

The Strominger-Yau-Zaslow conjecture: From torus fibrations to degenerations.

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ABSTRACT. We trace progress and thinking about the Strominger-Yau-Zaslow conjecture since its introduction in 1996. In particular, we aim to explain how the conjecture led to the algebro-geometric program developed by myself and Siebert, which aims to explain mirror symmetry by studying degenerations of Calabi-Yau manifolds. We end by outlining how tropical curves arise in the mirror symmetry story.

Introduction.

Up to the summer of 1996, there had been a number of spectacular successes in mirror symmetry. After the pioneering initial work of Candelas, de la Ossa, Greene and Parkes [6] which calculated the instanton predictions for the quintic three-fold, Batyrev [1] gave a powerful mirror symmetry construction for Calabi-Yau hypersurfaces in toric varieties, later generalized by Batyrev and Borisov [2] to complete intersections in toric varieties. Kontsevich [33] introduced his notion of stable maps of curves and studied their moduli, setting the stage for a new flowering of enumerative geometry. Eventually, this led to the first calculation of Gromov-Witten invariants for the quintic, by Givental [11] in March of 1996. In between, a great deal of the structure of mirror symmetry was elucidated by many researchers in both string theory and algebraic geometry.

On the other hand, at that time I had been primarily interested in the geometry of Calabi-Yau manifolds. Many of the results about mirror symmetry seemed to rely primarily on information about the ambient toric varieties; in particular, Givental's work and succeeding work by Lian, Liu and Yau [37], Bertram [5] and Gathmann [10] always performed calculations on the moduli space of stable maps into \mathbb{P}^4 rather than the quintic, and in the Batyrev-Borisov type constructions, while there was a clear combinatorial relationship between the ambient toric varieties, there was no apparent geometric relationship between the Calabi-Yaus themselves. As a result, I tended to avoid thinking about mirror symmetry precisely because of this lack of geometric understanding.

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This changed dramatically in June of 1996, when Strominger, Yau and Zaslow [49] released their paper “Mirror Symmetry is T -duality.” They made a remarkable proposal, based on recent ideas in string theory, that for the first time gave a geometric interpretation for mirror symmetry.

Let me summarize, very roughly, the physical argument here. Developments in string theory in the mid-1990s had introduced the notion of *Dirichlet branes*, or D -branes. These are submanifolds of space-time, with some additional data, which should serve as a boundary condition for open strings, i.e. we allow open strings to propagate with their endpoints constrained to lie on a D -brane. Remembering that space-time, according to string theory, looks like $\mathbb{R}^{1,3} \times X$, where $\mathbb{R}^{1,3}$ is ordinary space-time and X is a Calabi-Yau three-fold, we can split a D -brane into a product of a submanifold of $\mathbb{R}^{1,3}$ and one on X . It turned out, simplifying a great deal, that there were two particular types of submanifolds on X of interest: *holomorphic D -branes*, i.e. holomorphic submanifolds with a holomorphic line bundle, and *special Lagrangian D -branes*, which are *special Lagrangian submanifolds* with flat $U(1)$ bundle:

DEFINITION 0.1. Let X be an n -dimensional Calabi-Yau manifold with ω the Kähler form of a Ricci-flat metric on X and Ω a nowhere vanishing holomorphic n -form. Then a submanifold $M \subseteq X$ is *special Lagrangian* if it is Lagrangian, i.e. $\dim_{\mathbb{R}} M = \dim_{\mathbb{C}} X$ and $\omega|_M = 0$, and in addition $\text{Im } \Omega|_M = 0$.

The origins of mirror symmetry in physics suggest that if X and \check{X} are a mirror pair of Calabi-Yau manifolds, then string theory on a compactification of space-time using X should be the same as that using \check{X} , but with certain data interchanged. In the case of D -branes, the suggestion is that the moduli space of holomorphic D -branes on X should be isomorphic to the moduli space of special Lagrangian D -branes on \check{X} . Now X itself is the moduli space of points on X . So each point on X should correspond to a pair (M, ∇) , where $M \subseteq \check{X}$ is a special Lagrangian submanifold and ∇ is a flat connection on M .

A theorem of McLean [39] tells us that the tangent space to the moduli space of special Lagrangian deformations of a special Lagrangian submanifold $M \subseteq \check{X}$ is $H^1(M, \mathbb{R})$. Of course, the moduli space of flat $U(1)$ -connections modulo gauge equivalence on M is the torus $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$. In order for this moduli space to be of the correct dimension, we need $\dim H^1(M, \mathbb{R}) = n$, the complex dimension of X . This suggests that X consists of a family of tori which are dual to a family of special Lagrangian tori on \check{X} . An elaboration of this argument yields the following conjecture:

CONJECTURE 0.2. The Strominger-Yau-Zaslow conjecture. *If X and \check{X} are a mirror pair of Calabi-Yau n -folds, then there exists fibrations $f : X \rightarrow B$ and $\check{f} : \check{X} \rightarrow B$ whose fibres are special Lagrangian, with general fibre an n -torus. Furthermore, these fibrations are dual, in the sense that canonically $X_b = H^1(\check{X}_b, \mathbb{R}/\mathbb{Z})$ and $\check{X}_b = H^1(X_b, \mathbb{R}/\mathbb{Z})$ whenever X_b and \check{X}_b are non-singular tori.*

I will clarify this statement as we review the work of the past nine years; however, as I have stated this conjecture, it is likely to be false. On the other hand, there are weaker versions of the conjecture which probably are true. Even better, these weaker statements are probably within reach of modern-day technology (with a lot of hard work). Nevertheless, there has been a lot of good progress on precise versions of the above conjecture at the topological and symplectic level. In addition,

the conjecture has been successful at explaining many features of mirror symmetry, some of which are still heuristic and some of which are rigorous. It is my belief that a final satisfactory understanding of mirror symmetry will flow from the SYZ conjecture, even if results do not take the form initially suggested by it.

My main goal here is to explain the journey taken over the last nine years. I want to focus on explaining the evolution and development of the ideas, rather than focus on precise statements. Except in the first few sections, I will give few precise statements.

In those first sections, I will clarify the above statement of the conjecture, and show how it gives a satisfactory explanation of mirror symmetry in the so-called semi-flat case, i.e. the case when the metric along the special Lagrangian fibres is flat. This leads naturally to a discussion of affine manifolds, metrics on them, and the Legendre transform. These now appear to be the key structures underlying mirror symmetry.

We next take a look at the case when singular fibres appear. In this case we need to abandon the precise form of duality we developed in the semi-flat case and restrict our attention to topological duality. In the realm of purely topological duality, the SYZ conjecture has been entirely successful at explaining topological features of mirror symmetry for a large range of Calabi-Yau manifolds, including those produced by the Batyrev-Borisov construction for complete intersections in toric varieties.

Moving on, we take a look at Dominic Joyce's arguments demonstrating the problems with the strong form of the SYZ conjecture stated above. This forces us to recast the SYZ conjecture as a limiting statement. Mirror symmetry is always about the behaviour of Calabi-Yau manifolds near maximally unipotent degenerations. A limiting form of the SYZ conjecture suggests that one can find special Lagrangian tori on Calabi-Yau manifolds near a maximally unipotent degeneration, and as we approach the limit point in complex moduli space, we expect to see a larger portion of the Calabi-Yau manifold filled out by special Lagrangian tori. Unlike the original SYZ conjecture, though still difficult, this one looks likely to be accessible by current techniques.

This form of the conjecture then motivates a new round of questions. In this limiting picture, we expect the base B of the hypothetical special Lagrangian fibration to be the so-called Gromov-Hausdorff limit of a sequence of Calabi-Yau manifolds approaching the maximally unipotent degeneration. Gromov-Hausdorff convergence is a metric space concept, while maximally unipotent degeneration is an algebro-geometric, Hodge-theoretic concept. How do these two concepts relate? In the summer of 2000, Kontsevich suggested that the Gromov-Hausdorff limit will be, roughly, the dual intersection complex of the algebro-geometric degeneration, at least on a topological level. We explore this idea in §6.

On the other hand, how does this help us with mirror symmetry? Parallel to these developments on limiting forms of the SYZ conjecture, my coauthor Bernd Siebert had been studying degenerations of Calabi-Yau manifolds using logarithmic geometry with Stefan Schröer. Siebert noticed that mirror symmetry seemed to coincide with a combinatorial exchange of logarithmic data on the one side and polarizations on the other. Together, we realised that this approach to mirror symmetry meshed well with the limiting picture predicted by SYZ. Synthesizing these two approaches, we discovered an algebro-geometric version of the SYZ approach,

which I will describe here. The basic idea is to forget about special Lagrangian fibrations, and only keep track of the base of the fibration, which is an affine manifold. We show how polyhedral decompositions of affine manifolds give rise to degenerate Calabi-Yau varieties, and conversely how certain sorts of degenerations of Calabi-Yau varieties, which we call *toric degenerations*, give rise to affine manifolds as their dual intersection complex. Mirror symmetry is again explained by a discrete version of the Legendre transform much as in the semi-flat case.

We end with a discussion of the connection of tropical curves with this approach. The use of tropical curves in curve counting in two-dimensional toric varieties has been pioneered in work of Mikhalkin [40]; Nishinou and Siebert [42] generalized this work to higher dimensions using an approach directly inspired by the approach I discuss here. On the other hand, tropical curves have not yet been used for counting curves in Calabi-Yau manifolds, so I will end the paper by discussing how tropical curves arise naturally in our picture.

I would like to thank the organizers of the Seattle conference for running an excellent conference, and my coauthors Pelham Wilson and Bernd Siebert on SYZ related results; much of the work mentioned here came out of work with them.

1. First a topological observation.

Before doing anything else, let's ask a very basic question: why should dualizing torus fibrations interchange Hodge numbers of Calabi-Yau threefolds? If you haven't seen this, it's the first thing one should look at as it is particularly easy to see, if we make a few assumptions. Suppose we are given a pair of Calabi-Yau threefolds X and \check{X} with fibrations $f : X \rightarrow B$, $\check{f} : \check{X} \rightarrow B$ with the property that there is an open set $B_0 \subseteq B$ such that $f_0 : f^{-1}(B_0) \rightarrow B_0$ and $\check{f}_0 : \check{f}^{-1}(B_0) \rightarrow B_0$ are torus fibre bundles. So all the singular fibres of f and \check{f} lie over $\Gamma := B \setminus B_0$. (Note that unless $\chi(X) = 0$, there must be some singular fibres.) Finally, assume f_0 and \check{f}_0 are dual torus fibrations, i.e. f_0 can be identified with the torus fibration $R^1\check{f}_{0*}(\mathbb{R}/\mathbb{Z}) \rightarrow B_0$ and \check{f}_0 can be identified with the torus fibration $R^1f_{0*}(\mathbb{R}/\mathbb{Z}) \rightarrow B_0$. (This is a slight abuse of notation: by $R^1\check{f}_{0*}(\mathbb{R}/\mathbb{Z})$ we really mean the torus bundle obtained by taking the vector bundle associated to the local system $R^1\check{f}_{0*}\mathbb{R}$ and dividing out by the family of lattices $R^1\check{f}_{0*}\mathbb{Z}$.)

If V/Λ is a single torus with V an n -dimensional vector space and Λ a lattice in V , then $H^p(V/\Lambda, \mathbb{R}) \cong \bigwedge^p V^\vee$, while the dual torus, V^\vee/Λ^\vee , has $H^p(V^\vee/\Lambda^\vee, \mathbb{R}) \cong \bigwedge^p V$. If we choose an isomorphism $\bigwedge^n V \cong \mathbb{R}$, then we get an isomorphism $H^p(V/\Lambda, \mathbb{R}) \cong H^{n-p}(V^\vee/\Lambda^\vee, \mathbb{R})$. Similarly, in the relative setting for f_0 and \check{f}_0 , if we have an isomorphism $R^3f_{0*}\mathbb{R} \cong \mathbb{R}$, we obtain isomorphisms

$$R^p f_{0*}\mathbb{R} \cong R^{3-p}\check{f}_{0*}\mathbb{R}.$$

We now make a simplifying assumption. Let $i : B_0 \hookrightarrow B$ be the inclusion. We will say f is \mathbb{R} -simple if

$$i_* R^p f_{0*}\mathbb{R} \cong R^p f_*\mathbb{R}$$

for all p . (We can in general replace \mathbb{R} by any abelian group G , and then we say f is G -simple.) Of course, not all torus fibrations are \mathbb{R} -simple, but it turns out that the most interesting ones which occur in the topological form of SYZ are. So let's assume f and \check{f} are \mathbb{R} -simple. With this assumption, we obtain isomorphisms

$$(1) \quad R^p f_*\mathbb{R} \cong R^{3-p}\check{f}_*\mathbb{R}.$$

We can now use this to study the Leray spectral sequence for f and \check{f} .

Let's make an additional assumption that X and \check{X} are simply connected. So in particular B is simply connected. Let's assume B is a three-manifold. So we have the E_2 terms in the Leray spectral sequence for f :

$$\begin{array}{cccc}
 \mathbb{R} & 0 & 0 & \mathbb{R} \\
 H^0(B, R^2 f_* \mathbb{R}) & H^1(B, R^2 f_* \mathbb{R}) & H^2(B, R^2 f_* \mathbb{R}) & H^3(B, R^2 f_* \mathbb{R}) \\
 H^0(B, R^1 f_* \mathbb{R}) & H^1(B, R^1 f_* \mathbb{R}) & H^2(B, R^1 f_* \mathbb{R}) & H^3(B, R^1 f_* \mathbb{R}) \\
 \mathbb{R} & 0 & 0 & \mathbb{R}
 \end{array}$$

Since X is simply connected, $H^1(X, \mathbb{R}) = H^5(X, \mathbb{R}) = 0$, from which we conclude that $H^0(B, R^1 f_* \mathbb{R}) = H^3(B, R^2 f_* \mathbb{R}) = 0$. The same argument works for \check{f} , and then (1) gives $H^0(B, R^2 f_* \mathbb{R}) \cong H^0(B, R^1 \check{f}_* \mathbb{R}) = 0$ and similarly $H^3(B, R^1 f_* \mathbb{R}) = 0$. Finally, consider the possible non-zero maps for the spectral sequence:

$$\begin{array}{ccccccc}
 \mathbb{R} & & 0 & & 0 & & \mathbb{R} \\
 & \searrow & & & & & \\
 & & & & & & \\
 & & & & & & \\
 0 & & H^1(B, R^2 f_* \mathbb{R}) & & H^2(B, R^2 f_* \mathbb{R}) & & 0 \\
 & & & & & & \\
 0 & & H^1(B, R^1 f_* \mathbb{R}) & & H^2(B, R^1 f_* \mathbb{R}) & & 0 \\
 & & & & & & \\
 \mathbb{R} & & 0 & & 0 & & \mathbb{R}
 \end{array}$$

d_1 (arrow from \mathbb{R} to $H^2(B, R^2 f_* \mathbb{R})$)
 d_2 (arrow from $H^1(B, R^1 f_* \mathbb{R})$ to \mathbb{R})

We need one further assumption, which is again natural given the duality relationship of f_0 and \check{f}_0 : both f and \check{f} possess sections. (Actually, working over \mathbb{R} , we just need the existence of cohomology classes on X and \check{X} which evaluate to something non-zero on a fibre of f or \check{f} .) A section intersects each fibre non-trivially, and hence gives a section of $R^3 f_* \mathbb{R}$. Since such a section also represents a cohomology class on X , d_1 must be the zero map. Similarly, a fibre of f cannot be homologically trivial because it intersects the section non-trivially, and thus the map d_2 must be zero. So the spectral sequence degenerates at E_2 . In particular, we get $\mathbb{R}^{h^{1,1}} \cong H^2(X, \mathbb{R}) \cong H^1(B, R^1 f_* \mathbb{R}) \cong H^1(B, R^2 \check{f}_* \mathbb{R})$ and $\mathbb{R}^{h^{2,2}} \cong H^4(X, \mathbb{R}) \cong H^2(B, R^2 f_* \mathbb{R}) \cong H^2(B, R^1 \check{f}_* \mathbb{R})$, where $h^{p,q}$ are the Hodge numbers of X . Thus the third Betti number of \check{X} is $2 + h^{1,1} + h^{2,2} = 2(1 + h^{1,1})$, so we see $h^{1,1}(X) = h^{1,2}(\check{X})$ and $h^{1,2}(X) = h^{1,1}(\check{X})$.

So modulo some assumptions which would of course eventually have to be justified, it is clear, at least in the three-dimensional case, why the Hodge numbers are interchanged by duality.

More generally, in any dimension, one might hope that \mathbb{R} -simplicity implies $\dim_{\mathbb{R}} H^p(B, R^q f_* \mathbb{R}) = h^{p,q}$, and then a more general exchange of Hodge numbers becomes clear. See Theorem 8.10 for a related result.

REMARK 1.1. This argument can be refined over \mathbb{Z} to make new predictions about the behaviour of *integral* cohomology under mirror symmetry. In [14], Theorem 3.10, it was shown if f and \check{f} are \mathbb{Z} -simple and \mathbb{Q}/\mathbb{Z} -simple, f and \check{f} have

sections, and $H^1(X, \mathbb{Z}) = 0$, then

$$\begin{aligned} H^{even}(X, \mathbb{Z}[1/2]) &\cong H^{odd}(\check{X}, \mathbb{Z}[1/2]) \\ H^{odd}(X, \mathbb{Z}[1/2]) &\cong H^{even}(\check{X}, \mathbb{Z}[1/2]). \end{aligned}$$

There are problems in the argument with two-torsion, but it is likely the above isomorphisms hold over \mathbb{Z} . See [3] for evidence for this latter conjecture.

Enough speculation. Now let's get serious about the structure of special Lagrangian fibrations.

2. Moduli of special Lagrangian submanifolds

The first step in really understanding the SYZ conjecture is to examine the structures which arise on the base of a special Lagrangian fibration. These structures arise from McLean's theorem on the moduli space of special Lagrangian submanifolds [39], and these structures and their relationships were explained by Hitchin in [26]. We outline some of these ideas here. McLean's theorem says that the moduli space of deformations of a compact special Lagrangian submanifold of a compact Calabi-Yau manifold X is unobstructed, with tangent space at $M \subseteq X$ special Lagrangian canonically isomorphic to the space of harmonic 1-forms on M . This isomorphism is seen explicitly as follows. Let $\nu \in \Gamma(M, N_{M/X})$ be a normal vector field to M in X . Then $(\iota(\nu)\omega)|_M$ and $(\iota(\nu)\text{Im}\Omega)|_M$ are both seen to be well-defined forms on M : one needs to lift ν to a vector field but the choice is irrelevant because ω and $\text{Im}\Omega$ restrict to zero on M . McLean shows that the fact that M is special Lagrangian implies

$$\iota(\nu)\text{Im}\Omega = - * \iota(\nu)\omega$$

where $*$ denotes the Hodge star operator on M , and furthermore, ν corresponds to an infinitesimal deformation preserving the special Lagrangian condition if and only if $d(\iota(\nu)\omega) = d(\iota(\nu)\text{Im}\Omega) = 0$. This gives the correspondence between harmonic 1-forms and infinitesimal special Lagrangian deformations.

Let $f : X \rightarrow B$ be a special Lagrangian fibration with torus fibres, and assume for now that all fibres of f are non-singular. Then we obtain three structures on B : two affine structures and a metric, as we shall now see.

DEFINITION 2.1. Let B be an n -dimensional manifold. An *affine structure* on B is given by an atlas $\{(U_i, \psi_i)\}$ of coordinate charts $\psi_i : U_i \rightarrow \mathbb{R}^n$, whose transition functions $\psi_i \circ \psi_j^{-1}$ lie in $\text{Aff}(\mathbb{R}^n)$. We say the affine structure is *tropical* if the transition functions lie in $\mathbb{R}^n \rtimes GL(\mathbb{Z}^n)$, i.e. have integral linear part. We say the affine structure is *integral* if the transition functions lie in $\text{Aff}(\mathbb{Z}^n)$.

If an affine manifold B carries a Riemannian metric g , then we say the metric is *affine Kähler* or *Hessian* if g is locally given by $g_{ij} = \partial^2 K / \partial y_i \partial y_j$ for some convex function K and y_1, \dots, y_n affine coordinates.

Then we obtain the three structures as follows:

Affine structure 1. For a normal vector field ν to a fibre X_b of f , $(\iota(\nu)\omega)|_{X_b}$ is a well-defined 1-form on X_b , and we can compute its periods as follows. Let $U \subseteq B$ be a small open set, and suppose we have submanifolds $\gamma_1, \dots, \gamma_n \subseteq f^{-1}(U)$ which are families of 1-cycles over U and such that $\gamma_1 \cap X_b, \dots, \gamma_n \cap X_b$ form a basis

for $H_1(X_b, \mathbb{Z})$ for each $b \in U$. Consider the 1-forms $\omega_1, \dots, \omega_n$ on U defined by fibrewise integration:

$$\omega_i(\nu) = \int_{X_b \cap \gamma_i} \iota(\nu)\omega,$$

for ν a tangent vector on B at b , which we can lift to a normal vector field of X_b . We have $\omega_i = f_*(\omega|_{\gamma_i})$, and since ω is closed, so is ω_i . Thus there are locally defined functions y_1, \dots, y_n on U with $dy_i = \omega_i$. Furthermore, these functions are well-defined up to the choice of basis of $H_1(X_b, \mathbb{Z})$ and constants. Finally, they give well-defined coordinates, as follows from the fact that $\nu \mapsto \iota(\nu)\omega$ yields an isomorphism of $\mathcal{T}_{B,b}$ with $H^1(X_b, \mathbb{R})$ by McLean's theorem. Thus y_1, \dots, y_n define local coordinates of a tropical affine structure on B .

Affine structure 2. We can play the same trick with $\text{Im } \Omega$: choose submanifolds $\Gamma_1, \dots, \Gamma_n \subseteq f^{-1}(U)$ which are families of $n-1$ -cycles over U and such that $\Gamma_1 \cap X_b, \dots, \Gamma_n \cap X_b$ form a basis for $H^{n-1}(X_b, \mathbb{Z})$. We define λ_i by $\lambda_i = -f_*(\text{Im } \Omega|_{\Gamma_i})$, or equivalently,

$$\lambda_i(\nu) = - \int_{X_b \cap \Gamma_i} \iota(\nu) \text{Im } \Omega.$$

Again $\lambda_1, \dots, \lambda_n$ are closed 1-forms, with $\lambda_i = d\check{y}_i$ locally, and again $\check{y}_1, \dots, \check{y}_n$ are affine coordinates for a tropical affine structure on B .

The McLean metric. The Hodge metric on $H^1(X_b, \mathbb{R})$ is given by

$$g(\alpha, \beta) = \int_{X_b} \alpha \wedge * \beta$$

for α, β harmonic 1-forms, and hence induces a metric on B , which can be written as

$$g(\nu_1, \nu_2) = - \int_{X_b} \iota(\nu_1)\omega \wedge \iota(\nu_2) \text{Im } \Omega.$$

A crucial observation of Hitchin [26] is that these structures are related by the Legendre transform:

PROPOSITION 2.2. *Let y_1, \dots, y_n be local affine coordinates on B with respect to the affine structure induced by ω . Then locally there is a function K on B such that*

$$g(\partial/\partial y_i, \partial/\partial y_j) = \partial^2 K / \partial y_i \partial y_j.$$

Furthermore, $\check{y}_i = \partial K / \partial y_i$ form a system of affine coordinates with respect to the affine structure induced by $\text{Im } \Omega$, and if

$$\check{K}(\check{y}_1, \dots, \check{y}_n) = \sum \check{y}_i y_i - K(y_1, \dots, y_n)$$

is the Legendre transform of K , then

$$y_i = \partial \check{K} / \partial \check{y}_i$$

and

$$\partial^2 \check{K} / \partial \check{y}_i \partial \check{y}_j = g(\partial/\partial \check{y}_i, \partial/\partial \check{y}_j).$$

PROOF. Take families $\gamma_1, \dots, \gamma_n, \Gamma_1, \dots, \Gamma_n$ as above over an open neighbourhood U with the two bases being Poincaré dual, i.e. $(\gamma_i \cap X_b) \cdot (\Gamma_j \cap X_b) = \delta_{ij}$ for $b \in U$. Let $\gamma_1^*, \dots, \gamma_n^*$ and $\Gamma_1^*, \dots, \Gamma_n^*$ be the dual bases for $\Gamma(U, R^1 f_* \mathbb{Z})$ and

$\Gamma(U, R^{n-1}f_*\mathbb{Z})$ respectively. From the choice of γ_i 's, we get local coordinates y_1, \dots, y_n with $dy_i = \omega_i$, so in particular

$$\delta_{ij} = \omega_i(\partial/\partial y_j) = \int_{\gamma_i \cap X_b} \iota(\partial/\partial y_j)\omega,$$

so $\iota(\partial/\partial y_j)\omega$ defines the cohomology class γ_j^* in $H^1(X_b, \mathbb{R})$. Similarly, let

$$g_{ij} = - \int_{\Gamma_i \cap X_b} \iota(\partial/\partial y_j) \text{Im } \Omega;$$

then $-\iota(\partial/\partial y_j) \text{Im } \Omega$ defines the cohomology class $\sum g_{ij} \Gamma_i^*$ in $H^{n-1}(X_b, \mathbb{R})$, and $\lambda_i = \sum g_{ij} dy_j$. Thus

$$\begin{aligned} g(\partial/\partial y_j, \partial/\partial y_k) &= - \int_{X_b} \iota(\partial/\partial y_j)\omega \wedge \iota(\partial/\partial y_k) \text{Im } \Omega \\ &= g_{jk}. \end{aligned}$$

On the other hand, let $\check{y}_1, \dots, \check{y}_n$ be coordinates with $d\check{y}_i = \lambda_i$. Then

$$\partial\check{y}_i/\partial y_j = g_{ij} = g_{ji} = \partial\check{y}_j/\partial y_i,$$

so $\sum \check{y}_i dy_i$ is a closed 1-form. Thus there exists locally a function K such that $\partial K/\partial y_i = \check{y}_i$ and $\partial^2 K/\partial y_i \partial y_j = g(\partial/\partial y_i, \partial/\partial y_j)$. A simple calculation then confirms that $\partial \check{K}/\partial \check{y}_i = y_i$. On the other hand,

$$\begin{aligned} g(\partial/\partial \check{y}_i, \partial/\partial \check{y}_j) &= g\left(\sum_k \frac{\partial y_k}{\partial \check{y}_i} \frac{\partial}{\partial y_k}, \sum_l \frac{\partial y_l}{\partial \check{y}_j} \frac{\partial}{\partial y_l}\right) \\ &= \sum_{k,l} \frac{\partial y_k}{\partial \check{y}_i} \frac{\partial y_l}{\partial \check{y}_j} g(\partial/\partial y_k, \partial/\partial y_l) \\ &= \sum_{k,l} \frac{\partial y_k}{\partial \check{y}_i} \frac{\partial y_l}{\partial \check{y}_j} \frac{\partial \check{y}_k}{\partial y_l} \\ &= \frac{\partial y_j}{\partial \check{y}_i} = \frac{\partial^2 \check{K}}{\partial \check{y}_i \partial \check{y}_j}. \end{aligned}$$

□

Thus we introduce the notion of *Legendre transform* of an affine manifold with a multi-valued convex function.

DEFINITION 2.3. Let B be an affine manifold. A *multi-valued* function K on B is a collection of functions on an open cover $\{(U_i, K_i)\}$ such that on $U_i \cap U_j$, $K_i - K_j$ is affine linear. We say K is convex if the Hessian ($\partial^2 K_i/\partial y_j \partial y_k$) is positive definite for all i , in any, or equivalently all, affine coordinate systems y_1, \dots, y_n .

Given a pair (B, K) of affine manifold and convex multi-valued function, the *Legendre transform* of (B, K) is a pair (\check{B}, \check{K}) where \check{B} is an affine structure on the underlying manifold of B with coordinates given locally by $\check{y}_i = \partial K/\partial y_i$, and \check{K} is defined by

$$\check{K}_i(\check{y}_1, \dots, \check{y}_n) = \sum \check{y}_j y_j - K_i(y_1, \dots, y_n).$$

EXERCISE 2.4. Check that \check{K} is also convex, and that the Legendre transform of (\check{B}, \check{K}) is (B, K) .

3. Semi-flat mirror symmetry

Now let's forget about special Lagrangian fibrations for the moment. Instead, we see how the structures found on B give a toy version of mirror symmetry.

DEFINITION 3.1. Let B be a tropical affine manifold.

- (1) Define $\Lambda \subseteq \mathcal{T}_B$ to be the local system of lattices generated locally by $\partial/\partial y_1, \dots, \partial/\partial y_n$, where y_1, \dots, y_n are local affine coordinates. This is well-defined because transition maps are in $\mathbb{R}^n \rtimes GL_n(\mathbb{Z})$. Set

$$X(B) := \mathcal{T}_B/\Lambda;$$

this is a torus bundle over B . In addition, $X(B)$ carries a complex structure defined locally as follows. Let $U \subseteq B$ be an open set with affine coordinates y_1, \dots, y_n , so \mathcal{T}_U has coordinate functions y_1, \dots, y_n , $x_1 = dy_1, \dots, x_n = dy_n$. Then

$$q_j = e^{2\pi i(x_j + iy_j)}$$

gives a system of holomorphic coordinates on $\mathcal{T}_U/\Lambda|_U$, and the induced complex structure is independent of the choice of affine coordinates.

Later we will need a variant of this: for $\epsilon > 0$, set

$$X(B) := \mathcal{T}_B/\epsilon\Lambda;$$

this has a complex structure with coordinates given by

$$q_j = e^{2\pi i(x_j + iy_j)/\epsilon}.$$

(As we shall see later, the limit $\epsilon \rightarrow 0$ corresponds to a large complex structure limit.)

- (2) Define $\check{\Lambda} \subseteq \mathcal{T}_B^*$ to be the local system of lattices generated locally by dy_1, \dots, dy_n , with y_1, \dots, y_n local affine coordinates. Set

$$\check{X}(B) := \mathcal{T}_B^*/\check{\Lambda}.$$

Of course \mathcal{T}_B^* carries a canonical symplectic structure, and this symplectic structure descends to $\check{X}(B)$. □

We write $f : X(B) \rightarrow B$ and $\check{f} : \check{X}(B) \rightarrow B$ for these torus fibrations; these are clearly dual.

Now suppose in addition we have a Hessian metric g on B , with local potential function K . Then in fact both $X(B)$ and $\check{X}(B)$ become Kähler manifolds:

PROPOSITION 3.2. *$K \circ f$ is a (local) Kähler potential on $X(B)$, defining a Kähler form $\omega = 2i\partial\bar{\partial}(K \circ f)$. This metric is Ricci-flat if and only if K satisfies the real Monge-Ampère equation*

$$\det \frac{\partial^2 K}{\partial y_i \partial y_j} = \text{constant}.$$

PROOF. Working locally with affine coordinates (y_i) and complex coordinates $z_j = \frac{1}{2\pi i} \log q_j = x_j + iy_j$, we compute $\omega = 2i\partial\bar{\partial}(K \circ f) = \frac{i}{2} \sum \frac{\partial^2 K}{\partial y_j \partial y_k} dz_j \wedge d\bar{z}_k$ which is clearly positive. Furthermore, if $\Omega = dz_1 \wedge \dots \wedge dz_n$, then ω^n is proportional to $\Omega \wedge \bar{\Omega}$ if and only if $\det(\partial^2 K/\partial y_j \partial y_k)$ is constant. □

We write this Kähler manifold as $X(B, K)$.
Dually we have

PROPOSITION 3.3. *In local canonical coordinates y_i, \tilde{x}_i on \mathcal{T}_B^* , the functions $z_j = \tilde{x}_j + i\partial K/\partial y_j$ on \mathcal{T}_B^* induce a well-defined complex structure on $\check{X}(B)$, with respect to which the canonical symplectic form ω is a Kähler form of a metric. Furthermore this metric is Ricci-flat if and only if K satisfies the real Monge-Ampère equation*

$$\det \frac{\partial^2 K}{\partial y_j \partial y_k} = \text{constant}.$$

PROOF. It is easy to see that an affine linear change in the coordinates y_j (and hence an appropriate change in the coordinates \tilde{x}_j) results in a linear change of the coordinates z_j , so they induce a well-defined complex structure invariant under $\tilde{x}_j \mapsto \tilde{x}_j + 1$, and hence a complex structure on $\check{X}(B)$. Then one computes that

$$\omega = \sum d\tilde{x}_j \wedge dy_j = \frac{i}{2} \sum g^{jk} dz_j \wedge d\bar{z}_k$$

where $g_{ij} = \partial^2 K/\partial y_j \partial y_k$. Then the metric is Ricci-flat if and only if $\det(g^{jk}) = \text{constant}$, if and only if $\det(g_{jk}) = \text{constant}$. \square

As before, we call this Kähler manifold $\check{X}(B, K)$.
This motivates the definition

DEFINITION 3.4. An affine manifold with metric of Hessian form is a *Monge-Ampère manifold* if the local potential function K satisfies the Monge-Ampère equation $\det(\partial^2 K/\partial y_i \partial y_j) = \text{constant}$.

Monge-Ampère manifolds were first studied by Cheng and Yau in [7].

EXERCISE 3.5. *Show that the identification of \mathcal{T}_B and \mathcal{T}_B^* given by a Hessian metric induces a canonical isomorphism $X(B, K) \cong \check{X}(\check{B}, \check{K})$ of Kähler manifolds, where (\check{B}, \check{K}) is the Legendre transform of (B, K) .*

Finally, we note that a B -field can be introduced into this picture. To keep life relatively simple (so as to avoid having to pass to generalized complex structures [27], [23], [4]), we view the B -field as an element $\mathbf{B} \in H^1(B, \Lambda_{\mathbb{R}}/\Lambda)$, where $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Noting that a section of $\Lambda_{\mathbb{R}}/\Lambda$ over an open set U can be viewed as a section of $\mathcal{T}_U/\Lambda|_U$, such a section acts on $\mathcal{T}_U/\Lambda|_U$ via translation, and this action is in fact holomorphic with respect to the standard semi-flat complex structure. Thus a Čech 1-cocycle (U_{ij}, β_{ij}) representing \mathbf{B} allows us to reglue $X(B)$ via translations over the intersections U_{ij} . This gives a new complex manifold $X(B, \mathbf{B})$. If in addition there is a multi-valued potential function K defining a metric, these translations preserve the metric and yield a Kähler manifold $X(B, \mathbf{B}, K)$.

Thus the full toy version of mirror symmetry is as follows. The data consists of an affine manifold B with potential K and B -fields $\mathbf{B} \in H^1(B, \Lambda_{\mathbb{R}}/\Lambda)$, $\check{\mathbf{B}} \in H^1(B, \check{\Lambda}_{\mathbb{R}}/\check{\Lambda})$. Now it is not difficult to see, and you will have seen this already if you've done Exercise 3.5, that the local system $\check{\Lambda}$ defined using the affine structure on B is the same as the local system Λ defined using the affine structure on \check{B} . So we say the pair

$$(X(B, \mathbf{B}, K), \check{\mathbf{B}})$$

is mirror to

$$(X(\check{B}, \check{\mathbf{B}}, \check{K}), \mathbf{B}).$$

This provides a reasonably fulfilling picture of mirror symmetry in a simple context. Many more aspects of mirror symmetry can be worked out in this semi-flat context, see [36]. However, ultimately this only sheds limited insight into the general case. The only compact Calabi-Yau manifolds with semi-flat Ricci-flat metric which arise in this way are complex tori (shown by Cheng and Yau in [7]). To deal with more interesting cases, we need to allow singular fibres, and hence, singularities in the affine structure of B .

4. Affine manifolds with singularities

To deal with singular fibres, we define

DEFINITION 4.1. A *(tropical, integral) affine manifold with singularities* is a (C^0) manifold B with an open subset $B_0 \subseteq B$ which carries a (tropical, integral) affine structure, and such that $\Gamma := B \setminus B_0$ is a locally finite union of locally closed submanifolds of codimension ≥ 2 .

By way of example, let's explain how the Batyrev construction gives rise to a wide class of such manifolds. This construction is taken from [17], where a more combinatorially complicated version is given for complete intersections; see [24] and [25] for an alternative construction.

Let Δ be a reflexive polytope in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, where $M = \mathbb{Z}^n$; let N be the dual lattice, $\nabla \subseteq N_{\mathbb{R}}$ the dual polytope given by

$$\nabla := \{n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq -1 \text{ for all } m \in \Delta\}.$$

We assume $0 \in \Delta$ is the unique interior lattice point of Δ . Let Σ be the normal fan to Δ , consisting of cones over faces of ∇ , and similarly let $\check{\Sigma}$ be the normal fan to ∇ . Suppose we are given star subdivisions of ∇ and Δ , inducing subdivisions Σ' , $\check{\Sigma}'$ of the fans Σ , $\check{\Sigma}$, and in addition suppose that

$$\begin{aligned} h &: N_{\mathbb{R}} \rightarrow \mathbb{R} \\ \check{h} &: M_{\mathbb{R}} \rightarrow \mathbb{R} \end{aligned}$$

are (upper) strictly convex piecewise linear functions on the fans Σ' , $\check{\Sigma}'$ respectively. Also, let

$$\begin{aligned} \varphi &: N_{\mathbb{R}} \rightarrow \mathbb{R} \\ \check{\varphi} &: M_{\mathbb{R}} \rightarrow \mathbb{R} \end{aligned}$$

be the piecewise linear functions representing the anti-canonical classes of the toric varieties \mathbb{P}_{Δ} and \mathbb{P}_{∇} respectively; i.e. φ takes the value 1 on the primitive generator of each one-dimensional cone of Σ . Finally, assume that h and \check{h} are chosen so that $h' := h - \varphi$ and $\check{h}' = \check{h} - \check{\varphi}$ are (not necessarily strictly) convex functions.

Define, for any convex piecewise linear function \check{g} on the fan $\check{\Sigma}'$,

$$\nabla^{\check{g}} := \{n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq -\check{g}(m) \text{ for all } m \in M_{\mathbb{R}}\}.$$

In particular,

$$\nabla^{\check{h}} = \nabla^{\check{h}'} + \nabla^{\check{\varphi}} = \nabla^{\check{h}'} + \nabla,$$

where $+$ denotes Minkowski sum. Our goal will be to put an affine structure with singularities on $B = \partial \nabla^{\check{h}}$; however, this will require some choices.

Set

$$\tilde{\Delta} := \{(m, l) | m \in \Delta, \quad l \geq \check{h}'(m)\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}.$$

The assumption that \check{h}' is convex implies $\tilde{\Delta}$ is convex, but of course is non-compact. Let $\tilde{\Sigma}$ be the normal fan to $\tilde{\Delta}$ in $N_{\mathbb{R}} \oplus \mathbb{R}$. If $C(\sigma)$ denotes the cone over a set σ in $N_{\mathbb{R}} \oplus \mathbb{R}$, it is an exercise in convex sets to show that $\tilde{\Sigma}$ consists of three types of cones:

- (1) $C(\sigma \times \{0\})$ where σ is a face of ∇ ;
- (2) $C(\tau \times \{1\})$ where τ is a face of $\nabla^{\check{h}'}$;
- (3) $C(\sigma \times \{0\}) + C(\tau \times \{1\})$ where σ is a face of ∇ , τ is a face of $\nabla^{\check{h}'}$, and $\sigma + \tau$ is a face of $\nabla + \nabla^{\check{h}'} = \nabla^{\check{h}}$.

Note that if \check{h} is not integral, i.e. has non-integral slopes, then we don't expect $\tilde{\Sigma}$ to be a *rational* fan, though we will impose this condition in some contexts later.

Here is the choice we have to make in this construction.

DEFINITION 4.2. A subdivision of $\tilde{\Sigma}'$ of the fan $\tilde{\Sigma}$ is *good* if it satisfies the two properties

- (1) The fan $\{\sigma \cap (N_{\mathbb{R}} \times \{0\}) | \sigma \in \tilde{\Sigma}'\}$ coincides with Σ' .
- (2) Every one-dimensional cone of $\tilde{\Sigma}'$ not contained in $N_{\mathbb{R}} \times \{0\}$ is contained in $C(\nabla^{\check{h}'} \times \{1\})$.

□

If $\tilde{\Sigma}'$ is good, then cones in $\tilde{\Sigma}'$ can be divided up into three sorts, depending on which type of cone of $\tilde{\Sigma}$ they are contained in.

- (1) Cones of the form $C(\sigma \times \{0\})$ for $C(\sigma) \in \Sigma'$ with $\sigma \subseteq \partial\nabla$.
- (2) Cones contained in $C(\nabla^{\check{h}'} \times \{1\})$.
- (3) Cones of the form $C(\sigma \times \{0\}) + C(\tau \times \{1\})$ where $C(\sigma) \in \Sigma'$ for $\sigma \subseteq \partial\nabla$, $\tau \subseteq \partial\nabla^{\check{h}'}$, $\sigma + \tau$ contained in a face of $\nabla^{\check{h}}$.

Set

$$B := \partial\nabla^{\check{h}}.$$

The choice of the fan $\tilde{\Sigma}'$ yields a decomposition of B into a face-fitting collection of polytopes \mathcal{P} as follows. Identify $\nabla^{\check{h}} \subseteq N_{\mathbb{R}}$ with $\nabla^{\check{h}} \times \{1\} \subseteq N_{\mathbb{R}} \oplus \mathbb{R}$. Set

$$\mathcal{P} = \{\sigma \cap \partial(\nabla^{\check{h}} \times \{1\}) | \sigma \in \tilde{\Sigma}'\}.$$

By our description of the cones of $\tilde{\Sigma}'$, only cones of the third type contribute, and

$$(2) \quad (C(\sigma \times \{0\}) + C(\tau \times \{1\})) \cap (\partial(\nabla^{\check{h}} \times \{1\})) = (\sigma + \tau) \times \{1\}$$

when $C(\sigma \times \{0\}) + C(\tau \times \{1\}) \in \tilde{\Sigma}'$ is a cone of the third type.

Let $\text{Bar}(\mathcal{P})$ denote the first barycentric subdivision of \mathcal{P} and let $\Gamma \subseteq B$ be the union of all simplices of $\text{Bar}(\mathcal{P})$ not containing a vertex of \mathcal{P} (a zero-dimensional cell) or intersecting the interior of a maximal cell of \mathcal{P} . If we then set $B_0 := B \setminus \Gamma$, we can define an affine structure on B_0 as follows. B_0 has an open cover

$$\{W_{\sigma} | \sigma \in \mathcal{P} \text{ maximal}\} \cup \{W_v | v \in \mathcal{P} \text{ a vertex}\}$$

where $W_{\sigma} = \text{Int}(\sigma)$, the interior of σ , and

$$W_v = \bigcup_{\substack{\tau \in \text{Bar}(\mathcal{P}) \\ v \in \tau}} \text{Int}(\tau)$$

is the (open) star of v in $\text{Bar}(\mathcal{P})$. We define an affine chart

$$\psi_\sigma : W_\sigma \rightarrow \mathbb{A}^{n-1} \subseteq N_{\mathbb{R}}$$

given by the inclusion of W_σ in \mathbb{A}^{n-1} , the affine hyperplane containing σ . Also, take $\psi_v : W_v \rightarrow N_{\mathbb{R}}/\mathbb{R}v'$ to be the projection, where v can be written uniquely by (2) as $v' + v''$ with v' an integral point of ∇ and $v'' \in \nabla^{\check{h}'}$. One checks easily that for $v \in \sigma$, $\psi_\sigma \circ \psi_v^{-1}$ is affine linear with integral linear part (integrality follows from reflexivity of Δ !) so B is a tropical affine manifold with singularities.

DEFINITION 4.3. We say $\tilde{\Sigma}'$ is an *integral subdivision* of $\tilde{\Sigma}$ if the fan $\tilde{\Sigma}'$ is a *rational* polyhedral fan and every one-dimensional cone of $\tilde{\Sigma}'$ not contained in $N_{\mathbb{R}} \times \{0\}$ is generated by a *primitive integral* vector $(n, 1)$ with $n \in \nabla^{\check{h}'}$. In particular \check{h} must be an integral function.

If $\tilde{\Sigma}'$ is an integral subdivision, then B is in fact an integral affine manifold with singularities.

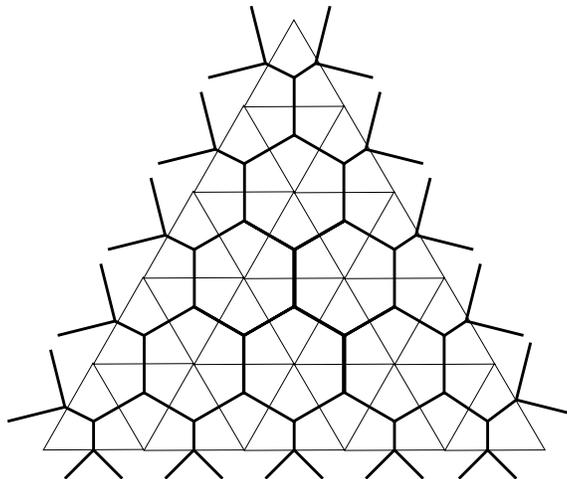
EXAMPLE 4.4. Let $\Delta \subseteq \mathbb{R}^4$ be the convex hull of the points

$$(-1, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1),$$

so ∇ is the convex hull of the points

$$(-1, -1, -1, -1), (4, -1, -1, -1), (-1, 4, -1, -1), (-1, -1, 4, -1), (-1, -1, -1, 4).$$

Take $\check{h} = \check{\varphi}$ and h a piecewise linear function defining an ample divisor on a crepant resolution of \mathbb{P}_Δ corresponding to some star triangulation of ∇ . In this case $B = \partial\nabla$. It is easy to see the affine structure on B_0 in fact extends across the interior of all three-dimensional faces of ∇ . This gives a smaller discriminant locus Γ which, given a nice regular triangulation of ∇ , looks like the following picture in a neighbourhood of a 2-face of ∇ : the light lines giving the triangulation and the dark lines the discriminant locus Γ .



In this picture, Γ is a trivalent graph, with two types of trivalent vertices. The ones along the edge are non-planar: the two additional legs of Γ drawn in this picture are contained in other two-faces of ∇ . However Γ is planar in the interior of this two-face.

In general, if the subdivisions Σ' and $\check{\Sigma}'$ of Σ and $\check{\Sigma}$ respectively represent maximal projective crepant partial resolutions of \mathbb{P}_Δ and \mathbb{P}_∇ , and $\dim B = 3$, then only these sorts of trivalent vertices occur. More specifically, one knows what the monodromy of the local systems Λ and $\check{\Lambda}$ on B_0 look like at these vertices. If $v \in \Gamma$ is a vertex contained in the interior of a two-face, then it is clear that the tangent space to that two-face is invariant under parallel transport, in a neighbourhood of v , of Λ . A more careful analysis yields that the monodromy matrices for Λ take the form, in a suitable basis,

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Here T_1, T_2, T_3 are given by parallel transport about loops around the three edges of Γ coming out of v . Of course, the monodromy of $\check{\Lambda}$ is the transpose inverse of these matrices. Similarly, if v is a vertex of Γ contained in an edge of $\nabla^{\check{h}}$, then the monodromy will take the form

$$T_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, T_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So we see that the monodromy of the two types of vertices are interchanged by looking at Λ and $\check{\Lambda}$.

One main result of [15] is

THEOREM 4.5. *If B is a three-dimensional tropical affine manifold with singularities such that Γ is trivalent and the monodromy of Λ at each vertex is one of the above two types, then $f_0 : X(B_0) \rightarrow B_0$ can be compactified to a topological fibration $f : X(B) \rightarrow B$. Dually, $\check{f}_0 : \check{X}(B_0) \rightarrow B_0$ can be compactified to a topological fibration $\check{f} : \check{X}(B) \rightarrow B$.*

We won't give any details here of how this is carried out, but it is not particularly difficult, as long as one restricts to the category of topological (not C^∞) manifolds. However, it is interesting to look at the singular fibres we need to add in this compactification.

If $b \in \Gamma$ is a point which is not a vertex of Γ , then $f^{-1}(b)$ is homeomorphic to $I_1 \times S^1$, where I_1 denotes a Kodaira type I_1 elliptic curve, i.e. a pinched torus.

If b is a vertex of Γ , with monodromy of the first type, then $f^{-1}(b) = S^1 \times S^1 \times S^1 / \sim$, with $(a, b, c) \sim (a', b', c')$ if $(a, b, c) = (a', b', c')$ or $a = a' = 1$, where S^1 is identified with the unit circle in \mathbb{C} . This is the three-dimensional analogue of a pinched torus, and $\chi(f^{-1}(b)) = +1$. We call this a *positive* fibre.

If b is a vertex of Γ , with monodromy of the second type, then $f^{-1}(b)$ can be described as $S^1 \times S^1 \times S^1 / \sim$, with $(a, b, c) \sim (a', b', c')$ if $(a, b, c) = (a', b', c')$ or $a = a' = 1, b = b'$, or $a = a', b = b' = 1$. The singular locus of this fibre is a figure eight, and $\chi(f^{-1}(b)) = -1$. We call this a *negative* fibre.

So we see a very concrete local consequence of SYZ duality: namely in the compactifications $X(B)$ and $\check{X}(B)$, the positive and negative fibres are interchanged. Of course, this results in the observation that Euler characteristic changes sign under mirror symmetry for Calabi-Yau threefolds.

EXAMPLE 4.6. Continuing with Example 4.4, it was proved in [15] that $\check{X}(B)$ is homeomorphic to the quintic and $X(B)$ is homeomorphic to the mirror quintic.

REMARK 4.7. Haase and Zharkov in [24] gave a different description of what is the same affine structure. Their construction has the advantage that it is manifestly dual. In other words, in our construction, we can interchange the role of Δ and ∇ to get two different affine manifolds, with B_Δ the affine manifold with singularities structure on $\partial\Delta^h$ and B_∇ the affine manifold with singularities structure on $\partial\nabla^h$. It is not obvious that these are “dual” affine manifolds, at least in the sense that $X(B_\nabla)$ is homeomorphic to $\check{X}(B_\Delta)$ and $\check{X}(B_\Delta)$ is homeomorphic to $X(B_\nabla)$. In the construction given above, this follows from the discrete Legendre transform we will discuss in §7. On the other hand, the construction I give here will arise naturally from the degeneration construction discussed later in this paper.

Ruan in [45] gave a description of *Lagrangian* torus fibrations for hypersurfaces in toric varieties using a symplectic flow argument, and his construction should coincide with a *symplectic* compactification of the symplectic manifolds $\check{X}(B_0)$. However, this symplectic compactification in three dimensions has not yet been constructed due to technical problems near the negative vertices, but see forthcoming work of Ricardo Castaño-Bernard and Diego Matessi. In addition, if these compactifications can be constructed, it has not yet been shown they coincide with a symplectic Calabi-Yau manifold.

I should also point out that the explicit compactifications mentioned in three dimensions can be carried out in all dimensions, and will be done so in [19]. We will show there in a much more general context that these compactifications are then homeomorphic to the expected Calabi-Yau manifolds.

5. The problems with the SYZ conjecture, and how to get around them

The previous section demonstrates that the SYZ conjecture gives a beautiful description of mirror symmetry at a purely topological level. This, by itself, can often be useful, but unfortunately is not strong enough to get at really interesting aspects of mirror symmetry, such as instanton corrections. For a while, though, many of us were hoping that the strong version of duality we have just seen would hold at the special Lagrangian level. This would mean that a mirror pair X, \check{X} would possess special Lagrangian torus fibrations $f : X \rightarrow B$ and $\check{f} : \check{X} \rightarrow B$ with codimension two discriminant locus, and the discriminant loci of f and \check{f} would coincide. These fibrations would then be dual away from the discriminant locus.

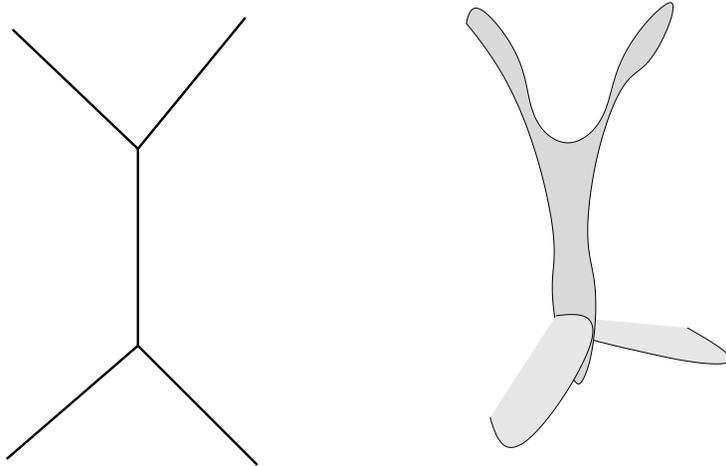
There are examples of special Lagrangian fibrations on non-compact toric varieties X with this behaviour. In particular, if $\dim X = n$ with a T^{n-1} action on X preserving the holomorphic n -form, and if X in addition carries a Ricci-flat metric which is invariant under this action, then X will have a very nice special Lagrangian fibration with codimension two discriminant locus. (See [16] and [12]). However, Dominic Joyce ([29] and other papers cited therein) began studying some three-dimensional S^1 -invariant examples, and discovered quite different behaviour. There is an argument that if a special Lagrangian fibration is C^∞ , then the discriminant locus will be (Hausdorff) codimension two. However, Joyce discovered examples which were not differentiable, but only piecewise differentiable, and furthermore, had a codimension one discriminant locus:

EXAMPLE 5.1. Define $F : \mathbb{C}^3 \rightarrow \mathbb{R} \times \mathbb{C}$ by $F(z_1, z_2, z_3) = (a, c)$ with $2a = |z_1|^2 - |z_2|^2$ and

$$c = \begin{cases} z_3 & a = z_1 = z_2 = 0 \\ z_3 - \bar{z}_1 \bar{z}_2 / |z_1| & a \geq 0, z_1 \neq 0 \\ z_3 - \bar{z}_1 \bar{z}_2 / |z_2| & a < 0. \end{cases}$$

It is easy to see that if $a \neq 0$, then $F^{-1}(a, c)$ is homeomorphic to $\mathbb{R}^2 \times S^1$, while if $a = 0$, then $F^{-1}(a, c)$ is a cone over T^2 : essentially, one copy of S^1 in $\mathbb{R}^2 \times S^1$ collapses to a point. In addition, all fibres of this map are special Lagrangian, and it is obviously only piecewise smooth. The discriminant locus is the entire plane given by $a = 0$.

This example forces a reevaluation of the strong form of the SYZ conjecture. In further work Joyce found evidence for a more likely picture for general special Lagrangian fibrations in three dimensions. The discriminant locus, instead of being a codimension two graph, will be a codimension one blob. Typically the union of the singular points of singular fibres will be a Riemann surface, and it will map to an amoeba shaped set in B , i.e. the discriminant locus looks like the picture on the right rather than the left, and will be a fattening of the old picture of a codimension two discriminant.



Joyce made some additional arguments to suggest that this fattened discriminant locus must look fundamentally different in a neighbourhood of the two basic types of vertices we saw in the previous section, with the two types of vertices expected to appear pretty much as depicted in the above picture. Thus the strong form of duality mentioned above, where we expect the discriminant loci of the special Lagrangian fibrations on a mirror pair to be the same, cannot hold. If this is the case, one needs to replace this strong form of duality with a weaker form.

It seems likely that the best way to rephrase the SYZ conjecture is in a limiting form. Mirror symmetry as we currently understand it has to do with degenerations of Calabi-Yau manifolds. Given a flat family $f : \mathcal{X} \rightarrow D$ over a disk D , with the fibre \mathcal{X}_0 over 0 singular and all other fibres n -dimensional Calabi-Yau manifolds, we say the family is *maximally unipotent* if the monodromy transformation $T : H^n(\mathcal{X}_t, \mathbb{Q}) \rightarrow H^n(\mathcal{X}_t, \mathbb{Q})$ ($t \in D$ non-zero) satisfies $(T - I)^{n+1} = 0$ but $(T - I)^n \neq 0$. It is a standard fact of mirror symmetry that mirrors should be associated to

maximally unipotent degenerations of Calabi-Yau manifolds. In particular, given two different maximally unipotent degenerations in a single complex moduli space for some Calabi-Yau manifold, one might obtain different mirror manifolds.

We recall the definition of Gromov-Hausdorff convergence, a notion of convergence of a sequence of metric spaces.

DEFINITION 5.2. Let (X, d_X) , (Y, d_Y) be two compact metric spaces. Suppose there exists maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ (not necessarily continuous) such that for all $x_1, x_2 \in X$,

$$|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \epsilon$$

and for all $x \in X$,

$$d_X(x, g \circ f(x)) < \epsilon,$$

and the two symmetric properties for Y hold. Then we say the Gromov-Hausdorff distance between X and Y is at most ϵ . The Gromov-Hausdorff distance $d_{GH}(X, Y)$ is the infimum of all such ϵ .

It follows from results of Gromov (see for example [43], pg. 281, Cor. 1.11) that the space of compact Ricci-flat manifolds with diameter $\leq C$ is precompact with respect to Gromov-Hausdorff distance, i.e. any sequence of such manifolds has a subsequence converging with respect to the Gromov-Hausdorff distance to a metric space. This metric space could be quite bad; this is quite outside the realm of algebraic geometry! Nevertheless, this raises the following natural question. Given a maximally unipotent degeneration of Calabi-Yau manifolds $\mathcal{X} \rightarrow D$, take a sequence $t_i \in D$ converging to 0, and consider a sequence $(\mathcal{X}_{t_i}, g_{t_i})$, where g_{t_i} is a choice of Ricci-flat metric chosen so that $Diam(g_{t_i})$ remains bounded. What is the Gromov-Hausdorff limit of $(\mathcal{X}_{t_i}, g_{t_i})$, or the limit of some convergent subsequence?

EXAMPLE 5.3. Consider a degenerating family of elliptic curves, with periods 1 and $\frac{1}{2\pi i} \log t$. If we take t approaching 0 along the positive real axis, then we can just view this as a family of elliptic curves \mathcal{X}_α with period 1 and $i\alpha$ with $\alpha \rightarrow \infty$. If we take the standard Euclidean metric g on \mathcal{X}_α , then the diameter of \mathcal{X}_α is unbounded. To obtain a bounded diameter, we replace g by g/α^2 ; equivalently, we can keep g fixed on \mathbb{C} but change the periods of the elliptic curve to $1/\alpha, i$. It then becomes clear that the Gromov-Hausdorff limit of such a sequence of elliptic curves is a circle \mathbb{R}/\mathbb{Z} .

This simple example motivates the first conjecture about maximally unipotent degenerations, conjectured independently by myself and Wilson on the one hand [22] and Kontsevich and Soibelman [34] on the other.

CONJECTURE 5.4. *Let $\mathcal{X} \rightarrow D$ be a maximally unipotent degeneration of simply-connected Calabi-Yau manifolds with full $SU(n)$ holonomy, $t_i \in D$ with $t_i \rightarrow 0$, and let g_i be a Ricci-flat metric on \mathcal{X}_{t_i} normalized to have fixed diameter C . Then a convergent subsequence of (\mathcal{X}_{t_i}, g_i) converges to a metric space (X_∞, d_∞) , where X_∞ is homeomorphic to S^n . Furthermore, d_∞ is induced by a Riemannian metric on $X_\infty \setminus \Gamma$, where $\Gamma \subseteq X_\infty$ is a set of codimension two.*

Here the topology of the limit depends on the nature of the non-singular fibres \mathcal{X}_t ; for example, if instead \mathcal{X}_t was hyperkähler, then we would expect the limit to be a projective space. Also, even in the case of full $SU(n)$ holonomy, if \mathcal{X}_t is not simply connected, we would expect limits such as \mathbb{Q} -homology spheres to arise.

Conjecture 5.4 is directly inspired by the SYZ conjecture. Suppose we had special Lagrangian fibrations $f_i : \mathcal{X}_{t_i} \rightarrow B_i$. Then as the maximally unipotent degeneration is approached, it is possible to see that the volume of the fibres of these fibrations go to zero. This would suggest these fibres collapse, hopefully leaving the base as the limit.

This conjecture was proved by myself and Wilson for K3 surfaces in [22]. The proof relies on a detailed analysis of the behaviour of Ricci-flat metrics in the limit, and also on the existence of explicit local models for Ricci-flat metrics near singular fibres of special Lagrangian fibrations.

The motivation for this conjecture from SYZ also provides a limiting form of the conjecture. There are any number of problems with trying to prove the existence of special Lagrangian fibrations on Calabi-Yau manifolds. Even the existence of a single special Lagrangian torus near a maximally unipotent degeneration is unknown, but we expect it should be easier to find them as we approach the maximally unipotent point. Furthermore, even if we find a special Lagrangian torus, we know that it moves in an n -dimensional family, but we don't know its deformations fill out the entire manifold. In addition, there is no guarantee that even if it does, we obtain a foliation of the manifold: nearby special Lagrangian submanifolds may intersect. (For an example, see [38].) So instead, we will just look at the moduli space of special Lagrangian tori.

Suppose, given $t_i \rightarrow 0$, that for t_i sufficiently close to zero, there is a special Lagrangian T^n whose homology class is invariant under monodromy, or more specifically, generates the space W_0 of the monodromy weight filtration (this is where we expect to find fibres of a special Lagrangian fibration associated to a maximally unipotent degeneration). Let $B_{0,i}$ be the moduli space of deformations of this torus; every point of $B_{0,i}$ corresponds to a smooth special Lagrangian torus in \mathcal{X}_{t_i} . This manifold then comes equipped with the McLean metric and affine structures defined in §2. One can then compactify $B_{0,i} \subseteq B_i$, (probably by taking the closure of $B_{0,i}$ in the space of special Lagrangian currents; the details aren't important here). This gives a series of metric spaces (B_i, d_i) with the metric d_i induced by the McLean metric. If the McLean metric is normalized to keep the diameter of B_i constant independent of i , then we can hope that (B_i, d_i) converges to a compact metric space (B_∞, d_∞) . Here then is the limiting form of SYZ:

CONJECTURE 5.5. *If (\mathcal{X}_{t_i}, g_i) converges to (X_∞, g_∞) and (B_i, d_i) is non-empty for large i and converges to (B_∞, d_∞) , then B_∞ and X_∞ are isometric up to scaling. Furthermore, there is a subspace $B_{\infty,0} \subseteq B_\infty$ with $\Gamma := B_\infty \setminus B_{\infty,0}$ of Hausdorff codimension 2 in B_∞ such that $B_{\infty,0}$ is a Monge-Ampère manifold, with the Monge-Ampère metric inducing d_∞ on $B_{\infty,0}$.*

Essentially what this is saying is that as we approach the maximally unipotent degeneration, we expect to have a special Lagrangian fibration on larger and larger subsets of \mathcal{X}_{t_i} . Furthermore, in the limit, the codimension one discriminant locus suggested by Joyce converges to a codimension two discriminant locus, and (the not necessarily Monge-Ampère, see [38]) Hessian metrics on $B_{0,i}$ converge to a Monge-Ampère metric.

The main point I want to get at here is that it is likely the SYZ conjecture is only “approximately” correct, and one needs to look at the limit to have a hope of proving anything. On the other hand, the above conjecture seems likely to be

accessible by currently understood techniques, though with a lot of additional work, and I wouldn't be surprised to see it proved in the next few years.

How do we do mirror symmetry using this modified version of the SYZ conjecture? Essentially, we would follow these steps:

- (1) We begin with a maximally unipotent degeneration of Calabi-Yau manifolds $\mathcal{X} \rightarrow D$, along with a choice of polarization. This gives us a Kähler class $[\omega_t] \in H^2(\mathcal{X}_t, \mathbb{R})$ for each $t \in D \setminus 0$, represented by ω_t the Kähler form of a Ricci-flat metric g_t .
- (2) Identify the Gromov-Hausdorff limit of a sequence $(\mathcal{X}_{t_i}, r_i g_{t_i})$ with $t_i \rightarrow 0$, and r_i a scale factor which keeps the diameter of \mathcal{X}_{t_i} constant. The limit will be, if the above conjectures work, an affine manifold with singularities B along with a Monge-Ampère metric.
- (3) Perform a Legendre transform to obtain a new affine manifold with singularities \check{B} , though with the same metric.
- (4) Try to construct a compactification of $X_\epsilon(\check{B}_0)$ for small $\epsilon > 0$ to obtain a complex manifold $X_\epsilon(\check{B})$. This will be the mirror manifold.

Actually, we need to elaborate on this last step a bit more. The problem is that while we expect that it should be possible in general to construct symplectic compactifications of the symplectic manifold $\check{X}(B_0)$ (and hence get the mirror as a symplectic manifold), we don't expect to be able to compactify $X_\epsilon(\check{B}_0)$ as a complex manifold. Instead, the expectation is that a small deformation of $X_\epsilon(\check{B}_0)$ is necessary before it can be compactified. Furthermore, this small deformation is critically important in mirror symmetry: *it is this small deformation which provides the B-model instanton corrections.*

Because of the importance of this last issue, it has already been studied by several authors: Fukaya in [9] has studied the problem directly using heuristic ideas, while Kontsevich and Soibelman [35] have modified the problem of passing from an affine manifold to a complex manifold by instead producing a non-Archimedean space. We will return to these issues later in this paper, when I discuss my own work with Siebert which has been partly motivated by the same problem. Because this last item is so important, let's give it a name:

QUESTION 5.6 (The reconstruction problem, Version I). *Given a tropical affine manifold with singularities B , construct a complex manifold $X_\epsilon(B)$ which is a compactification of a small deformation of $X_\epsilon(B_0)$.*

I do not wish to dwell further on this version of the SYZ conjecture here, because it lies mostly in the realm of analysis and differential geometry and the behaviour of Ricci-flat metrics, and will give us little insight into what makes subtler aspects of traditional mirror symmetry work: for example, how exactly do instanton corrections arise? So we move on to explain how the limiting form of SYZ inspired a more algebro-geometric form of SYZ, which in turn avoids all analytic problems and holds out great promise for understanding fundamental mysteries of mirror symmetry.

6. Gromov-Hausdorff limits, algebraic degenerations, and mirror symmetry

We now have two notions of limit: the familiar algebro-geometric notion of a flat degenerating family $\mathcal{X} \rightarrow D$ over a disk on the one hand, and the Gromov-Hausdorff limit on the other. Kontsevich had an important insight (see [34]) into the connection between these two. In this section I will give a rough idea of how and why this works.

Very roughly speaking, the Gromov-Hausdorff limit $(\mathcal{X}_{t_i}, g_{t_i})$ as $t_i \rightarrow 0$ should coincide, topologically, with the dual intersection complex of the singular fibre \mathcal{X}_0 . More precisely, in a relatively simple situation, suppose $f : \mathcal{X} \rightarrow D$ is relatively minimal (in the sense of Mori) and normal crossings, with \mathcal{X}_0 having irreducible components X_1, \dots, X_m . The dual intersection complex of \mathcal{X}_0 is the simplicial complex with vertices v_1, \dots, v_m , and which contains a simplex $\langle v_{i_0}, \dots, v_{i_p} \rangle$ if $X_{i_0} \cap \dots \cap X_{i_p} \neq \emptyset$.

Let us explain roughly why this should be, first by looking at a standard family of degenerating elliptic curves with periods 1 and $\frac{n}{2\pi i} \log t$ for n a positive integer. Such a family over the punctured disk is extended to a family over the disk by adding an I_n (a cycle of n rational curves) fibre over the origin.

Taking a sequence $t_i \rightarrow 0$ with t_i real and positive gives a sequence of elliptic curves of the form $X_{\epsilon_i}(B)$ where: $B = \mathbb{R}/n\mathbb{Z}$ and $\epsilon_i = -\frac{2\pi}{\ln t_i}$. In addition, the metric on $X_{\epsilon_i}(B)$, properly scaled, comes from the constant Hessian metric on B . So we wish to explain how B is related to the geometry near the singular fibre. To this end, let X_1, \dots, X_n be the irreducible components of \mathcal{X}_0 ; these are all \mathbb{P}^1 's. Let P_1, \dots, P_n be the singular points of \mathcal{X}_0 .

We'll consider two sorts of open sets in \mathcal{X} . For the first type, choose a coordinate z on X_i , with P_i given by $z = 0$ and P_{i+1} given by $z = \infty$. Let $U_i \subseteq D_i$ be the open set $\{z | \delta \leq |z| \leq 1/\delta\}$ for some small fixed δ . Then one can find a neighbourhood \tilde{U}_i of U_i in \mathcal{X} such that \tilde{U}_i is biholomorphic to $U_i \times D_\rho$ for $\rho > 0$ sufficiently small, D_ρ a disk of radius ρ in \mathbb{C} , and $f|_{\tilde{U}_i}$ is the projection onto D_ρ .

On the other hand, each P_i has a neighbourhood \tilde{V}_i in \mathcal{X} biholomorphic to a polydisk $\{(z_1, z_2) \in \mathbb{C}^2 | |z_1| \leq \delta', |z_2| \leq \delta'\}$ on which f takes the form $z_1 z_2$.

If δ and δ' are chosen correctly, then for t sufficiently close to zero,

$$\{\tilde{V}_i \cap \mathcal{X}_t | 1 \leq i \leq n\} \cup \{\tilde{U}_i \cap \mathcal{X}_t | 1 \leq i \leq n\}$$

form an open cover of \mathcal{X}_t . Now each of the sets in this open cover can be written as $X_\epsilon(U)$ for some U a one-dimensional (non-compact) affine manifold and $\epsilon = -2\pi/\ln |t|$. If U is an open interval $(a, b) \subseteq \mathbb{R}$, then $X_\epsilon(U)$ is biholomorphic to the annulus

$$\{z \in \mathbb{C} | e^{-2\pi b/\epsilon} \leq |z| \leq e^{-2\pi a/\epsilon}\}$$

as $q = e^{2\pi i(x+iy)/\epsilon}$ is a holomorphic coordinate on $X_\epsilon((a, b))$. Thus

$$\tilde{U}_i \cap \mathcal{X}_t \cong X_\epsilon \left(\left(\frac{\epsilon \ln \delta}{2\pi}, -\frac{\epsilon \ln \delta}{2\pi} \right) \right)$$

with $\epsilon = -2\pi/\ln |t|$. As $t \rightarrow 0$, the interval $(\epsilon \ln \delta/2\pi, -\epsilon \ln \delta/2\pi)$ shrinks to a point. So $\tilde{U}_i \cap \mathcal{X}_t$ is a smaller and smaller open subset of \mathcal{X}_t as $t \rightarrow 0$ when we view things in this way. This argument suggests that every irreducible component should be associated to a point on B .

Now look at $\tilde{V}_i \cap \mathcal{X}_t$. This is

$$\begin{aligned} \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|, |z_2| < \delta', z_1 z_2 = t\} &\cong \{z \in \mathbb{C} \mid |t|/\delta' \leq |z| \leq \delta'\} \\ &\cong X_\epsilon \left(\frac{-\epsilon}{2\pi} \ln \delta', \frac{\epsilon}{2\pi} (\ln \delta' - \ln |t|) \right) \end{aligned}$$

with $\epsilon = -2\pi/\ln |t|$. This interval approaches the unit interval $(0, 1)$ as $t \rightarrow 0$. So the open set $\tilde{V}_i \cap \mathcal{X}_t$ ends up being a large portion of \mathcal{X}_t . We end up with \mathcal{X}_t , for small t , being a union of open sets of the form $X_\epsilon((i + \epsilon', i + 1 - \epsilon'))$ (i.e. $\tilde{V}_i \cap \mathcal{X}_\epsilon$) and $X_\epsilon((i - \epsilon'', i + \epsilon''))$ (i.e. $\tilde{U}_i \cap \mathcal{X}_t$) for ϵ', ϵ'' sufficiently small. These should glue, at least approximately, to give $X_\epsilon(B)$. So we see that irreducible components of \mathcal{X}_0 seem to coincide with points on B , but intersections of components coincide with lines. In this way we see the dual intersection complex emerge.

Let us make one more observation before beginning with rigorous results in the next section. Suppose more generally we had a *Gorenstein toroidal crossings* degeneration of Calabi-Yau manifolds $f : \mathcal{X} \rightarrow D$. This means that every point $x \in \mathcal{X}$ has a neighbourhood isomorphic to an open set in an affine Gorenstein toric variety, with f given locally by a monomial which vanishes exactly to order 1 on each codimension one toric stratum. This is a generalization of the notion of normal crossings, see [48]. Very roughly, the above argument suggests that each irreducible component of the central fibre will correspond to a point of the Gromov-Hausdorff limit. The following exercise shows what kind of contribution to B to expect from a point $x \in \mathcal{X}_0$ which is a zero-dimensional stratum in \mathcal{X}_0 .

EXERCISE 6.1. *Suppose there is a point $x \in \mathcal{X}_0$ which has a neighbourhood isomorphic to a neighbourhood of a dimension zero torus orbit of an affine Gorenstein toric variety Y_x . Such an affine variety is specified as follows. Set $M = \mathbb{Z}^n$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ as in §4. Then there is a lattice polytope $\sigma \subseteq M_{\mathbb{R}}$, $n = \dim \mathcal{X}_t$, $C(\sigma) := \{(rm, r) \mid m \in \sigma, r \geq 0\} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$, $P := C(\sigma)^\vee \cap (N \oplus \mathbb{Z})$ the monoid determined by the dual of the cone $C(\sigma)$, and finally, $Y_x = \text{Spec } \mathbb{C}[P]$, and f coincides with the monomial $z^{(0,1)}$.*

Let us now take a small neighbourhood of x of the form

$$\tilde{U}_\delta = \{y \in \text{Spec } \mathbb{C}[P] \mid |z^p| < \delta \text{ for all } p \in P\}.$$

This is an open set as the condition $|z^p| < \delta$ can be tested on a finite generating set for P , provided that $\delta < 1$. Then show that for a given t , $|t| < 1$ and $\epsilon = -2\pi/\log |t|$, if

$$\sigma_t := \{m \in M_{\mathbb{R}} \mid \langle p, (m, 1) \rangle > \frac{\log \delta}{\log |t|} \text{ for all } p \in P\},$$

then

$$f^{-1}(t) \cap \tilde{U}_\delta \cong X_\epsilon(\sigma_t).$$

Note that

$$\sigma := \{m \in M_{\mathbb{R}} \mid \langle p, (m, 1) \rangle \geq 0 \text{ for all } p \in P\},$$

so σ_t is an open subset of σ , and as $t \rightarrow 0$, σ_t converges to the interior of σ . \square

This observation will hopefully motivate the basic construction of the next section.

7. Toric degenerations, the intersection complex and its dual

We now return to rigorous statements. I would like to explain the basic ideas behind the program launched in [20]. While I will use the previous sections as motivation, this work actually got its start when Siebert began a program of studying mirror symmetry via degenerations of Calabi-Yau manifolds. Work of Schröer and Siebert [47], [48] led Siebert to the idea that log structures on degenerations of Calabi-Yau manifolds would allow one to view mirror symmetry as an operation performed on degenerate Calabi-Yau varieties. Siebert observed that at a combinatorial level, mirror symmetry exchanged data pertaining to the log structure and a polarization. This will be explained more clearly in the following section, where we introduce log structures. Together, Siebert and I realised that the combinatorial data he was considering could be encoded naturally in the dual intersection complex of the degeneration, and that mirror symmetry then corresponded to a discrete Legendre transform on the dual intersection complex. It then became apparent that this approach provided an algebro-geometrization of the SYZ conjecture.

Here I will explain this program from the opposite direction, starting with the motivation of the previous section for introducing the dual intersection complex, and then work backwards until we arrive naturally at log structures. Much of the material in this section comes from [20], §4.

DEFINITION 7.1. Let $f : \mathcal{X} \rightarrow D$ be a proper flat family of relative dimension n , where D is a disk and \mathcal{X} is a complex analytic space (not necessarily non-singular). We say f is a *toric degeneration* of Calabi-Yau varieties if

- (1) \mathcal{X}_t is an irreducible normal Calabi-Yau variety with only canonical singularities for $t \neq 0$. (The reader may like to assume \mathcal{X}_t is smooth for $t \neq 0$).
- (2) If $\nu : \tilde{\mathcal{X}}_0 \rightarrow \mathcal{X}_0$ is the normalization, then $\tilde{\mathcal{X}}_0$ is a disjoint union of toric varieties, the conductor locus $C \subseteq \tilde{\mathcal{X}}_0$ is reduced, and the map $C \rightarrow \nu(C)$ is unramified and generically two-to-one. The square

$$\begin{array}{ccc} C & \longrightarrow & \tilde{\mathcal{X}}_0 \\ \downarrow & & \downarrow \nu \\ \nu(C) & \longrightarrow & \mathcal{X}_0 \end{array}$$

is cartesian and cocartesian.

- (3) \mathcal{X}_0 is a reduced Gorenstein space and the conductor locus C restricted to each irreducible component of $\tilde{\mathcal{X}}_0$ is the union of all toric Weil divisors of that component.
- (4) There exists a closed subset $Z \subseteq \mathcal{X}$ of relative codimension ≥ 2 such that Z satisfies the following properties: Z does not contain the image under ν of any toric stratum of $\tilde{\mathcal{X}}_0$, and for any point $x \in \mathcal{X} \setminus Z$, there is a neighbourhood \tilde{U}_x (in the analytic topology) of x , an $n+1$ -dimensional affine toric variety Y_x , a regular function f_x on Y_x given by a monomial, and a commutative diagram

$$\begin{array}{ccc} \tilde{U}_x & \xrightarrow{\psi_x} & Y_x \\ \downarrow f|_{\tilde{U}_x} & & \downarrow f_x \\ D' & \xrightarrow{\varphi_x} & \mathbb{C} \end{array}$$

where ψ_x and φ_x are open embeddings and $D' \subseteq D$. Furthermore, f_x vanishes precisely once on each toric divisor of Y_x .

EXAMPLE 7.2. Take \mathcal{X} to be defined by the equation $tf_4 + z_0z_1z_2z_3 = 0$ in $\mathbb{P}^3 \times D$, where D is a disk with coordinate t and f_4 is a general homogeneous quartic polynomial on \mathbb{P}^3 . It is easy to see that \mathcal{X} is singular at the locus

$$\{t = f_4 = 0\} \cap \text{Sing}(\mathcal{X}_0).$$

As \mathcal{X}_0 is the coordinate tetrahedron, the singular locus of \mathcal{X}_0 consists of the six coordinate lines of \mathbb{P}^3 , and \mathcal{X} has four singular points along each such line, for a total of 24 singular points. Take $Z = \text{Sing}(\mathcal{X})$. Then away from Z , the projection $\mathcal{X} \rightarrow D$ is normal crossings, which yields condition (4) of the definition of toric degeneration. It is easy to see all other conditions are satisfied.

EXAMPLE 7.3. Let $\Delta \subseteq M_{\mathbb{R}}$ be a reflexive polytope with dual $\nabla \subseteq N_{\mathbb{R}}$. Choose *integral* strictly convex piecewise linear functions $h : N_{\mathbb{R}} \rightarrow \mathbb{R}$, $\check{h} : M_{\mathbb{R}} \rightarrow \mathbb{R}$ as in §4. Consider the family in $\mathbb{P}_{\Delta} \times D$ defined by

$$(3) \quad z^0 + t \sum_{m \in \Delta \cap M} a_m t^{\check{h}(m)} z^m = 0.$$

Here z^m denotes the section of $\mathcal{O}_{\mathbb{P}_{\Delta}}(1)$ determined by the lattice point $m \in \Delta \cap M$ and $a_m \in k$ is a general choice of coefficient. So z^0 is the section which vanishes once on each toric boundary component of \mathbb{P}_{Δ} .

Now for most choices of h , this does not define a toric degeneration: the singularities are too bad. However, there is a natural toric variety in which to describe this degeneration. Let $X(\tilde{\Sigma})$ denote the toric variety defined by the fan $\tilde{\Sigma}$, as defined in §4. Then $\tilde{\Delta}$ is the Newton polytope of a line bundle \mathcal{L} on $X(\tilde{\Sigma})$, and terms of the form $t^{\check{h}(m)} z^m$ can be interpreted as sections of this line bundle, corresponding to $(m, \check{h}(m)) \in \tilde{\Delta} \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$. In addition, there is a natural map $X(\tilde{\Sigma}) \rightarrow \mathbb{A}^1$ defined by projection onto \mathbb{R} , and this map defines the regular function t . Thus (3) defines a hypersurface $\mathcal{X}_{\Delta} \subseteq X(\tilde{\Sigma})$ and t defines a map $\mathcal{X}_{\Delta} \rightarrow \mathbb{A}^1$. This is a toric degeneration.

Given a good subdivision $\tilde{\Sigma}'$ of Σ' such that $\tilde{\Sigma}'$ is an integral subdivision of Σ' as in Definition 4.3, we obtain a partial resolution $X(\tilde{\Sigma}') \rightarrow X(\tilde{\Sigma})$. Then the proper transform of \mathcal{X}_{Δ} in $X(\tilde{\Sigma}')$, \mathcal{X}'_{Δ} , yields another toric degeneration $\mathcal{X}'_{\Delta} \rightarrow \mathbb{A}^1$. For proofs, see [17], where the construction is generalized to complete intersections in toric varieties. \square

Given a toric degeneration $f : \mathcal{X} \rightarrow D$, we can build the *dual intersection complex* (B, \mathcal{P}) of f , as follows. Here B is an integral affine manifold with singularities, and \mathcal{P} is a *polyhedral decomposition* of B , a decomposition of B into lattice polytopes. In fact, we will construct B as a union of lattice polytopes. Specifically, let the normalisation of \mathcal{X}_0 , $\tilde{\mathcal{X}}_0$, be written as a disjoint union $\coprod X_i$ of toric varieties X_i , $\nu : \tilde{\mathcal{X}}_0 \rightarrow \mathcal{X}_0$ the normalisation. The *strata* of \mathcal{X}_0 are the elements of the set

$$\text{Strata}(\mathcal{X}_0) = \{\nu(S) \mid S \text{ is a toric stratum of } X_i \text{ for some } i\}.$$

Here by toric stratum we mean the closure of a $(\mathbb{C}^*)^n$ orbit.

Let $\{x\} \in \text{Strata}(\mathcal{X}_0)$ be a zero-dimensional stratum. Applying Definition 7.1 (4) to a neighbourhood of x , there is a toric variety Y_x such that in a neighbourhood of x , $f : \mathcal{X} \rightarrow D$ is locally isomorphic to $f_x : Y_x \rightarrow \mathbb{C}$, where f_x is given by a monomial. Now the condition that f_x vanishes precisely once along each toric

divisor of Y_x is the statement that Y_x is Gorenstein, and as such, it arises as in Exercise 6.1. Indeed, let M, N be as usual, with $\text{rank } M = \dim \mathcal{X}_0$. Then there is a lattice polytope $\sigma_x \subseteq M_{\mathbb{R}}$ such that $C(\sigma_x) = \{(rm, r) | m \in \sigma, r \geq 0\}$ is the cone defining the toric variety Y_x . As we saw in Exercise 6.1, a small neighbourhood of x in \mathcal{X} should contribute a copy of σ_x to B , which provides the motivation for our construction. We can now describe how to construct B by gluing together the polytopes

$$\{\sigma_x | \{x\} \in \text{Strata}(\mathcal{X}_0)\}.$$

We will do this in the case that every irreducible component of \mathcal{X}_0 is in fact itself normal so that $\nu : X_i \rightarrow \nu(X_i)$ is an isomorphism. The reader may be able to imagine the more general construction. With this normality assumption, there is a one-to-one inclusion reversing correspondence between faces of σ_x and elements of $\text{Strata}(\mathcal{X}_0)$ containing x . We can then identify faces of σ_x and $\sigma_{x'}$ if they correspond to the same strata of \mathcal{X}_0 . Some argument is necessary to show that this identification can be done via an integral affine transformation, but again this is not difficult.

Making these identifications, one obtains B . One can then prove

LEMMA 7.4. *If \mathcal{X}_0 is complex n dimensional, then B is an real n dimensional manifold.*

See [20], Proposition 4.10 for a proof.

Now so far B is just a topological manifold, constructed by gluing together lattice polytopes. Let

$$\mathcal{P} = \{\sigma \subseteq B | \sigma \text{ is a face of } \sigma_x \text{ for some zero-dimensional stratum } x\}.$$

There is a one-to-one inclusion reversing correspondence between strata of \mathcal{X}_0 and elements of \mathcal{P} .

It only remains to give B an affine structure with singularities. Let $\text{Bar}(\mathcal{P})$ be the first barycentric subdivision of \mathcal{P} , and let Γ be the union of simplices in $\text{Bar}(\mathcal{P})$ not containing a vertex of \mathcal{P} or intersecting the interior of a maximal cell of \mathcal{P} . If we then set $B_0 := B \setminus \Gamma$, we can define an affine structure on B_0 as follows. B_0 has an open cover

$$\{W_\sigma | \sigma \in \mathcal{P} \text{ maximal}\} \cup \{W_v | v \in \mathcal{P} \text{ a vertex}\}$$

where $W_\sigma = \text{Int}(\sigma)$, the interior of σ , and

$$W_v = \bigcup_{\substack{\tau \in \text{Bar}(\mathcal{P}) \\ v \in \tau}} \text{Int}(\tau)$$

is the (open) star of v in $\text{Bar}(\mathcal{P})$, just as in §4.

We now define charts. As a maximal cell of \mathcal{P} is of the form $\sigma_x \subseteq M_{\mathbb{R}}$, this inclusion induces a natural affine chart $\psi_\sigma : \text{Int}(\sigma) \rightarrow M_{\mathbb{R}}$. On the other hand, a vertex v of \mathcal{P} corresponds to a codimension 0 stratum of \mathcal{X}_0 , i.e. to an irreducible component X_i for some i . Because this is a compact toric variety, it is defined by a complete fan Σ_i in $M_{\mathbb{R}}$. Furthermore, there is a one-to-one correspondence between p -dimensional cones of Σ_i and p -dimensional cells of \mathcal{P} containing v as a vertex, as they both correspond to strata of \mathcal{X}_0 contained in X_i . There is then a continuous map

$$\psi_v : W_v \rightarrow M_{\mathbb{R}}$$

which takes $W_v \cap \sigma$, for any $\sigma \in \mathcal{P}$ containing v as a vertex, into the corresponding cone of Σ_i *integral affine linearly*. Such a map is uniquely determined by the combinatorial correspondence and the requirement that it be integral affine linear on each cell. It is then obvious these charts define an integral affine structure on B_0 . Thus we have constructed (B, \mathcal{P}) .

EXAMPLE 7.5. Let $f : \mathcal{X} \rightarrow D$ be a degeneration of elliptic curves to an I_n fibre. Then B is the circle $\mathbb{R}/n\mathbb{Z}$, decomposed into n line segments of length one.

EXAMPLE 7.6. Continuing with Example 7.3, the dual intersection complex constructed from the toric degenerations $\mathcal{X}'_{\Delta} \rightarrow \mathbb{A}^1$ is the affine manifold with singularities structure with polyhedral decomposition (B, \mathcal{P}) constructed on $B = \partial\nabla^h$ in §4. This is not particularly difficult to show. Again, for the proof and more general complete intersection case, see [17].

Is the dual intersection complex the right affine manifold with singularities? The following theorem provides evidence for this, and gives the connection between this construction and the SYZ conjecture.

THEOREM 7.7. *Let $\mathcal{X} \rightarrow D$ be a toric degeneration, with dual intersection complex (B, \mathcal{P}) . Then there is an open set $U \subseteq B$ such that $B \setminus U$ retracts onto the discriminant locus Γ of B , such that \mathcal{X}_t contains an open subset \mathcal{U}_t which is isomorphic as complex manifolds to a small deformation of a twist of $X_{\epsilon}(U)$, where $\epsilon = O(-1/\ln|t|)$.*

We will not be precise here about what we mean by small deformation; by twist, we mean a twist of the complex structure of $X_{\epsilon}(U)$ by a B -field. See [18] for a much more precise statement; the above statement is meant to give a feel for what is true. The proof, along with much more precise statements, will eventually appear in [19].

If $\mathcal{X} \rightarrow D$ is a *polarized* toric degeneration, i.e. if there is a relatively ample line bundle \mathcal{L} on \mathcal{X} , then we can construct another affine manifold with singularities and polyhedral decomposition $(\check{B}, \check{\mathcal{P}})$, which we call the *intersection complex*, as follows.

For each irreducible component X_i of \mathcal{X}_0 , $\mathcal{L}|_{X_i}$ is an ample line bundle on a toric variety. Let $\sigma_i \subseteq N_{\mathbb{R}}$ denote the Newton polytope of this line bundle. There is then a one-to-one inclusion preserving correspondence between strata of \mathcal{X}_0 contained in X_i and faces of σ_i . We can then glue together the σ_i 's in the obvious way: if Y is a codimension one stratum of \mathcal{X}_0 , it is contained in two irreducible components X_i and X_j , and defines faces of σ_i and σ_j . These faces are affine isomorphic because they are both the Newton polytope of $\mathcal{L}|_Y$, and we can then identify them in the canonical way. Thus we obtain a topological space \check{B} with a polyhedral decomposition $\check{\mathcal{P}}$. We give it an affine structure with singularities in a similar manner as before. Again, let Γ be the union of simplices in $\text{Bar}(\check{\mathcal{P}})$ not containing a vertex of $\check{\mathcal{P}}$ or intersecting the interior of a maximal cell of $\check{\mathcal{P}}$. Setting $\check{B}_0 := \check{B} \setminus \Gamma$, this again has an open cover

$$\{W_{\sigma} | \sigma \in \check{\mathcal{P}} \text{ maximal}\} \cup \{W_v | v \in \check{\mathcal{P}} \text{ a vertex}\}.$$

As usual, as W_{σ} is the interior of σ , it comes along with a canonical affine structure. On the other hand, a vertex v of $\check{\mathcal{P}}$ corresponds to a dimension zero stratum x of \mathcal{X}_0 , and associated to x is the polytope $\sigma_x \subseteq M_{\mathbb{R}}$. Let $\check{\Sigma}_x$ be the normal fan to

σ_x in $N_{\mathbb{R}}$. Then there is a one-to-one inclusion preserving correspondence between cones in $\tilde{\Sigma}_x$ and strata of \mathcal{X}_0 containing x . This correspondence allows us to define a chart

$$\check{\psi}_v : W_v \rightarrow N_{\mathbb{R}}$$

which takes $W_v \cap \check{\sigma}$, for any $\check{\sigma} \in \check{\mathcal{P}}$ containing v as a vertex, into the corresponding cone of $\tilde{\Sigma}_x$ in an integral affine linear way. This gives the manifestly integral affine structure on \check{B}_0 , and hence defines the intersection complex $(\check{B}, \check{\mathcal{P}})$.

Analogously to Theorem 7.7, we expect

CONJECTURE 7.8. *Let $\mathcal{X} \rightarrow D$ be a polarized toric degeneration, with intersection complex $(\check{B}, \check{\mathcal{P}})$. Let ω_t be a Kähler form on \mathcal{X}_t representing the first Chern class of the polarization. Then there is an open set $\check{U} \subseteq \check{B}$ such that $\check{B} \setminus \check{U}$ retracts onto the discriminant locus Γ of \check{B} , such that \mathcal{X}_t is a symplectic compactification of $\check{X}(\check{U})$ for any t .*

I don't expect this to be particularly difficult: it should be amenable to the techniques of Ruan [46], but has not been carried out.

The relationship between the intersection complex and the dual intersection complex can be made more precise by introducing multi-valued piecewise linear functions, in analogy with the multi-valued convex functions of Definition 2.3:

DEFINITION 7.9. Let B be an affine manifold with singularities with polyhedral decomposition \mathcal{P} . Then a multi-valued piecewise linear function φ on B is a collection of continuous functions on an open cover $\{(U_i, \varphi_i)\}$ such that φ_i is affine linear on each cell of \mathcal{P} intersecting U_i , and on $U_i \cap U_j$, $\varphi_i - \varphi_j$ is affine linear. Furthermore, for any $\sigma \in \mathcal{P}$, in a neighbourhood of each point $x \in U_i \cap \text{Int}(\sigma)$, there is an affine linear function ψ such that $\varphi_i - \psi$ is zero on σ .

To explain this last condition, and to clarify additional structure on B , let us examine a property of the polyhedral decomposition \mathcal{P} of B when (B, \mathcal{P}) is a dual intersection complex.

Consider any p -dimensional cell $\sigma \in \mathcal{P}$. This corresponds to an $n-p$ -dimensional stratum $X_\sigma \subseteq \mathcal{X}_0$, and as such, it is a toric variety defined by a fan Σ_σ in \mathbb{R}^p . Now for any vertex v of σ , X_σ is a toric stratum of the irreducible component X_v of \mathcal{X}_0 . Thus Σ_σ can be obtained as a *quotient fan* of Σ_v . In other words, there is a p -dimensional cone K_σ of Σ_v corresponding to σ such that

$$\Sigma_\sigma = \Sigma_v(K_\sigma) := \{(K + \mathbb{R}K_\sigma)/\mathbb{R}K_\sigma \mid K \in \Sigma_v, K \supseteq K_\sigma\}.$$

In particular, there is an open neighbourhood $U_{v,\sigma}$ of $\text{Int}(K_\sigma)$ and an integral linear map $S_{v,\sigma} : U_{v,\sigma} \rightarrow \mathbb{R}^p$ such that

$$\{S_{v,\sigma}^{-1}(K) \mid K \in \Sigma_\sigma\} = \{U_{v,\sigma} \cap K \mid K \in \Sigma_v, K \supseteq K_\sigma\}.$$

These neighbourhoods $U_{v,\sigma}$ can be viewed as open subsets of B , if taken sufficiently small, and if they are chosen to cover $\text{Int}(\sigma)$, then the maps $S_{v,\sigma}$ patch to give an integral affine submersion $S_\sigma : U_\sigma \rightarrow \mathbb{R}^p$, where $U_\sigma = \bigcup_{v \in \sigma} U_{v,\sigma}$ is an open neighbourhood of $\text{Int}(\sigma)$. This map has the property that

$$\{S_\sigma^{-1}(K) \mid K \in \Sigma_\sigma\} = \{U_\sigma \cap \tau \mid \tau \supseteq \sigma, \tau \in \mathcal{P}\}.$$

In general we call a polyhedral decomposition *toric* if for all $\sigma \in \mathcal{P}$ there is always such an integral affine linear map $S_\sigma : U_\sigma \rightarrow \mathbb{R}^p$ from a neighbourhood U_σ of $\text{Int}(\sigma)$ and a fan Σ_σ in \mathbb{R}^p with the above property. (See [20], Definition 1.22 for

a perhaps too precise definition of toric polyhedral decompositions. The definition there is complicated by allowing cells to be self-intersecting, or equivalently, allowing irreducible components of \mathcal{X}_0 to be non-normal.) We can think of the fan Σ_σ as being the fan structure of \mathcal{P} transverse to σ at a point in the interior of σ . The main point for a dual intersection complex is that this fan structure is determined by X_σ , and this is independent of the choice of the point in the interior of σ .

Now let us return to piecewise linear functions. Suppose we are given a polarized toric degeneration $\mathcal{X} \rightarrow D$. We in fact obtain a piecewise linear function φ on the dual intersection complex (B, \mathcal{P}) as follows. Restricting to any toric stratum X_σ , $\mathcal{L}|_{X_\sigma}$ is determined completely by an integral piecewise linear function $\bar{\varphi}_\sigma$ on Σ_σ , well-defined up to a choice of linear function. Pulling back this piecewise linear function via S_σ to U_σ , we obtain a collection of piecewise linear functions $\{(U_\sigma, \varphi_\sigma) | \sigma \in \mathcal{P}\}$. The fact that $(\mathcal{L}|_{X_\tau})|_{X_\sigma} = \mathcal{L}|_{X_\sigma}$ for $\tau \subseteq \sigma$ implies that on overlaps φ_σ and φ_τ differ by at most a linear function. So $\{(U_\sigma, \varphi_\sigma)\}$ defines a multi-valued piecewise linear function. The last condition in the definition of multi-valued piecewise linear function then reflects the need for the function to be locally a pull-back of a function via S_σ in a neighbourhood of σ . In fact, given any multi-valued piecewise linear function φ on (B, \mathcal{P}) with \mathcal{P} a toric polyhedral decomposition of B , φ is determined by functions $\bar{\varphi}_\sigma$ on Σ_σ for $\sigma \in \mathcal{P}$, via pull-back by S_σ .

If \mathcal{L} is ample, then the piecewise linear function determined by $\mathcal{L}|_{X_\sigma}$ is strictly convex. So we say a multi-valued piecewise linear function is *strictly convex* if $\bar{\varphi}_\sigma$ is strictly convex for each $\sigma \in \mathcal{P}$.

Now suppose we are given abstractly a triple $(B, \mathcal{P}, \varphi)$ with B an integral affine manifold with singularities with a toric polyhedral decomposition \mathcal{P} , and φ a strictly convex multi-valued piecewise linear function on B . Then we construct the *discrete Legendre transform* $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ of $(B, \mathcal{P}, \varphi)$ as follows.

\check{B} will be constructed by gluing together Newton polytopes. If we view, for v a vertex of \mathcal{P} , the fan Σ_v as living in $M_{\mathbb{R}}$, then the Newton polytope of $\bar{\varphi}_v$ is

$$\check{v} = \{x \in N_{\mathbb{R}} | \langle x, y \rangle \geq -\bar{\varphi}_v(y) \quad \forall y \in M_{\mathbb{R}}\}.$$

There is a one-to-one order reversing correspondence between faces of \check{v} and cells of \mathcal{P} containing v . Furthermore, if σ is the smallest cell of \mathcal{P} containing two vertices v and v' , then the corresponding faces of \check{v} and \check{v}' are integral affine isomorphic, as they are both isomorphic to the Newton polytope of $\bar{\varphi}_\sigma$. Thus we can glue \check{v} and \check{v}' along this common face. After making all these identifications, we obtain a cell complex $(\check{B}, \check{\mathcal{P}})$, which is really just the dual cell complex of (B, \mathcal{P}) . Of course, we have some additional information, namely an affine structure on the interior of each maximal cell of $\check{\mathcal{P}}$. To give \check{B} an integral affine structure with singularities, one proceeds as usual, using this affine structure along with an identification of a neighbourhood of each vertex of $\check{\mathcal{P}}$ with the normal fan of the corresponding maximal cell of \mathcal{P} .

Finally, the function φ has a discrete Legendre transform $\check{\varphi}$ on $(\check{B}, \check{\mathcal{P}})$. We have no choice but to define $\check{\varphi}$ in a neighbourhood of a vertex $\check{\sigma} \in \check{\mathcal{P}}$ dual to a maximal cell $\sigma \in \mathcal{P}$ to be a piecewise linear function whose Newton polytope is σ , i.e.

$$\bar{\varphi}_{\check{\sigma}}(y) = -\inf\{\langle y, x \rangle | x \in \sigma \subseteq M_{\mathbb{R}}\}.$$

This gives $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$, the discrete Legendre transform of $(B, \mathcal{P}, \varphi)$. If B is \mathbb{R}^n , then this coincides with the classical notion of a discrete Legendre transform. The discrete Legendre transform has several relevant properties:

- The discrete Legendre transform of $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ is $(B, \mathcal{P}, \varphi)$.
- If we view the underlying topological spaces B and \check{B} as being identified by being the underlying space of dual cell complexes, then $\Lambda_{B_0} \cong \check{\Lambda}_{\check{B}_0}$ and $\check{\Lambda}_{\check{B}_0} \cong \Lambda_{B_0}$, where the subscript denotes which affine structure is being used to define Λ or $\check{\Lambda}$.

This hopefully makes it clear that the discrete Legendre transform is a suitable replacement for the duality provided to us by the Legendre transform of §2.

Finally, it leads to what we may think of as an *algebraic-geometric SYZ procedure*. In analogy with the procedure suggested in §5, we follow these steps:

- (1) We begin with a toric degeneration of Calabi-Yau manifolds $\mathcal{X} \rightarrow D$ with an ample polarization.
- (2) Construct $(B, \mathcal{P}, \varphi)$ from this data, as explained above.
- (3) Perform the discrete Legendre transform to obtain $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$.
- (4) Try to construct a polarized degeneration of Calabi-Yau manifolds $\check{\mathcal{X}} \rightarrow D$ whose dual intersection complex is $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$.

EXAMPLE 7.10. The discrete Legendre transform enables us to reproduce Batyrev duality. Returning to the construction of §4 and Example 7.3, choosing a strictly convex piecewise linear function on $\check{\Sigma}'$ corresponding to a line bundle \mathcal{L} induces a polarization of \mathcal{X}'_{Δ} . This then gives us a strictly convex multi-valued piecewise linear function $\varphi_{\mathcal{L}}$ on (B, \mathcal{P}) , hence a discrete Legendre transform $(\check{B}, \check{\mathcal{P}}, \check{\varphi}_{\mathcal{L}})$. In [17] I showed that this is the dual intersection complex associated to some choice of subdivision $\check{\Sigma}$ of $\check{\Sigma}$ obtained by interchanging the roles of ∇ and Δ in the construction of §4. As an exercise, you can check the following for yourself. If we take $h = \varphi$, $\check{h} = \check{\varphi}$, then we obtain $B = \partial\nabla$, $\check{B} = \partial\Delta$ with polyhedral decomposition \mathcal{P} and $\check{\mathcal{P}}$ consisting of all proper faces of ∇ and Δ respectively. The anti-canonical polarizations on \mathbb{P}_{Δ} and \mathbb{P}_{∇} induce multi-valued piecewise linear functions $\varphi, \check{\varphi}$ on B and \check{B} respectively. Then show $(B, \mathcal{P}, \varphi)$ and $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ are discrete Legendre transforms of each other.

Thus Batyrev (and Batyrev-Borisov) duality is a special case of this construction.

The only step missing in this mirror symmetry algorithm is the last:

QUESTION 7.11 (The reconstruction problem, Version II). *Given $(B, \mathcal{P}, \varphi)$, is it possible to construct a polarized toric degeneration $\mathcal{X} \rightarrow D$ whose dual intersection complex is $(B, \mathcal{P}, \varphi)$?*

It is fairly obvious how to reconstruct the central fibre \mathcal{X}_0 of a degeneration from the data $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$, and we will see this explicitly in §8. One could naively hope that this reducible variety has good deformation theory and it can be smoothed. However, in general its deformation theory is ill-behaved.

As initially observed in the normal crossings case by Kawamata and Namikawa in [32], one needs to put some additional structure on \mathcal{X}_0 before it has good deformation theory. This structure is a *log structure*, and introducing log structures allows us to study many aspects of mirror symmetry directly on the degenerate fibre itself.

We shall do this in the next section, but first, let me address the question of how general this mirror symmetry construction might be:

CONJECTURE 7.12. *If $f : \mathcal{X} \rightarrow D$ is a large complex structure limit degeneration, then f is birationally equivalent to a toric degeneration $f' : \mathcal{X}' \rightarrow D$.*

The condition of being a large complex structure limit, as defined by Morrison in [41], is a stronger one than maximally unipotent. Why should I imagine something like this to be true? Well, fantasizing freely, we would expect that after choosing a polarization and Ricci-flat metric on fibres \mathcal{X}_t , we have a sequence (\mathcal{X}_t, g_t) converging to B an affine manifold with singularities. Now in general an affine manifold with singularities need not arise as the dual intersection complex of a toric degeneration, first of all because it need not have a toric polyhedral decomposition. For example, even in two dimensions there are orbifold singularities (corresponding to singular elliptic fibres which are not semi-stable) which do not arise in dual intersection complexes of toric degenerations, yet can occur as the base of a special Lagrangian fibration on a K3 surface. However, the *general* base does arise as the dual intersection complex of a toric degeneration in the K3 case. The hope is that the condition of large complex structure limit forces the singularities of B to be “sufficiently general” so that one can construct a nice toric polyhedral decomposition \mathcal{P} on B , and from this construct a toric degeneration. Presumably, this toric degeneration will be, if picked correctly, birational to the original one.

This argument of course is rather hand-wavy, but I believe it provides some moral expectation that there might be a large class of degenerations for which our method applies. I note that one can prove the conjecture in the case of K3 surfaces.

We now come to the technical heart of the program laid out in [20]. Some aspects of this program are quite technical, so the goal here is to explain the highlights of [20] as simply possible.

8. Log structures

We first introduce the log structures of Fontaine-Illusie and Kato ([28], [31]).

DEFINITION 8.1. A log structure on a scheme (or analytic space) X is a (unital) homomorphism

$$\alpha_X : \mathcal{M}_X \rightarrow \mathcal{O}_X$$

of sheaves of (multiplicative and commutative) monoids inducing an isomorphism $\alpha_X^{-1}(\mathcal{O}_X^\times) \rightarrow \mathcal{O}_X^\times$. The triple $(X, \mathcal{M}_X, \alpha_X)$ is then called a *log space*. We often write the whole package as X^\dagger .

A morphism of log spaces $F : X^\dagger \rightarrow Y^\dagger$ consists of a morphism $\underline{F} : X \rightarrow Y$ of underlying spaces together with a homomorphism $F^\# : \underline{F}^{-1}(\mathcal{M}_Y) \rightarrow \mathcal{M}_X$ commuting with the structure homomorphisms:

$$\alpha_X \circ F^\# = \underline{F}^* \circ \alpha_Y.$$

The key examples:

EXAMPLES 8.2. (1) Let X be a scheme and $Y \subseteq X$ a closed subset of codimension one. Denote by $j : X \setminus Y \rightarrow X$ the inclusion. Then the inclusion

$$\alpha_X : \mathcal{M}_X = j_*(\mathcal{O}_{X \setminus Y}^\times) \cap \mathcal{O}_X \rightarrow \mathcal{O}_X$$

of the sheaf of regular functions with zeroes contained in Y is a log structure on X . This is called a *divisorial log structure* on X .

(2) A *prelog structure*, i.e. an arbitrary homomorphism of sheaves of monoids $\varphi : \mathcal{P} \rightarrow \mathcal{O}_X$, defines an associated log structure \mathcal{M}_X by

$$\mathcal{M}_X = (\mathcal{P} \oplus \mathcal{O}_X^\times) / \{(p, \varphi(p)^{-1}) \mid p \in \varphi^{-1}(\mathcal{O}_X^\times)\}$$

and $\alpha_X(p, h) = h \cdot \varphi(p)$.

(3) If $f : X \rightarrow Y$ is a morphism of schemes and $\alpha_Y : \mathcal{M}_Y \rightarrow \mathcal{O}_Y$ is a log structure on Y , then the prelog structure $f^{-1}(\mathcal{M}_Y) \rightarrow \mathcal{O}_X$ defines an associated log structure on X , the *pull-back log structure*.

(4) In (1) we can pull back the log structure on X to Y using (3). Thus in particular, if $\mathcal{X} \rightarrow D$ is a toric degeneration, the inclusion $\mathcal{X}_0 \subseteq \mathcal{X}$ gives a log structure on \mathcal{X} and an induced log structure on \mathcal{X}_0 . Similarly the inclusion $0 \in D$ gives a log structure on D and an induced one on 0 . Here $\mathcal{M}_0 = \mathbb{C}^\times \oplus \mathbb{N}$, where \mathbb{N} is the (additive) monoid of natural (non-negative) numbers, and

$$\alpha_0(h, n) = \begin{cases} h & n = 0 \\ 0 & n \neq 0. \end{cases}$$

0^\dagger is usually called the standard log point.

We then have log morphisms $\mathcal{X}^\dagger \rightarrow D^\dagger$ and $\mathcal{X}_0^\dagger \rightarrow 0^\dagger$.

(5) If $\sigma \subseteq M_{\mathbb{R}} = \mathbb{R}^n$ is a rational convex polyhedral cone, $\sigma^\vee \subseteq N_{\mathbb{R}}$ the dual cone, let $P = \sigma^\vee \cap N$: this is a monoid. The affine toric variety defined by σ can be written as $X = \text{Spec } \mathbb{C}[P]$. We then have a pre-log structure induced by the homomorphism of monoids

$$P \rightarrow \mathbb{C}[P]$$

given by $p \mapsto z^p$. There is then an associated log structure on X . This is in fact the same as the log structure induced by $\partial X \subseteq X$, where ∂X is the toric boundary of X , i.e. the union of toric divisors of X .

If $p \in P$, then the monomial z^p defines a map $f : X \rightarrow \text{Spec } \mathbb{C}[\mathbb{N}]$ ($= \text{Spec } \mathbb{C}[t]$) which is a log morphism with the log structure on $\text{Spec } \mathbb{C}[\mathbb{N}]$ induced similarly by $\mathbb{N} \rightarrow \mathbb{C}[\mathbb{N}]$. The fibre $X_0 = \text{Spec } \mathbb{C}[P]/(z^p)$ is a subscheme of X , and there is an induced log structure on X_0 , and a map $X_0^\dagger \rightarrow 0^\dagger$ as in (4). f is an example of a *log smooth* morphism. Essentially all log smooth morphisms are étale locally of this form (if \mathbb{N} is replaced by a more general monoid). See [30] for details.

Condition (4) of Definition 7.1 in fact implies that locally, away from Z , \mathcal{X}^\dagger and \mathcal{X}_0^\dagger are of the above form. So we should view $\mathcal{X}^\dagger \rightarrow D^\dagger$ as log smooth away from Z , and from the log point of view, \mathcal{X}_0^\dagger can be treated much like a non-singular scheme away from X . We will see this explicitly below when we talk about differentials. \square

On a log scheme X^\dagger there is always an exact sequence

$$1 \longrightarrow \mathcal{O}_X^\times \xrightarrow{\alpha^{-1}} \mathcal{M}_X \longrightarrow \overline{\mathcal{M}}_X \longrightarrow 0,$$

where we write the quotient sheaf of monoids $\overline{\mathcal{M}}_X$ additively. We call $\overline{\mathcal{M}}_X$ the *ghost sheaf* of the log structure. I like to view $\overline{\mathcal{M}}_X$ as specifying the combinatorial information associated to the log structure. For example, if X^\dagger is induced by the Cartier divisor $Y \subseteq X$ with X normal, then the stalk $\overline{\mathcal{M}}_{X,x}$ at $x \in X$ is the group of effective Cartier divisors on a neighbourhood of x supported on Y .

EXERCISE 8.3. Show that in Example 8.2, (5), $\overline{\mathcal{M}}_{X,x} = P$ if $\dim \sigma = n$ and x is the unique zero-dimensional torus orbit of X . More generally,

$$\overline{\mathcal{M}}_{X,x} = (\tau^\vee / \tau^\perp) \cap (N/N \cap \tau^\perp) = \text{Hom}_{\text{monoid}}(\tau \cap M, \mathbb{N}),$$

when $x \in X$ is in the torus orbit corresponding to a face τ of σ . In particular, τ can be recovered as $\text{Hom}_{\text{monoid}}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}^+)$, where $\mathbb{R}_{\geq 0}^+$ is the additive monoid of non-negative real numbers. \square

Another important fact for us is that if $f : Y \rightarrow X$ is a morphism with X carrying a log structure, and Y is given the pull-back log structure, then $\overline{\mathcal{M}}_Y = f^{-1}\overline{\mathcal{M}}_X$. In the case that $\overline{\mathcal{M}}_X$ is induced by an inclusion of $Y \subseteq X$, $\overline{\mathcal{M}}_X$ is supported on Y , so we can equate $\overline{\mathcal{M}}_X$ and $\overline{\mathcal{M}}_Y$, the ghost sheaves for the divisorial log structure on X and its restriction to Y . Putting this together with Exercise 8.3 and the definition of dual intersection complex, we see that given a toric degeneration $\mathcal{X} \rightarrow D$ the dual intersection complex *completely determines* the ghost sheaf $\overline{\mathcal{M}}_{\mathcal{X}} = \overline{\mathcal{M}}_{\mathcal{X}_0}$ off of Z . We in fact take the view that anyway the log structure on Z is not particularly well-behaved, and we always ignore it on Z . In fact, given a log structure $\mathcal{M}_{\mathcal{X}_0 \setminus Z}$ on $\mathcal{X}_0 \setminus Z$, this defines a push-forward log structure $\mathcal{M}_{\mathcal{X}_0} := j_*\mathcal{M}_{\mathcal{X}_0 \setminus Z}$. There is an induced map $\alpha : \mathcal{M}_{\mathcal{X}_0} \rightarrow \mathcal{O}_{\mathcal{X}_0}$, as $j_*\mathcal{O}_{\mathcal{X}_0 \setminus Z} = \mathcal{O}_{\mathcal{X}_0}$ because \mathcal{X}_0 is Cohen-Macaulay. Thus in what follows, if we have determined a log structure on $\mathcal{X}_0 \setminus Z$, we just as well get a log structure on \mathcal{X}_0 and will not concern ourselves with the behaviour of this log structure along Z .

All this gives the necessary hint for working backwards, to go from (B, \mathcal{P}) to \mathcal{X}_0^\dagger . Suppose we are given an integral affine manifold with singularities B with toric polyhedral decomposition \mathcal{P} . At each vertex v of \mathcal{P} , \mathcal{P} locally looks like a fan Σ_v , defining a toric variety X_v . For every edge $\omega \in \mathcal{P}$ with endpoints v and w , ω defines a ray in both fans Σ_v and Σ_w , hence toric divisors $D_\omega^v \subseteq X_v$, $D_\omega^w \subseteq X_w$. The condition that \mathcal{P} is a toric polyhedral decomposition tells us that D_ω^v and D_ω^w are isomorphic toric varieties, and we can choose a torus equivariant isomorphism $s_\omega : D_\omega^v \rightarrow D_\omega^w$ for each edge ω . If we choose these gluing maps to satisfy a certain compatibility condition on codimension two strata (we leave this to the reader to write down this simple compatibility condition), then we can glue together the X_v 's to obtain, in general, an algebraic space we write as $X_0(B, \mathcal{P}, s)$, where $s = (s_\omega)$ is the collection of gluing maps. (In [20], we describe the gluing data in a slightly different, but equivalent, way). We call s *closed gluing data*. This is how we construct a potential central fibre of a toric degeneration.

Now $X_0(B, \mathcal{P}, s)$ cannot be a central fibre of a toric degeneration unless it carries a log structure of the correct sort. There are many reasons this may not happen. First, if s is poorly chosen, there may be zero-dimensional strata of $X_0(B, \mathcal{P}, s)$ which do not have neighbourhoods locally étale isomorphic to the toric boundary of an affine toric variety; this is a minimum prerequisite. As a result, we have to restrict attention to closed gluing data induced by what we call *open gluing data*. Explicitly, each maximal cell $\sigma \in \mathcal{P}$ defines an affine toric variety $U(\sigma)$ given by the cone $C(\sigma) \subseteq M_{\mathbb{R}} \oplus \mathbb{R}$, assuming we view $\sigma \subseteq M_{\mathbb{R}}$ as a lattice polytope. Let $V(\sigma) \subseteq U(\sigma)$ be the toric boundary. It turns out, as we show in [20], that a necessary condition for $X_0(B, \mathcal{P}, s)$ to be the central fibre of a toric degeneration is that it is obtained by dividing out $\coprod_{\sigma \in \mathcal{P}_{\max}} V(\sigma)$ by an étale equivalence relation. In other words, we are gluing together the $V(\sigma)$'s to obtain an algebraic space, and those étale equivalence relations which produce algebraic spaces of the form

$X_0(B, \mathcal{P}, s)$ are easily determined. This is carried out in detail in [20], §2. The construction there appears technically difficult because of the necessity of dealing with algebraic spaces, but is basically straightforward. The basic point is that if $\sigma_1, \sigma_2 \in \mathcal{P}$ are two maximal cells, with $\sigma_1 \cap \sigma_2 = \tau$, then τ determines faces of the cones $C(\sigma_1)$ and $C(\sigma_2)$, hence open subsets $U_i(\tau) \subseteq U(\sigma_i)$, with toric boundaries $V_i(\tau) \subseteq V(\sigma_i)$. Now in general there is no *natural* isomorphism between $U_1(\tau)$ and $U_2(\tau)$: this is a problem when $\sigma_1 \cap \sigma_2 \cap \Gamma \neq \emptyset$, where Γ is as usual the singular locus of B . However, crucially $V_1(\tau)$ and $V_2(\tau)$ are naturally isomorphic, and we can choose compatible equivariant isomorphisms to obtain *open gluing data*. Choosing open gluing data allows us to define the étale equivalence relation: we are just gluing any two sets $V(\sigma_1), V(\sigma_2)$ via the chosen isomorphism between $V_1(\tau)$ and $V_2(\tau)$. Any choice of open gluing data s gives rise in this way to an algebraic space $X_0(B, \mathcal{P}, s)$, and to any choice of open gluing data there is associated closed gluing data s' such that $X_0(B, \mathcal{P}, s) \cong X_0(B, \mathcal{P}, s')$.

The advantage of using open gluing data is that each $V(\sigma)$ for $\sigma \in \mathcal{P}_{\max}$ carries a log structure induced by the divisorial log structure $V(\sigma) \subseteq U(\sigma)$. Unfortunately, these log structures are not identified under the open gluing maps, precisely because of a lack of a natural isomorphism between the $U_i(\tau)$'s cited above. However, the ghost sheaves of the log structures are isomorphic. So the ghost sheaves $\overline{\mathcal{M}}_{V(\sigma)}$ glue to give a ghost sheaf of monoids $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}$. Summarizing what we have said so far: (this is a combination of results of [20], §§2,4)

THEOREM 8.4. *Given (B, \mathcal{P}) , if s is closed gluing data, and $\mathcal{X}_0 = X_0(B, \mathcal{P}, s)$ is the central fibre of a toric degeneration $\mathcal{X} \rightarrow D$ with dual intersection complex (B, \mathcal{P}) , then s is induced by open gluing data and $\overline{\mathcal{M}}_{\mathcal{X}_0}|_{\mathcal{X}_0 \setminus Z} \cong \overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}|_{\mathcal{X}_0 \setminus Z}$.*

This is as far as we can get with the combinatorics. The next point is to attempt to construct $\mathcal{M}_{X_0(B, \mathcal{P}, s)}$. The idea is that $\mathcal{M}_{X_0(B, \mathcal{P}, s)}$ is an extension of $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}$ by $\mathcal{O}_{X_0(B, \mathcal{P}, s)}^\times$, so we are looking for some subsheaf of the sheaf

$$\mathcal{E}xt^1(\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}^{\text{gp}}, \mathcal{O}_{X_0(B, \mathcal{P}, s)}^\times).$$

Here the superscript gp denotes the Grothendieck group of the monoid. Any extension of $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}^{\text{gp}}$ by $\mathcal{O}_{X_0(B, \mathcal{P}, s)}^\times$ gives rise to a sheaf of groups $\mathcal{M}_{X_0(B, \mathcal{P}, s)}^{\text{gp}}$ surjecting onto $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}^{\text{gp}}$, and the inverse image of $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)} \subseteq \overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}^{\text{gp}}$ is a sheaf of monoids $\mathcal{M}_{X_0(B, \mathcal{P}, s)}$. Of course, one also needs a map $\alpha : \mathcal{M}_{X_0(B, \mathcal{P}, s)} \rightarrow \mathcal{O}_{X_0(B, \mathcal{P}, s)}$, and this complicates things a bit. To make a long story short, we can identify a subsheaf of extensions which yield genuine log structures. A section of this subsheaf determines a log structure on $X_0(B, \mathcal{P}, s)$ with the correct ghost sheaf. However, this is not precisely what we want. What we really want is a log structure on $X_0(B, \mathcal{P}, s)$ along with a log morphism $X_0(B, \mathcal{P}, s)^\dagger \rightarrow 0^\dagger$ which is log smooth. (We will address the question of the bad set $Z \subseteq \mathcal{X}_0$ shortly.) We call such a structure a *log smooth structure* on $X_0(B, \mathcal{P}, s)$. It turns out these structures are given by certain sections of $\mathcal{E}xt^1(\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}^{\text{gp}}/\bar{\rho}, \mathcal{O}_X^\times)$, where $\bar{\rho}$ is the canonical section of $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}$ whose germ at $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s), \eta} = \mathbb{N}$ is 1 for η a generic point of an irreducible component of $X_0(B, \mathcal{P}, s)$. So in fact, we can identify a subsheaf of $\mathcal{E}xt^1(\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}^{\text{gp}}/\bar{\rho}, \mathcal{O}_X^\times)$, which we call $\mathcal{L}\mathcal{S}_{X_0(B, \mathcal{P}, s)}$, whose sections determine a log structure on $X_0(B, \mathcal{P}, s)$ and a log smooth morphism $X_0(B, \mathcal{P}, s)^\dagger \rightarrow 0^\dagger$, i.e. a log smooth structure.

The technical heart of [20] is an explicit calculation of the sheaf $\mathcal{L}\mathcal{S}_{X_0(B, \mathcal{P}, s)}$. This is carried out locally in [20], Theorem 3.22, where the sheaf is calculated on the (étale) open subsets $V(\sigma)$ of $X_0(B, \mathcal{P}, s)$, and globally in [20], Theorem 3.24. I will not state the precise results, but go into detail in a special case, which illustrates the most important features of the theory.

EXAMPLE 8.5. Suppose $X_0(B, \mathcal{P}, s)$ is normal crossings, i.e. every cell of \mathcal{P} is affine isomorphic to a standard simplex. Then we have the local \mathcal{T}^1 sheaf,

$$\mathcal{T}^1 = \mathcal{E}xt_{X_0(B, \mathcal{P}, s)}^1(\Omega_{X_0(B, \mathcal{P}, s)/k}^1, \mathcal{O}_{X_0(B, \mathcal{P}, s)}).$$

This is a line bundle on $S = \text{Sing}(X_0(B, \mathcal{P}, s))$. Then one can show $\mathcal{L}\mathcal{S}_{X_0(B, \mathcal{P}, s)}$ is the \mathcal{O}_S^\times -torsor associated to \mathcal{T}^1 .

This brings us back to Friedman's condition of d -semistability [8]. A variety with normal crossings is d -semistable if $\mathcal{T}^1 \cong \mathcal{O}_S$. Thus we recover Kawamata and Namikawa's result [32] showing that $X_0(B, \mathcal{P}, s)$ carries a normal crossings log structure over 0^\dagger if and only if $X_0(B, \mathcal{P}, s)$ is d -semistable. This is because, of course, the \mathcal{O}_S^\times -torsor associated to \mathcal{T}^1 has a section if and only if $\mathcal{T}^1 \cong \mathcal{O}_S$.

Now Theorem 3.24 of [20] tells us that in general $\mathcal{L}\mathcal{S}_{X_0(B, \mathcal{P}, s)}$ is not a trivial \mathcal{O}_S^\times -torsor. The sheaf depends continuously on s , but discretely on monodromy of the singularities of B .

Let's explain the latter point explicitly if $\dim B = 2$. The irreducible components of S are in one-to-one correspondence with one-dimensional cells of \mathcal{P} . If $\omega \in \mathcal{P}$ is such an edge, suppose it contains one singularity of B such that Λ has monodromy $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ in a suitable basis around a loop around the singularity. Then \mathcal{T}^1 restricted to the one-dimensional stratum $X_\omega \cong \mathbb{P}^1$ of $X_0(B, \mathcal{P}, s)$ is $\mathcal{O}_{\mathbb{P}^1}(n)$.

To make this statement completely accurate, one needs to define n so that it is independent of the choice of basis and loop. To do this, one chooses a loop which is counterclockwise with respect to the orientation determined by the chosen basis of $\Lambda_b = \mathcal{T}_{B,b}$, where $b \in B$ is the base-point of the loop.

If all the n 's appearing are positive, then for some choices of gluing data s , we may hope to have a section t of \mathcal{T}^1 which vanishes only at a finite set of points Z . If Z does not contain a toric stratum (i.e. a triple point) then we obtain a log structure on $X_0(B, \mathcal{P}, s) \setminus Z$ of the desired sort, hence a log structure on $X_0(B, \mathcal{P}, s)$ (log smooth off of Z) by push-forward. We then have

PROPOSITION 8.6. *In the situation of this example, with $\dim B = 2$ and*

$$t \in \Gamma(X_0(B, \mathcal{P}, s), \mathcal{T}^1)$$

a section vanishing on a finite set Z not containing a triple point, there exists a smoothing $\mathcal{X} \rightarrow D$ of $X_0(B, \mathcal{P}, s)$ such that the singular locus of \mathcal{X} is $Z \subseteq \mathcal{X}_0 = X_0(B, \mathcal{P}, s)$, and the induced log morphism $\mathcal{X}_0^\dagger \rightarrow 0^\dagger$ coincides with $X_0(B, \mathcal{P}, s)^\dagger \rightarrow 0^\dagger$ determined by t .

The proof of this is a rather simple application of Friedman's or Kawamata and Namikawa's results.

On the other hand, if $n < 0$ for some singular point of B , we run into problems, and there is in fact no smoothing of $X_0(B, \mathcal{P}, s)$. This should not be surprising for the following reason. If $n = -1$, it turns out we would have to compactify the torus fibration $X(B_0)$ by adding a strange sort of I_1 fibre over such a singular

point. An I_1 fibre is an immersed sphere, and the intersection multiplicity of the two sheets at the singular point of the fibre is $+1$ for an ordinary I_1 fibre. However, when the monodromy is given by $n = -1$, the intersection multiplicity is -1 . This does not occur for a special Lagrangian T^2 -fibration, so it is not surprising we can't construct a smoothing in this case. \square

If $\dim B = 2$ and $n > 0$ for all singularities on B , then we say B is *positive*. One can generalize this notion of positive to higher dimensional B with polyhedral decompositions, see [20], Definition 1.54. Positivity of B is a necessary condition for $X_0(B, \mathcal{P}, s)$ to appear as the central fibre of a toric degeneration. All the examples of §4 are positive; this in fact follows from the convexity of reflexive polytopes, and the positive condition can be viewed as a type of convexity statement.

Passing back to the general case now, with no restriction on the dimension of B or the shape of the cells of \mathcal{P} , it follows from [20], Theorem 3.24, that $\mathcal{L}\mathcal{S}_{X_0(B, \mathcal{P}, s)}$ is a subsheaf of *sets* of a coherent sheaf we call $\mathcal{L}\mathcal{S}_{\text{pre}, X_0(B, \mathcal{P}, s)}^+$. This sheaf is a direct sum $\bigoplus_{\substack{\omega \in \mathcal{P} \\ \dim \omega = 1}} \mathcal{N}_\omega$, where \mathcal{N}_ω is a line bundle on the toric stratum of $X_0(B, \mathcal{P}, s)$ corresponding to ω . Furthermore, as in the two-dimensional normal crossings case, \mathcal{N}_ω is a semi-ample line bundle if B is positive.

A section $t \in \Gamma(X_0(B, \mathcal{P}, s), \mathcal{L}\mathcal{S}_{\text{pre}, X_0(B, \mathcal{P}, s)}^+)$ which is a section of $\mathcal{L}\mathcal{S}_{X_0(B, \mathcal{P}, s)}$ outside of the zero set Z of t determines a log smooth structure on $X_0(B, \mathcal{P}, s) \setminus Z$. In particular, if Z does not contain any toric stratum, we are in good shape. We then obtain a log morphism $X_0(B, \mathcal{P}, s)^\dagger \rightarrow 0^\dagger$ which is log smooth away from Z . We call such a structure a *log Calabi-Yau space*.

Let's review: given data

- s open gluing data;
- $t \in \Gamma(X_0(B, \mathcal{P}, s), \mathcal{L}\mathcal{S}_{\text{pre}, X_0(B, \mathcal{P}, s)}^+)$, with t a section of $\mathcal{L}\mathcal{S}_{X_0(B, \mathcal{P}, s)}$ over $X_0(B, \mathcal{P}, s) \setminus Z$ for some set Z which does not contain any toric stratum of $X_0(B, \mathcal{P}, s)$.

we obtain $X_0(B, \mathcal{P}, s)^\dagger \rightarrow 0^\dagger$.

Conversely, we show in [20] that if $\mathcal{X} \rightarrow D$ is a toric degeneration, then $\mathcal{X}_0^\dagger \rightarrow 0^\dagger$ is obtained in this way from the dual intersection complex (B, \mathcal{P}) from some choice of data s and t . To complete this picture, it remains to answer

QUESTION 8.7 (The reconstruction problem, Version III). *Suppose (B, \mathcal{P}) is positive.*

- (1) *What are the possible choices of s and t which yield log Calabi-Yau spaces?*
- (2) *Given $X_0(B, \mathcal{P}, s)^\dagger \rightarrow 0^\dagger$, when does it arise as the central fibre of a toric degeneration $\mathcal{X} \rightarrow D$?*

As we have sketched it, this question is now the refined version of our basic reconstruction problem Question 7.11.

The choice of the data s and t determine the moduli of log Calabi-Yau spaces arising from a given dual intersection complex. So far we haven't even made the claim that this moduli space is non-empty, and for general choice of (B, \mathcal{P}) , I do not know if this is the case or not, though it is non-empty if $\dim B = 2$ or 3 . However, one would like a more explicit description of this moduli space in any event. In general the moduli space is a scheme, but it may be singular (an example is given in [20], Example 4.28). Some additional hypotheses are necessary to solve

this problem. To motivate the necessary hypothesis, let's go back to §1, where we introduced the notion of simplicity. We saw that the basic topology of mirror symmetry works only when the fibration is simple. So maybe we should expect the current construction to work better when we have simplicity.

There is one technical problem with this: the definition of simplicity assumes the existence of a torus fibration $f : X \rightarrow B$. Instead, we want to define simplicity entirely in terms of B itself. Unfortunately, the solution to this is rather technical, and produces a definition which is very difficult to absorb (Definition 1.60 of [20]). Let us just say here that if B is simple in this new sense and $X(B_0) \rightarrow B_0$ was compactified in a sensible manner to a topological torus fibration $f : X(B) \rightarrow B$, then f would be simple in the sense of §1.

However, once we accept this, life simplifies a great deal. Extraordinarily, the a priori very complicated moduli space of log Calabi-Yau spaces with a given dual intersection complex has a very simple description when B is simple! One very difficult main result of [20], (Theorem 5.4) is

THEOREM 8.8. *Given (B, \mathcal{P}) positive and simple, the set of log Calabi-Yau spaces with dual intersection complex (B, \mathcal{P}) , modulo isomorphism preserving B , is $H^1(B, i_*\Lambda \otimes \mathbb{C}^\times)$. An isomorphism is said to preserve B if it induces the identity on the dual intersection complex.*

So the moduli space is an algebraic torus (or a disjoint union of algebraic tori) of dimension equal to $\dim_k H^1(B, i_*\Lambda \otimes \mathbb{C})$.

Note that this is the expected dimension predicted by the SYZ conjecture. Indeed, if a smoothing of $X_0(B, \mathcal{P}, s)^\dagger$ exists and it was a topological compactification $X(B)$ of $X_0(B)$, with a torus fibration $f : X(B) \rightarrow B$ extending $f_0 : X(B_0) \rightarrow B_0$, then $R^{n-1}f_{0*}\mathbb{R} \cong \Lambda_{\mathbb{R}}$, so by simplicity, $R^{n-1}f_*\mathbb{R} \cong i_*\Lambda_{\mathbb{R}}$. The discussion of §1 suggests that $\dim H^1(B, R^{n-1}f_*\mathbb{R})$ is $h^{1,n-1}$ of the smoothing, which is of course the dimension of the complex moduli space of the smoothing.

This argument can be made rigorous by introducing *log differentials*.

DEFINITION 8.9. Let $\pi : X^\dagger \rightarrow S^\dagger$ be a morphism of logarithmic spaces. A *log derivation* on X^\dagger over S^\dagger with values in an \mathcal{O}_X -module \mathcal{E} is a pair (D, Dlog) , where $D : \mathcal{O}_X \rightarrow \mathcal{E}$ is an ordinary derivation of X/S and $\text{Dlog} : \mathcal{M}_X^{\text{gp}} \rightarrow \mathcal{E}$ is a homomorphism of abelian sheaves with $\text{Dlog} \circ \pi^\# = 0$; these fulfill the following compatibility condition

$$D(\alpha_X(m)) = \alpha_X(m) \cdot \text{Dlog}(m),$$

for all $m \in \mathcal{M}_X$.

We denote by $\Theta_{X^\dagger/S^\dagger}$ the sheaf of log derivations of X^\dagger over S^\dagger with values in \mathcal{O}_X . We set $\Omega_{X^\dagger/S^\dagger}^1 = \text{Hom}_{\mathcal{O}_X}(\Theta_{X^\dagger/S^\dagger}, \mathcal{O}_X)$.

This generalizes the more familiar notion of differentials with logarithmic poles along a normal crossings divisor. If $Y \subseteq X$ is a normal crossings divisor inducing a log structure on X , then $\Omega_{X^\dagger/k}^1$ is the sheaf of differentials with logarithmic poles along Y , and $\Omega_{Y^\dagger/k}^1$ is the restriction of this sheaf to Y . In general, $\Omega_{X^\dagger/S^\dagger}^1$ is locally free if π is log smooth. As a result, one can do deformation theory in the log category for log smooth morphisms (see [30]). This is one of the principle reasons for introducing log geometry into our picture.

If $X_0(B, \mathcal{P}, s)^\dagger \rightarrow 0^\dagger$ is a log Calabi-Yau space, then the morphism is log smooth off of Z . Define

$$\begin{aligned}\Theta_{X_0(B, \mathcal{P}, s)}^p &:= j_* \bigwedge^p \Theta_{(X_0(B, \mathcal{P}, s)^\dagger \setminus Z)/0^\dagger} \\ \Omega_{X_0(B, \mathcal{P}, s)}^p &:= j_* \bigwedge^p \Omega_{(X_0(B, \mathcal{P}, s)^\dagger \setminus Z)/0^\dagger}\end{aligned}$$

where $j : X_0(B, \mathcal{P}, s) \setminus Z \rightarrow X_0(B, \mathcal{P}, s)$ is the inclusion.

Then one has

THEOREM 8.10. *Suppose (B, \mathcal{P}) is positive and simple, and suppose we are given a log Calabi-Yau space $X_0(B, \mathcal{P}, s)^\dagger \rightarrow 0^\dagger$ which occurs as the central fibre of a toric degeneration $\mathcal{X} \rightarrow D$ whose general fibre \mathcal{X}_t is non-singular. Then for $q = 0, 1, n-1$ and n with $n = \dim B$, we have isomorphisms*

$$\begin{aligned}H^p(B, i_* \bigwedge^q \Lambda \otimes k) &\cong H^p(X_0(B, \mathcal{P}, s), \Theta_{X_0(B, \mathcal{P}, s)}^q) \cong H^p(\mathcal{X}_t, \Theta_{\mathcal{X}_t}^q) \\ H^p(B, i_* \bigwedge^q \check{\Lambda} \otimes k) &\cong H^p(X_0(B, \mathcal{P}, s), \Omega_{X_0(B, \mathcal{P}, s)}^q) \cong H^p(\mathcal{X}_t, \Omega_{\mathcal{X}_t}^q)\end{aligned}$$

where $\Theta_{\mathcal{X}_t}^q$ and $\Omega_{\mathcal{X}_t}^q$ are the ordinary sheaves of holomorphic poly-vector fields and holomorphic differentials on a smooth fibre \mathcal{X}_t .

The proof of this result will be given in a forthcoming paper with Siebert. The result holds for all q when additional hypotheses are assumed, essentially saying the mirror to \mathcal{X}_t is non-singular. Note in particular, since Λ and $\check{\Lambda}$ are interchanged under discrete Legendre transform, we get the interchange of ordinary Hodge numbers from this result. In the more general situation where the Calabi-Yaus arising are singular, one might speculate about the relationship between these groups and stringy Hodge numbers.

9. The cone picture and the fan picture

This section is purely philosophical. In most of our discussion in §§7 and 8, we focused on the dual intersection complex, and in particular, focused on the question of constructing a degeneration from its dual intersection complex. Since our primary goal was to solve the reconstruction problem Question 5.6, and as the dual intersection complex is related to the complex structure (Theorem 7.7) it seems natural to focus on the dual intersection complex. We will see in the next section that this intuition may not always be correct. So far, the intersection complex only seemed to arise when talking about mirror symmetry. However, mirror symmetry instructs us to view both sides of the picture on the same footing. When we construct a degenerate Calabi-Yau space from a dual intersection complex, we say we are in the *fan picture*, while if we construct a degenerate Calabi-Yau space from an intersection complex, we say we are in the *cone picture*.

More precisely, we have seen how given an integral affine manifold with singularities with toric polyhedral decomposition (B, \mathcal{P}) , then an additional choice of open gluing data s specifies a space $X_0(B, \mathcal{P}, s)$, along with a sheaf of monoids $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}$. Some additional data may specify a log structure on $X_0(B, \mathcal{P}, s)$ with this ghost sheaf. The irreducible components of $X_0(B, \mathcal{P}, s)$ are defined using fans, given by the fan structure of \mathcal{P} at each vertex of \mathcal{P} . This is why we call this side the fan picture.

On the other hand, given (B, \mathcal{P}) we can also construct a *projective* scheme $\check{X}_0(B, \mathcal{P}, \check{s})$ given suitable gluing data \check{s} . The irreducible components of this scheme

are in one-to-one correspondence with the maximal cells of \mathcal{P} ; given such a maximal cell σ , viewing it as a lattice polytope in \mathbb{R}^n determines a projective toric variety, and $\tilde{X}_0(B, \mathcal{P}, \check{s})$ is obtained by gluing together these projective toric varieties using the data \check{s} . This is not quite the same data as occurred in the fan picture, because we also need to glue the line bundles, and this is additional data. The reason for calling this side the cone picture is that each irreducible component can be described as follows. Given $\sigma \subseteq M_{\mathbb{R}}$, let $P_\sigma = C(\sigma) \cap (M \oplus \mathbb{Z})$. Then the corresponding projective toric variety is $\text{Proj } \mathbb{C}[P_\sigma]$, where $\mathbb{C}[P_\sigma]$ is graded using the projection of P_σ onto \mathbb{Z} . Hence the irreducible components and strata arise from cones over elements of \mathcal{P} .

We summarize the duality between the cone and fan pictures:

Fan picture	
Gluing data yields	$X_0(B, \mathcal{P}, s), \overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}$
$\sigma \in \mathcal{P}, \dim \sigma = p$	An $n - p$ -dimensional stratum of $X_0(B, \mathcal{P}, s)$
φ multi-valued convex PL function	An ample line bundle on $X_0(B, \mathcal{P}, s)$
$H^p(B, i_* \wedge^q \Lambda \otimes k)$	$H^p(X_0(B, \mathcal{P}, s), \Theta_{X_0(B, \mathcal{P}, s)}^q)$
Cone picture	
Gluing data yields	$\tilde{X}_0(B, \mathcal{P}, \check{s}),$ ample line bundle
$\sigma \in \mathcal{P}, \dim \sigma = p$	A p -dimensional stratum of $\tilde{X}_0(B, \mathcal{P}, \check{s})$
φ multi-valued convex PL function	A sheaf of monoids $\overline{\mathcal{M}}_{\tilde{X}_0(B, \mathcal{P}, \check{s})}$
$H^p(B, i_* \wedge^q \Lambda \otimes k)$	$H^p(\tilde{X}_0(B, \mathcal{P}, \check{s}), \Omega_{\tilde{X}_0(B, \mathcal{P}, \check{s})}^q)$

(Some restrictions may apply to gluing data on both sides in order for φ to yield the desired data.)

In particular, mirror symmetry interchanges discrete information about the log structure (i.e. $\overline{\mathcal{M}}_{X_0(B, \mathcal{P}, s)}$) and discrete information about the polarization (i.e. the class of the line bundle on each irreducible component).

10. Tropical curves

So far we have seen only the most elementary aspects of mirror symmetry emerge from this algebro-geometric version of SYZ, e.g. the interchange of Hodge numbers. However, the real interest in this approach lies in hints that it will provide a natural explanation for rational curve counting in mirror symmetry. If we follow the philosophy of the previous section, we need to identify structures on affine manifolds with singularities which in one of the two pictures has to do with rational curves and in the other picture has to do with periods. I believe the correct structure to study is that of tropical curves on affine manifolds with singularities B . See [40],[44] for an introduction to tropical curves in \mathbb{R}^n . Here, we can take B to be tropical, rather than integral; hence the name.

DEFINITION 10.1. Let B be a tropical affine manifold with singularities with discriminant locus Δ . Let Γ be a weighted, connected finite graph, with its set of vertices and edges denoted by $\Gamma^{[0]}$ and $\Gamma^{[1]}$ respectively, with weight function $w_\Gamma : \Gamma^{[1]} \rightarrow \mathbb{N} \setminus \{0\}$. A parametrized tropical curve in B is a continuous map $h : \Gamma \rightarrow B$ satisfying the following conditions:

- (1) For every edge $E \subseteq \Gamma$, $h|_{\text{Int}(E)}$ is an embedding, $h^{-1}(B_0)$ is dense in $\text{Int}(E)$, and there is a primitive section $u \in \Gamma(\text{Int}(E), h^*(i_*\Lambda))$ which is tangent to $h(\text{Int}(E))$ at every point of $h(\text{Int}(E)) \cap B_0$.
- (2) For every vertex $v \in \Gamma^{[0]}$, let $E_1, \dots, E_m \in \Gamma^{[1]}$ be the edges adjacent to v . Let u_i be the section of $h^*(i_*\Lambda)|_{\text{Int}(E_i)}$ promised by (1), chosen to point away from v . This defines germs $u_i \in h^*(i_*\Lambda)_v = (i_*\Lambda)_{h(v)}$.
 - (a) If $h(v) \in B_0$, the following *balancing condition* holds in $\Lambda_{h(v)}$:

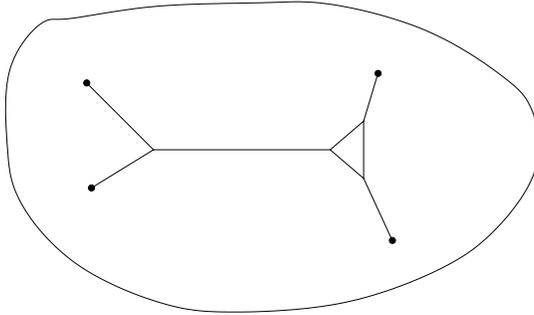
$$\sum_{j=1}^m w_\Gamma(E_j) u_j = 0.$$
 - (b) If $h(v) \notin B_0$, then the following balancing condition is satisfied in $(i_*\Lambda)_{h(v)}$:

$$\sum_{j=1}^m w_\Gamma(E_j) u_j = 0 \pmod{(i_*\check{\Lambda})_{h(v)}^\perp \cap (i_*\Lambda)_{h(v)}}.$$

The latter group is interpreted as follows. Let $b \in B_0$ be a point near $h(v)$, and identify, via parallel transport along a path between $h(v)$ and b , the groups $(i_*\Lambda)_{h(v)}$ and $(i_*\check{\Lambda})_{h(v)}$ with local monodromy invariant subgroups of Λ_b and $\check{\Lambda}_b$ respectively. Then $(i_*\check{\Lambda})_{h(v)}^\perp$ is a subgroup of Λ_b , and the intersection makes sense. It is independent of the choice of b and path.

So tropical curves behave away from the discriminant locus of B much as the tropical curves of [40],[44] do, but they may have legs terminating on the discriminant locus. As we are interested in the case that B is compact, we do not want legs which go off to ∞ . I warn the reader, however, that this definition is provisional, and the behaviour in (2) (b) may not be exactly what we want.

Here we see a tropical elliptic curve, the solid dots being points of the discriminant locus. The legs terminating at these points must be in a monodromy invariant direction.



Now let us connect this to the question of counting curves. In the situation of a degeneration, $\varphi: \mathcal{X} \rightarrow D$, it is natural to consider families of maps of curves:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \pi \downarrow & & \downarrow \varphi \\ D & \xrightarrow{g} & D \end{array}$$

Here g may be a ramified covering, and π is a flat morphism with reduced one-dimensional fibres. In the case of interest, $f|_{\mathcal{C}_t} : \mathcal{C}_t \rightarrow \mathcal{X}_t$ should be a stable map of curves for $t \neq 0$. Let us assume that \mathcal{C}_t is a non-singular curve for $t \neq 0$. In the logarithmic context, it is then natural to put the log structure induced by the divisor $\mathcal{C}_0 \subseteq \mathcal{C}$ on \mathcal{C} , and so get a diagram

$$\begin{array}{ccc} \mathcal{C}^\dagger & \xrightarrow{f} & \mathcal{X}^\dagger \\ \pi \downarrow & & \downarrow \varphi \\ D^\dagger & \xrightarrow{g} & D^\dagger \end{array}$$

of log morphisms. Restricting to the central fibre, we obtain a diagram

$$\begin{array}{ccc} \mathcal{C}_0^\dagger & \xrightarrow{f} & \mathcal{X}_0^\dagger \\ \pi \downarrow & & \downarrow \varphi \\ 0^\dagger & \xrightarrow{g} & 0^\dagger \end{array}$$

This suggests that we should build up a theory of stable log maps and log Gromov-Witten invariants. I will say little about this here, as this rapidly gets quite technical. There is work in progress of Siebert on this subject. This point of view has already been used in [42] for counting curves in toric varieties, so some more hints of this approach can be found there. Instead, I wish to sketch how such a diagram yields a tropical curve.

To do so, consider a situation where π is normal crossings, and the induced map $\mathcal{C}_0^\dagger \rightarrow \mathcal{X}_0^\dagger$ has no infinitesimal log automorphisms over 0^\dagger . (This is the log equivalent of the notion of stable map). Let (B, \mathcal{P}) be the dual intersection complex of the log Calabi-Yau space \mathcal{X}_0^\dagger . We can define the *dual intersection graph* of $f : \mathcal{C}_0^\dagger \rightarrow \mathcal{X}_0^\dagger$, which will be a parameterized tropical curve on B . I will only do the case here when the image of f is disjoint from the set $Z \subseteq \mathcal{X}$ of Definition 7.1, (4); otherwise there are some technicalities to worry about.

First we build Γ . Let C_1, \dots, C_m be the irreducible components of \mathcal{C}_0 . Assume these components are normal for ease of describing this construction. Set $\Gamma^{[0]} = \{v_1, \dots, v_m\}$. On the other hand, $\Gamma^{[1]}$ will contain an edge $\overline{v_i v_j}$ joining v_i and v_j whenever $C_i \cap C_j \neq \emptyset$.

To define $h : \Gamma \rightarrow B$, we first describe the image of each vertex. Let X_{σ_i} be the minimal stratum of \mathcal{X}_0 containing $f(C_i)$, where $\sigma_i \in \mathcal{P}$. Let η_i be the generic point of C_i , $\xi_i = f(\eta_i)$. Then we have an induced map $f^\# : \mathcal{M}_{\mathcal{X}_0, \xi_i} \rightarrow \mathcal{M}_{\mathcal{C}_0, \eta_i}$, as f is a log morphism. This induces a diagram on stalks of ghost sheaves

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\mathcal{C}_0, \eta_i} & \xleftarrow{\overline{f}^\#} & \overline{\mathcal{M}}_{\mathcal{X}_0, \xi_i} \\ \pi^\# \uparrow & & \uparrow \overline{\varphi}^\# \\ \overline{\mathcal{M}}_0 & \xleftarrow{\overline{g}^\#} & \overline{\mathcal{M}}_0 \end{array}$$

Now $\overline{\mathcal{M}}_0 = \mathbb{N}$ (see Example 8.2, (4)) and $\overline{\mathcal{M}}_{\mathcal{C}_0, \eta_i} = \mathbb{N}$ since π is normal crossings. On the other hand, $\pi^\#$ is the identity and if g is a branched cover of degree d , then

$g^\#$ is multiplication by d . By Exercise 8.3,

$$\overline{\mathcal{M}}_{\mathcal{X}_0, \xi_i} = \text{Hom}(C(\sigma_i) \cap (M \oplus \mathbb{Z}), \mathbb{N}).$$

But

$$\text{Hom}_{\text{monoid}}(\text{Hom}_{\text{monoid}}(C(\sigma_i) \cap (M \oplus \mathbb{Z}), \mathbb{N}), \mathbb{N}) = C(\sigma_i) \cap (M \oplus \mathbb{Z}),$$

so $\bar{f}^\#$ is determined by an element (m, r) of $C(\sigma_i) \cap (M \oplus \mathbb{Z})$. Now

$$\bar{f}^\#(\bar{\varphi}^\#(1)) = \bar{f}^\#(0, 1) = \langle (m, r), (0, 1) \rangle = r$$

while

$$\bar{\pi}^\#(\bar{g}^\#(1)) = \bar{\pi}^\#(d) = d.$$

Thus $r = d$, and $m/d \in \sigma_i$. We define $h(v_i) = m/d$. This is a point of σ_i which is contained in B .

If $C_i \cap C_j \neq \emptyset$, there is a minimal stratum $X_{\sigma_{i,j}}$ containing $C_i \cap C_j$. Of course $X_{\sigma_{i,j}} \subseteq X_{\sigma_i} \cap X_{\sigma_j}$. In particular, $\sigma_{i,j}$ contains σ_i and σ_j . We take $h(\overline{v_i v_j})$ to be the straight line joining $h(v_i)$ and $h(v_j)$ inside $\sigma_{i,j}$. Furthermore, if $\sigma_{i,j} \subseteq M_{\mathbb{R}}$ is embedded as a lattice polytope, let m_{ij} be a primitive lattice element parallel to $m_i - m_j$, and we take $w_\Gamma(\overline{v_i v_j})$ to be defined by the equation

$$w_\Gamma(\overline{v_i v_j}) m_{ij} = \#(C_i \cap C_j)(m_i - m_j)/d.$$

PROPOSITION 10.2. *h is a parametrized tropical curve.*

We do not give a proof here. The case where X_{σ_i} is always an irreducible component of \mathcal{X}_0 is essentially covered in [42]. Instead, we'll do another extremal case, which exhibits some interesting features of log geometry.

EXAMPLE 10.3. Suppose a component C_1 of \mathcal{C}_0 and all components C_2, \dots, C_t intersecting C_1 are mapped by f to a zero dimensional stratum X_σ of \mathcal{X}_0 . Without loss of generality we can assume

$$\mathcal{X}_0 = V(\sigma) = \text{Spec } \mathbb{C}[C(\sigma)^\vee \cap (N \oplus \mathbb{Z})]/(z^{(0,1)})$$

as defined in §8. Thus h maps v_1, \dots, v_t into points $m_1/d, \dots, m_t/d \in \sigma$. Let us understand why the balancing condition holds at m_1/d . Let $U \subseteq \mathcal{C}_0$ be an open neighbourhood of C_1 which only intersects C_1, \dots, C_t , so h is constant on U as an ordinary morphism (but not as a log morphism). Restrict the log structure on \mathcal{C}_0 to U . We have an exact sequence

$$1 \longrightarrow \mathcal{O}_U^\times \longrightarrow \mathcal{M}_U^{\text{gp}} \xrightarrow{p} \overline{\mathcal{M}}_U^{\text{gp}} \longrightarrow 0.$$

Taking global sections, we get

$$1 \longrightarrow \Gamma(U, \mathcal{O}_U^\times) \longrightarrow \Gamma(U, \mathcal{M}_U^{\text{gp}}) \xrightarrow{p} \Gamma(U, \overline{\mathcal{M}}_U^{\text{gp}}) = \mathbb{Z}^t \xrightarrow{q} \text{Pic } U.$$

A section $s \in \Gamma(U, \overline{\mathcal{M}}_U^{\text{gp}})$ defines an \mathcal{O}_U^\times -torsor $p^{-1}(s)$, whose class in the Picard group of U is $q(s)$. It is an easy exercise in log geometry to show that if s is the i th standard basis vector for \mathbb{Z}^t , then $q(s) = \mathcal{O}_C(-C_i)|_U$. Note $\deg \mathcal{O}_C(-C_i)|_{C_1} = -\#C_1 \cap C_i$ for $i = 2, \dots, t$ and $\deg \mathcal{O}_C(-C_i)|_{C_1} = \sum_{i=2}^t \#C_1 \cap C_i$ for $i = 1$ as $C_1 \cdot \mathcal{C}_0 = 0$ in \mathcal{C} .

Now observe that $f^\#$ acting on the sheaves of monoids induces a diagram

$$\begin{array}{ccccc} \mathcal{O}_{V(\sigma),x}^\times \oplus \overline{\mathcal{M}}_{V(\sigma),x}^{\text{gp}} & \xrightarrow{\cong} & \mathcal{M}_{V(\sigma),x}^{\text{gp}} & \xrightarrow{f^\#} & \Gamma(U, \mathcal{M}_U^{\text{gp}}) \\ & & \downarrow & & \downarrow p \\ N \oplus \mathbb{Z} & \xrightarrow{\cong} & \overline{\mathcal{M}}_{V(\sigma),x}^{\text{gp}} & \xrightarrow{\bar{f}^\#} & \Gamma(U, \overline{\mathcal{M}}_U^{\text{gp}}) \xrightarrow{\cong} \mathbb{Z}^t. \end{array}$$

Now the map $\bar{f}^\#$, by construction, is given by

$$(n, r) \in N \oplus \mathbb{Z} \mapsto ((n, r), (m_i, d))_{i=1, \dots, t}.$$

On the other hand, in order for $\bar{f}^\#$ to lift to $f^\#$, the \mathcal{O}_U^\times torsor $p^{-1}(\bar{f}^\#(n, r))$ must have a section for every $(n, r) \in N \oplus \mathbb{Z}$, i.e. must be trivial in the Picard group. This implies

$$\deg \bigotimes_{i=1}^t (\mathcal{O}_C(-C_i)|_{C_1})^{\otimes \langle (n, r), (m_i, d) \rangle} = 0,$$

or

$$\sum_{i=2}^t (\#C_1 \cap C_i) \langle (n, r), (m_i, d) - (m_1, d) \rangle = 0$$

for all $(n, r) \in N \oplus \mathbb{Z}$. But this is equivalent to

$$\sum_{i=2}^t (\#C_1 \cap C_i) (m_i/d - m_1/d) = 0,$$

which is the balancing condition. \square

Following the logic of mirror symmetry, this suggests that tropical curves on the cone side should have to do with periods. It is only recently that an understanding of this has begun to emerge, and unfortunately, I do not have space or time to elaborate on this. Let me say that at the time of this writing, Siebert and I appear to have a complete solution to Question 8.7, (2). In this solution, we construct explicit deformations of a log Calabi-Yau space, order by order. Formally, our construction looks somewhat similar to that of Kontsevich and Soibelman [35] for constructing non-Archimedean K3 surfaces from affine manifolds, and we apply a key lemma of [35]. However, Kontsevich and Soibelman work on what we would call the fan side, while we work on the cone side. This may be surprising given that all of our discussions involving the strategy of building log Calabi-Yau spaces was done on the fan side. However, if we take the mirror philosophy seriously, and we want to see tropical curves appear in a description of a smoothing, we need to work on the cone side. It turns out to be extremely natural. In fact, all tropical rational curves play a role in our construction. While it is not yet clear exactly how these curves influence periods, it is clear they have something to do with them. Once this is fully understood, I believe we will finally have a firm understanding of a geometric explanation of mirror symmetry.

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