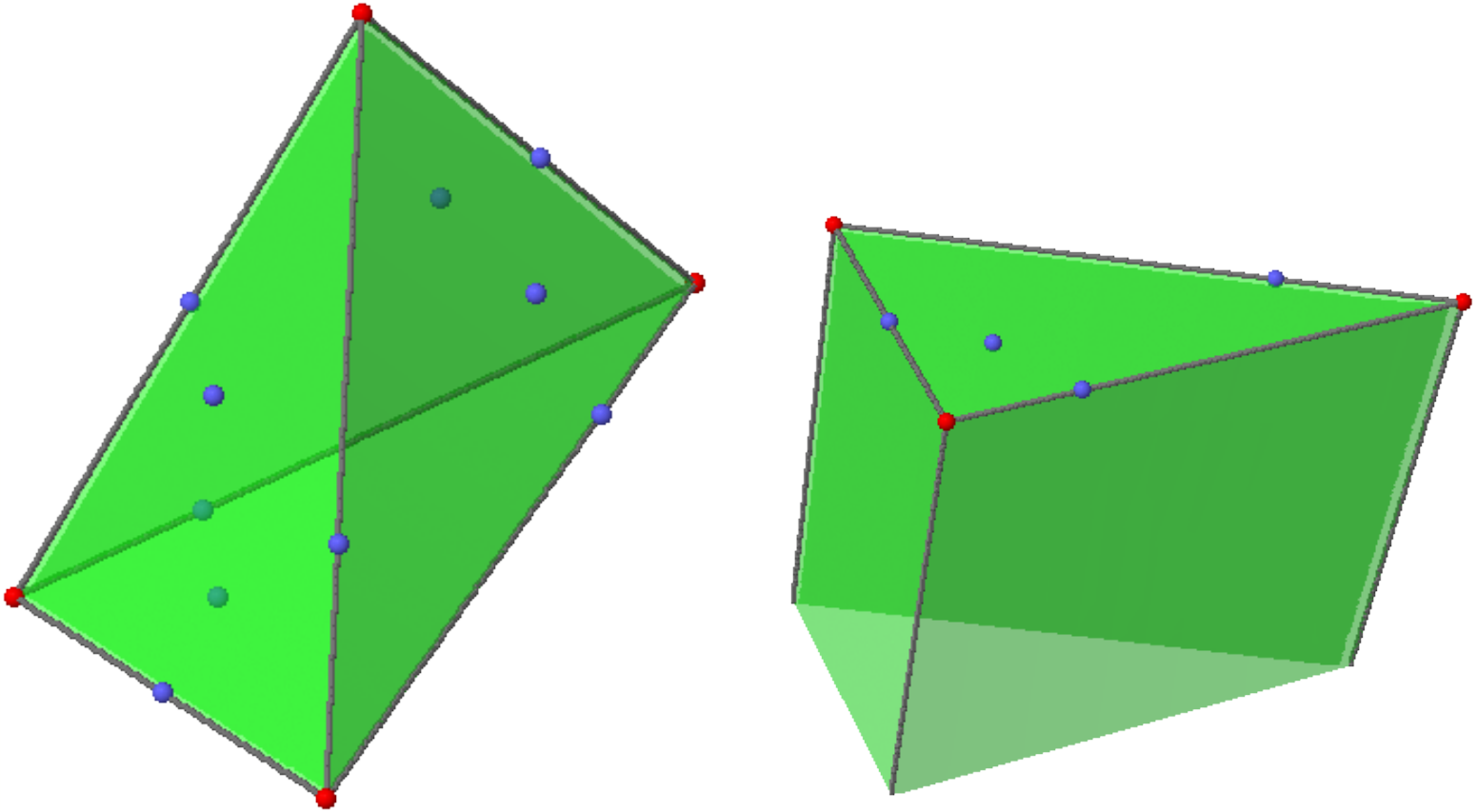
Tropical Manifolds from Reflexive Polytopes

- Let Δ be a reflexive polytope.
- Let $\mathcal{P} = \partial\Delta$.
- The fan structures are given by projections along the vertices.



Constructing Discriminant Locus

- Fan structures at vertices define an affine structure on B away from the closed *discriminant locus* Δ of codimension two.
- For each bounded $\tau \in \mathcal{P}$, s.t. $\dim \tau \neq 0, n$, choose $a_\tau \in \operatorname{Int} \tau$:



Constructing Discriminant Locus

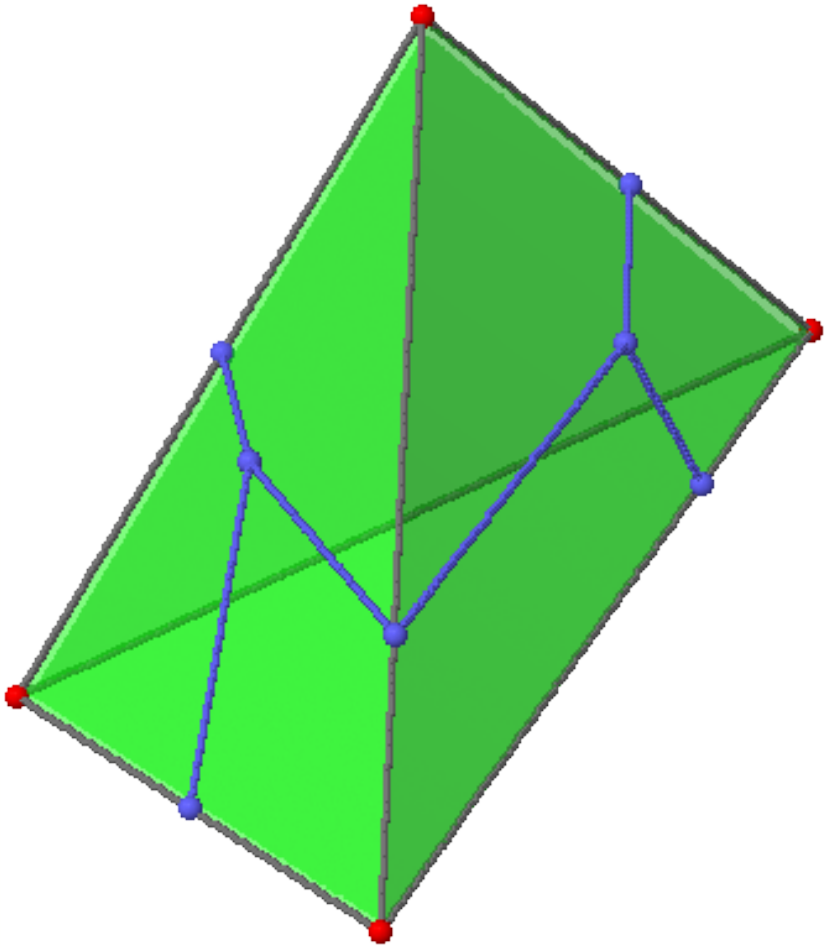
- For each unbounded τ, s.t. $\dim \tau \neq n$, choose a $0 \neq a_\tau \in \Lambda_{\tau,\mathbb{R}}$, s.t. $a_\tau + \tau \subset \tau$.
- For each chain $\tau_1 \subset \cdots \subset \tau_{n-1}$ with $\dim \tau_i = i$ and τ_i bounded for $i \leqslant r$, where $r \geqslant 1$, let

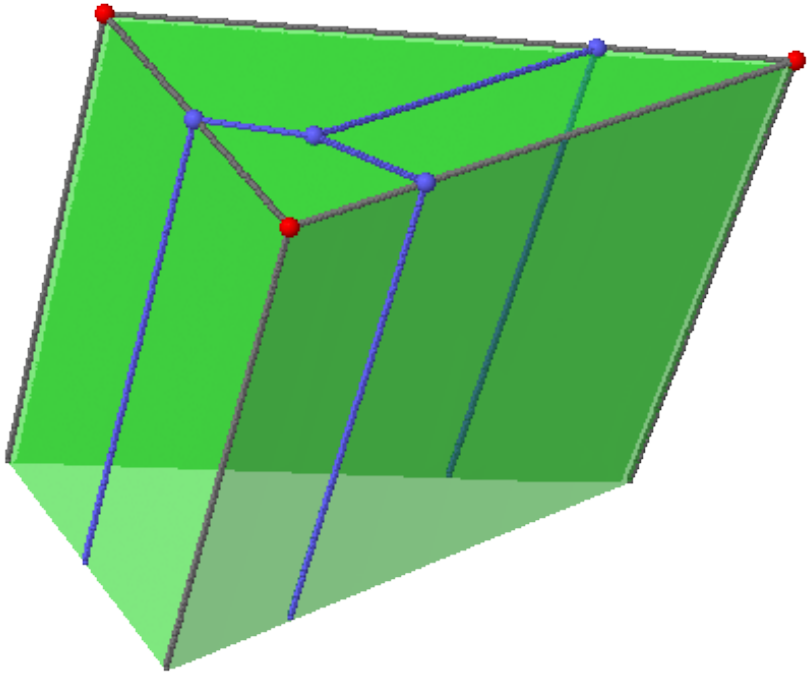
$$\Delta_{\tau_1 \ldots \tau_{n-1}} = \mathrm{Conv}\,\{a_{\tau_i} : 1 \leqslant i \leqslant r\} + \sum_{i>r} \mathbb{R}_{\geqslant 0} \cdot a_{\tau_i} \subset \tau_{n-1}.$$

- Let Δ be the union of such polyhedra.

Constructing Discriminant Locus

(Only “visible” half of Δ is shown for the bounded case.)





Remarks on Discriminant Locus

- If $\varrho \in \mathcal{P}$ and $\dim \varrho = n - 1$, the connected components of $\varrho \setminus \Delta$ are in bijection with the vertices of ϱ.
- Interiors of top dimensional cells and fan structures at vertices define an affine structure on $B \setminus \Delta$ (fan structures give charts via the exponential maps).
- This defines a flat connection on $T_{B \setminus \Delta}$ which we may use for parallel transport.
- There is some flexibility in constructing Δ.
- "Generic" discriminant locus: coordinates of a_τ are "as algebraically independent as possible." Then Δ contains no rational points.

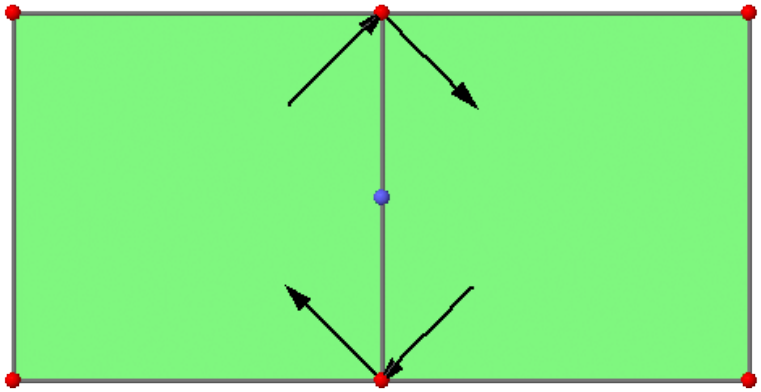
Local Monodromy

Choose the following data:

- $\omega \in \mathcal{P}$ — a bounded edge (i.e. a one-dimensional cell)
- v^+ and v^- — vertices of ω
- $\varrho \in \mathcal{P}$, $\dim \varrho = n-1, \varrho \not\subset \partial B, \omega \subset \varrho$
- σ^+ and σ^- — top dimensional cells containing ϱ

Follow the change of affine charts given by

- the fan structure at v^+
- the polyhedral structure of σ^+
- the fan structure at v^-
- the polyhedral structure of σ^-
- again the fan structure at v^+



Local Monodromy

- We obtain a transformation $T_{\omega\varrho} \in \mathrm{SL}(\Lambda_{v^+})$, where Λ_{v^+} is the lattice of integral tangent vectors to B at the point v^+. (Not the integral vector fields along the zero-dimensional face v^+!)
- Let $d_\omega \in \Lambda_\omega \subset \Lambda_{v^+}$ be the primitive vector from v^+ to v^-.
- Let $\check{d}_\varrho \in \Lambda_\varrho^\perp \subset \Lambda_{v^+}^*$ be the primitive vector s.t. $\langle \check{d}_\varrho, \sigma^+ \rangle \geqslant 0$.
- Then $T_{\omega\varrho}(m) = m + \varkappa_{\omega\varrho} \langle m, \check{d}_\varrho \rangle d_\omega$ for some constant $\varkappa_{\omega\varrho}$.
- $\varkappa_{\omega\varrho}$ is independent on the ordering of $v^\pm$ and $\sigma^\pm$.
- If $m \in \Lambda_\varrho$, then $T_{\omega\varrho}(m) = m$.
- For any m we have $T_{\omega\varrho}(m) - m \in \Lambda_\varrho$.
- $\varkappa_{\omega\varrho} \geqslant 0$ for "geometrically meaningful manifolds".
- If all $\varkappa_{\omega\varrho} \geqslant 0$, B is called *positive*.
- The affine structure extends to a neighborhood of $\tau \in \mathcal{P}$ iff $\varkappa_{\omega\varrho} = 0$ for all ω and ϱ s.t. $\omega \subset \tau \subset \varrho$.

Functions on Tropical Manifolds

Definition

An *affine function* on an open set $U \subset B$ is a continuous map $U \to \mathbb{R}$ that is affine on $U \setminus \Delta$.

Definition

A *piecewise-linear (PL) function* on U is a continuous map $\varphi : U \to \mathbb{R}$ s.t. for all fan structures $S_\tau : U_\tau \to N_{\tau,\mathbb{R}}$ along $\tau \in \mathcal{P}$ we have $\varphi|_{U \cap U_\tau} = \lambda + S_\tau^*(\varphi_\tau)$ for some affine function $\lambda : U_\tau \to \mathbb{R}$ and a function $\varphi_\tau : N_{\tau,\mathbb{R}} \to \mathbb{R}$ which is PL w.r.t. the fan Σ_τ.

(This definition ensures that φ is "good enough" near Δ.)

Multivalued PL Functions and Polarizations

Definition

A *multivalued piecewise-linear (MPL) function* φ on U is a collection of PL functions $\{\varphi_i\}$ on an open cover $\{U_i\}$ of U s.t. φ_i's differ by affine functions on overlaps.

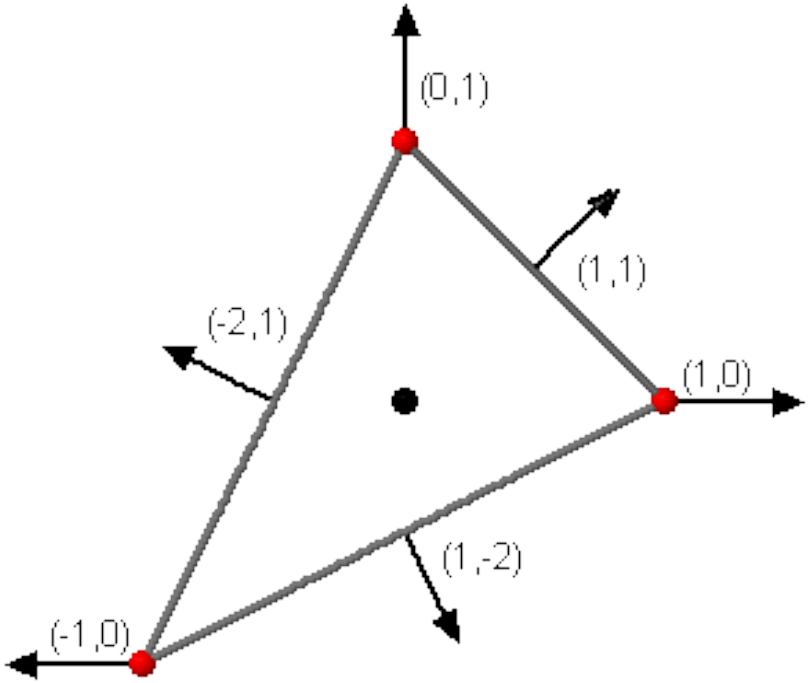
MPL functions can be given by specifying maps $\varphi_\tau : N_{\tau,\mathbb{R}} \to \mathbb{R}$.
All of the above definitions can be restricted to integral functions in the obvious way.

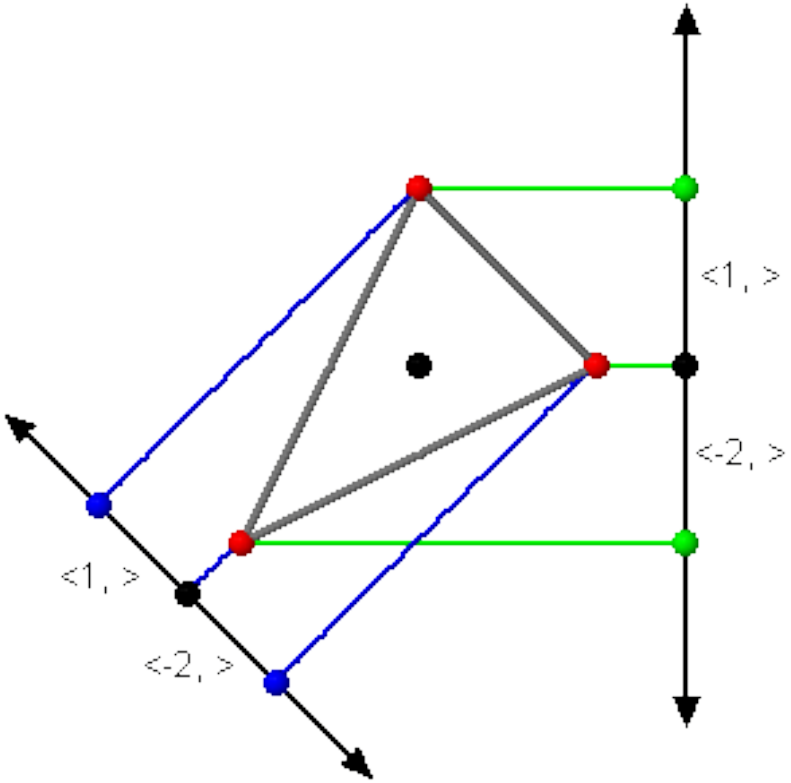
Definition

If all local PL representatives of an integral MPL function φ are strictly convex, φ is a *polarization* of $(B, \mathcal{P})$ and $(B, \mathcal{P}, \varphi)$ is a *polarized integral tropical manifold*.
(If $\partial B \neq \varnothing$, we also require that $|\Sigma_\tau|$ is convex for every $\tau \in \mathcal{P}$.)

Example of a Polarization

- Suppose $(B, \mathcal{P})$ is constructed from a reflexive polytope Δ.
- Let ψ be a PL function on the fan generated by Δ, s.t. $\psi|_{\partial\Delta} \equiv 1$.
- For each vertex v choose an integral affine function ψ_v s.t. $\psi_v(v) = 1 = \psi(v)$ and let $\varphi_v = \psi - \psi_v$ on U_v.





Discrete Legendre Transform

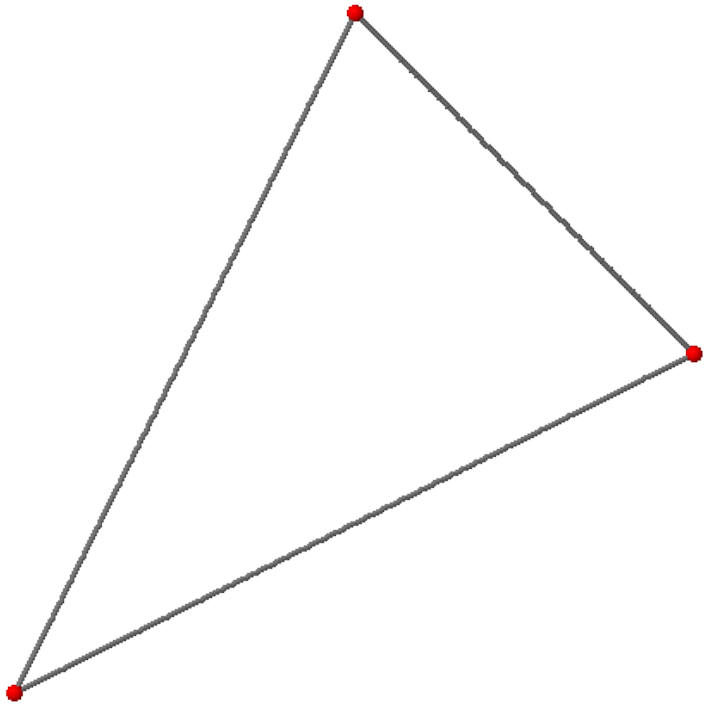
- The discrete Legendre transform is a duality transformation of the set of polarized (positive) integral tropical manifolds $(B, \mathcal{P}, \varphi) \leftrightarrow (\check{B}, \check{\mathcal{P}}, \check{\varphi})$.
- As a category, $\check{\mathcal{P}}$ is the opposite of $\mathcal{P}$.
- The functor $\check{F} : \check{\mathcal{P}} \to$ **LPoly** is given by $\check{F}(\check{\tau}) = \mathrm{Newton}(\varphi_\tau)$, where $\check{\tau} = \tau$ as objects and $\mathrm{Newton}(\varphi_\tau)$ is the Newton polyhedron of φ_τ:

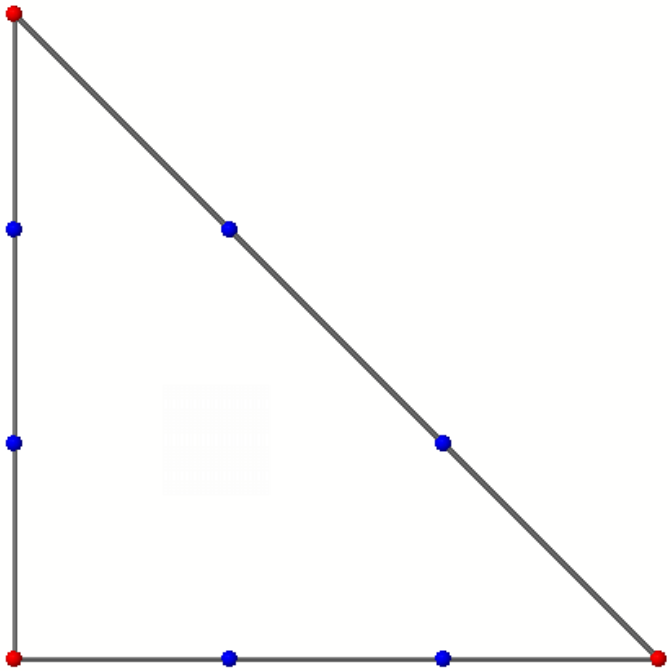
$$\mathrm{Newton}(\varphi_\tau) = \left\{ x \in N^*_{\tau,\mathbb{R}} : \varphi_\tau + x \geqslant 0 \right\}.$$

- For the fans from our example we get:

Discrete Legendre Transform

- These data are enough to construct $\check{B}$ as a topological manifold.
- It also can be obtained as the dual cell complex of $(B, \mathcal{P})$ using the barycentric subdivision, which gives a homeomorphism between B and $\check{B}$, but is not a polyhedral complex.
- In our example we get the boundary of the polar reflexive polygon:





Discrete Legendre Transform

- We still need to give fan structures and a polarization.
- Let $\sigma \in \mathcal{P}$, $\dim \sigma = n$. Then $\check{\sigma}$ is a vertex.
- Let $\Sigma_{\check{\sigma}}$ be the normal fan of σ in $\Lambda^*_{\sigma,\mathbb{R}}$.
- Using the parallel transport, we can identify $\Lambda^*_{\sigma,\mathbb{R}}$ with $\Lambda^*_{v,\mathbb{R}} = T^*_{B,v}$ for a vertex v of σ.
- Let $\widetilde{\sigma} \subset T_{B,v}$ be the inverse image of σ under the exponential map.
- Let $\check{\varphi}_{\check{\sigma}} : |\Sigma_{\check{\sigma}}| \to \mathbb{R}$ be given by $\check{\varphi}_{\check{\sigma}}(m) = -\inf(m(\widetilde{\sigma}))$.
- If $\Sigma_{\check{\sigma}}$ is incomplete, extend $\check{\varphi}_{\check{\sigma}}$ to $\Lambda^*_{\tau,\mathbb{R}}$ by infinity.
- A different choice of v corresponds to the translation of $\widetilde{\sigma}$ by an integral vector and the change of $\check{\varphi}_{\check{\sigma}}$ by an integral affine function.
- In our example we get the following (same as if we repeated construction with vectors!):

Cone Identification

- Consider the cones

$$\left\{m \in \Lambda^*_{v,\mathbb{R}} : m(\widetilde{\sigma}) \geqslant 0\right\} \in \Sigma_{\check{\sigma}}, \quad \left\{m \in N^*_{v,\mathbb{R}} : m(dS_v(\widetilde{\sigma})) \geqslant 0\right\} \in \Sigma^*_v.$$

- dS_v is an integral affine identification of cones corresponding to cells containing v ("tangent wedges") in $T_{B,v} \simeq \Lambda_{v,\mathbb{R}}$ and $N_{v,\mathbb{R}}$.
- Cones above can be identified via $dS^*_v : N^*_{v,\mathbb{R}} \to \Lambda^*_{v,\mathbb{R}}$.
- These maps induce the fan structure at $\check{\sigma}$.
- This gives the polarized integral tropical manifold $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$.
- Can take $\check{\Delta} = \Delta$ (using homeomorphism $B \to \check{B}$).
- Discrete Legendre transform of a positive manifolds is positive.

The End

Thank you!