Lecture 3

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Goal of the Day

The goal of today is to find an answer for our old friend

$\mathbf{Q_d}$: *What is the number of rational curves of degree d through $3d-1$ points in the plane?*

We will tackle this question by introducing moduli spaces of stable maps, and we will sketch the proof of Kontsevich using Gromov-Witten invariants. Before we do so though, I want to go back to Q_3, where I told you the answer was 12, and present a classical proof of this fact. Hopefully the amount of cleverness needed for this proof will convince you of the need for a new idea to approach the general question.

Sketch of Classical Proof for 12 Rational Cubics

Since we know that passing through 8 points corresponds to 8 linear conditions, we need to show that being a rational (aka nodal) cubic cuts a hypersurface of degree 12 in the $\mathbb{P}^9$ parameterizing cubics in $\mathbb{P}^2$.

We therefore consider a general line (with coordinate t) in the space of cubics: it has the form

$$f(x,y) + tg(x,y) = 0, \tag{1}$$

where f and g are polynomials of degree 3. Figure 1 illustrates the situation. On the right hand side we (attempted to) draw the total space $\mathcal{S}$ of the family over the t-line. This means, we consider the surface in $\mathbb{P}^1 \times \mathbb{P}^2$ cut out by equation (1). Or, another way to think of it, the fiber over a particular point $\bar{t}$ is precisely the cubic $\{f(x,y) + \bar{t}g(x,y) = 0\}$ living in the $\mathbb{P}^2$-plane $t = \bar{t}$.

We now compute the Euler characteristic of the total space of $\mathcal{S}$ in two different ways, and use this to compute the number of nodal cubics in this family.

Global description: $\mathcal{S}$ is "almost" equal to $\mathbb{P}^2$, because, for any point $P \in \mathbb{P}^2$ different from the 9 points of intersection of f and g, there is exactly one cubic in the family containing P. Those 9 points, on the other hand, are

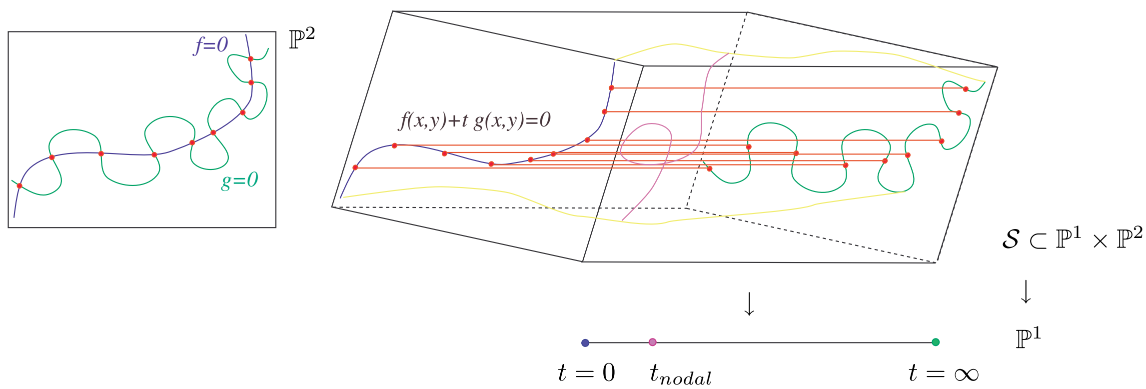

Figure 1: A general line in the space of cubics obtained as the linear span of $f = 0$ and $g = 0$. On the right hand side, $\mathcal{S}$ is the total space of the family. Notice that this surface contains 9 "horizontal" lines.

contained in every single cubic of the family, giving rise to the 9 horizontal lines drawn in the picture. We therefore see that:

$$\mathcal{S} = \mathbb{P}^2 \smallsetminus 9\text{points} \sqcup 9\mathbb{P}^1$$

(those with a little bit of experience in algebraic geometry will have recognized $\mathcal{S}$ as the blow-up of $\mathbb{P}^2$ at the 9 points above). Therefore

$$\mathcal{X}(\mathcal{S}) = 3 - 9 + 18 = 12 \tag{2}$$

Fiberwise description: now consider the family S fiber by fiber. The general fiber is a smooth cubic, which is a torus and has Euler Characteristic 0. There are a number n_{nod} of nodal cubics, which contribute 1 to the Euler Characteristic. I.e.

$$\mathcal{X}(\mathcal{S}) = n_{nod} \tag{3}$$

And equating (2) and (3) gives precisely what we want: there are 12 nodal cubics in the family!

Moduli Spaces of Rational Stable Maps

Seen how much cleverness was required to solve Q_3 this way, we are going to radically change our point of view. Instead of thinking of a rational curve of degree d as of a curve of degree d that happens to have enough nodes as to be rational, we think of it as the image of a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ of degree d.

Problem 1. *Describe the moduli space of maps $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ of degree d[1] . Find that its dimension is $3d-1$!*

Problem 2. *Introduce marks in the picture. Realize that each mark increases the dimension by* 1*.*

As usual, this moduli space is not very interesting, and further it is not compact. And, as usual, it is the compactification that makes things a lot more interesting.

Definition 1. *An **n-pointed rational stable map** is a map $\varphi : C \to \mathbb{P}^2$, where:*

1. *C is a n-marked tree of projective lines.*
2. *Every twig in C mapped to a point must have at least three special points on it.*

Problem 3. *Realize that condition* 2 *is equivalent to asking that the map has only finitely many automorphisms. Since I haven't told you what an automorphism of a map is, this might be a bit tricky...however I will leave as part of the exercise figuring out what the natural concept of an automorphism might be in this case.*

Fact/Definition: *The **moduli space of rational stable maps of degree** d **to** $\mathbb{P}^2$ **with** n **marks*** (in short $\overline{M}_{0,n}(\mathbb{P}^2, d)$) is a smooth[2] compactification of the moduli spaces of n-pointed maps from a smooth $\mathbb{P}^1$.

Natural Maps

There are natural maps between moduli spaces of stable maps:

evaluation maps: there are as many of these maps as there are marks.

$$\begin{array}{rccc} ev_i : & \overline{M}_{0,n}(\mathbb{P}^2, d) & \to & \mathbb{P}^2 \\ & (C, \varphi, P_1, \ldots, P_n) & \mapsto & \varphi(P_i) \end{array}$$

forgetting points:

$$\begin{array}{rccc} \mathrm{forg}_i : & \overline{M}_{0,n}(\mathbb{P}^2, d) & \to & \overline{M}_{0,n-1}(\mathbb{P}^2, d) \\ & (C, \varphi, P_1, \ldots, P_n) & \mapsto & (C, \varphi, P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n) \end{array}$$

[1] A given geometric map can have more than one algebraic expression! This introduces an equivalence relation that you have to keep in account when answering this question.

[2] This is special to genus 0 and the target being a "convex" variety.

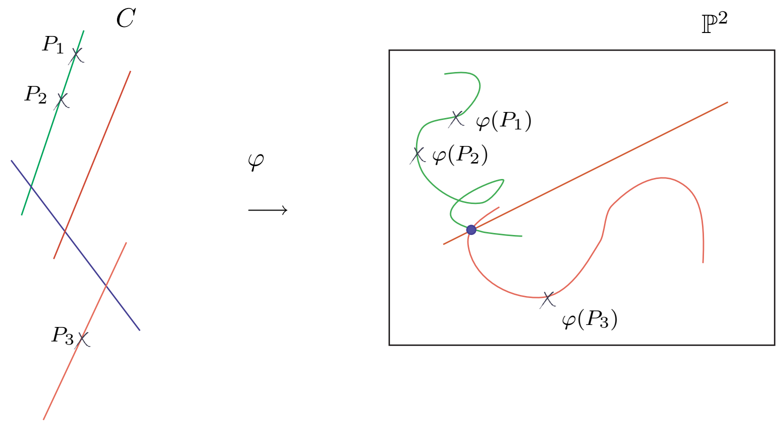

Figure 2: A rational stable map of degree d to $\mathbb{P}^2$

forgetting the map:

$$\begin{aligned} f: \quad \overline{M}_{0,n}(\mathbb{P}^2, d) &\to \overline{M}_{0,n} \\ (C, \varphi, P_1, \ldots, P_n) &\mapsto (C, P_1, \ldots, P_n) \end{aligned}$$

Problem 4. *What I just wrote is true generically, but there are cases in which you need to contract twigs and such to make things well defined. Make all of this rigorous.*

The boundary

The boundary can be described in terms of moduli spaces of maps of smaller degree. But in this case, we can't just take products, as we want to make sure that the points corresponding to the node "end up" in the same place on the target (see Figure 3). Therefore we have to take a fiber product with respect to the appropriate evaluation morphisms.

In the example of Figure 3, the boundary stratum is isomorphic to:

$$B \cong \overline{M}_{0,2\cup\{\bullet\}}(\mathbb{P}^2, d_1) \times_{ev_\bullet \times ev_\star} \overline{M}_{0,1\cup\{\star\}}(\mathbb{P}^2, d_2)$$

Remark. Recall that taking a fiber product is equivalent to intersecting the ordinary product with the pullback of the diagonal, i.e. :

$$\overline{M}_{0,2\cup\{\bullet\}}(\mathbb{P}^2, d_1) \times_{ev_\bullet \times ev_\star} \overline{M}_{0,1\cup\{\star\}}(\mathbb{P}^2, d_2) = \overline{M}_{0,2\cup\{\bullet\}}(\mathbb{P}^2, d_1) \times \overline{M}_{0,1\cup\{\star\}}(\mathbb{P}^2, d_2) \cap (ev_\bullet \times ev_\star)^{-1}(\Delta_{\mathbb{P}^2 \times \mathbb{P}^2})$$

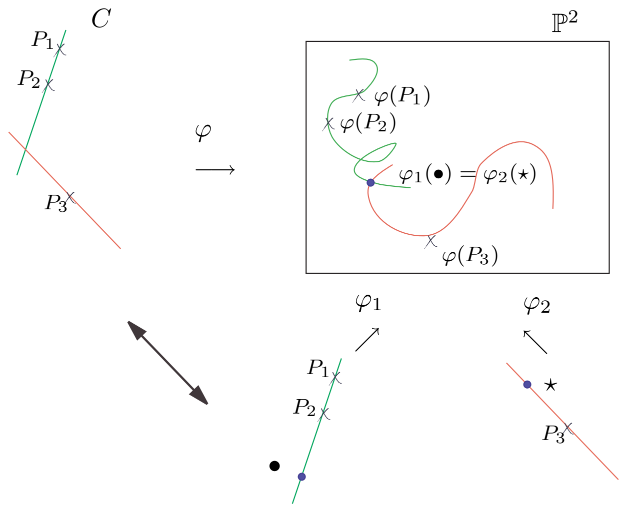

Figure 3: A boundary stratum.

Gromov-Witten Invariants

Finally we are ready to define our heroes: **Gromov-Witten invariants**. These are simply top intersections of special classes on moduli spaces of stable maps: take a closed subvariety α of the target space, and consider:

$$ev_i^*(\alpha).$$

I.e., all maps from pointed curves such that the i-th mark lands in α! We call this is a Gromov-Witten class.

Problem 5. *Show that a Gromov-Witten class has codimension in the moduli space of stable maps equal to the codimension of α in the target space.*

We define a **Gromov-Witten invariant** to be an intersection of Gromov-Witten classes that consists of a finite number of points. We denote it:

$$\langle \alpha_1 \dots \alpha_n \rangle_{0,d}^{\mathbb{P}^2} := \int_{\overline{M}_{0,n}(\mathbb{P}^2,d)} ev_1^*(\alpha_1) \cap \dots \cap ev_n^*(\alpha_n),$$

where the integral sign simply represents "counting the number of such points". The invariant is 0 if the intersection of the classes is either empty or of positive dimension.

Some properties of Gromov-Witten Invariants

Here are some basic properties of Gromov-Witten invariants.

Degree 0: the only (possibly) nonzero degree 0 invariants are those with exactly 3 mark points and sum of the codimensions of the three classes equal to the dimension of the target. In that case.

$$\langle \alpha_1\alpha_2\alpha_3\rangle^X_{0,0} = \alpha_1 \cap \alpha_2 \cap \alpha_3$$

Fundamental class insertions: any Gromov-Witten invariant containing a fundamental class insertion vanishes, unless it is of degree 0 and three pointed, in which case:

$$\langle \alpha_1\alpha_2 1\rangle^X_{0,0} = \alpha_1 \cap \alpha_2$$

Writing what we just said in a formula:

$$\langle \alpha_1\alpha_2 \ldots \alpha_{n-1} 1\rangle^X_{0,d} = 0$$

Divisor equation: if one of the insertions is a hypersurface D of degree e, then

$$\langle D\alpha_2 \ldots \alpha_{n-1} 1\rangle^X_{0,d} = de\langle \alpha_2 \ldots \alpha_{n-1} 1\rangle^X_{0,d}$$

Kontsevich's Proof

Believe it or not, we know enouhg about Gromov-Witten invariants to answer our question Q_d. Throughout this section, we call P (the class of) a generic point in $\mathbb{P}^2$, ℓ (the class of) a generic line in $\mathbb{P}^2$, 1 the fundamental class of $\mathbb{P}^2$. Also, we denote N_d the answer to Q_d, i.e.

$\mathbf{N_d}$: number of rational curves of degree d through $3d-1$ points in $\mathbb{P}^2$.

We can interpret N_d as a Gromov-Witten invariant:

$$N_d = \langle \underbrace{P \ldots P}_{3d-1 \text{ times}} \rangle^{\mathbb{P}^2}_{0,d}$$

So what? We still do not know how to compute it...well, wait just one more second. Kontsevich's genius was to...break the symmetry a bit, and break one of the points into two lines, so as to consider:

$$\mathfrak{C} = ev_1^*(\ell) \cap ev_2^*(\ell) \cap ev_3^*(P) \cap \ldots \cap ev_{3d}^*(P)$$

Counting dimensions, we see that $\mathfrak{C}$ is a curve in $\overline{M}_{0,3d}(\mathbb{P}^2, d)$. We are now going to intersect this curve with two equivalent hypersurfaces, and extract from equating the result a recursion that computes N_d.

WDVV

Recall our forgetful morphisms from a while ago...now we are going to use them. We are going to forget a bunch of marks (all of them minus 4), and we are going to forget the map. All together we obtain:

$$F : \overline{M}_{0,3d}(\mathbb{P}^2, d) \longrightarrow \overline{M}_{0,4} = \mathbb{P}^1$$

We consider the hypersurface $F^{-1}(\text{point}) \subset \overline{M}_{0,3d}(\mathbb{P}^2, d)$. Since any two points in $\mathbb{P}^1$ are equivalent, we can really choose any point we want. We are going to choose two special points, corresponding to the boundary divisors in Figure 4. By doing so, we obtain:

$$\mathfrak{C} \cap F^{-1}(Q_1) = \mathfrak{C} \cap F^{-1}(Q_2) \tag{4}$$

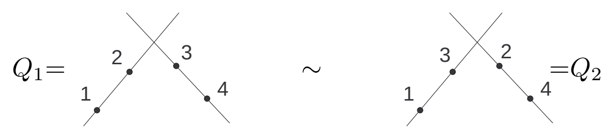

Figure 4: Two equivalent points in $\overline{M}_{0,4}$

All we have left to do is now interpret what (4) means. On the left hand side we have to restrict our attention to boundary divisors that have the first two marks on one twig, the third and fourth on the other. On the right hand side, 1 and 3 are together, and so are 2 and 4.

Recall the structure of the boundary: we have to take fiber products over the evaluation morphisms of two moduli spaces of maps of degrees adding to d, where our original set of marks has been partitioned in two, and then we have to add one mark on each twig that will become the node.

By mentioning the fact that $\Delta_{\mathbb{P}^2 \times \mathbb{P}^2}$ is equivalent to $P \times 1 + \ell \times \ell + 1 \times P$, we finally can write (4) as follows:

left hand side:

$$\sum_{d_1+d_2=d} \Big(\langle \ell\ell *** 1 \rangle^{\mathbb{P}^2}_{0,d_1} \langle P *** PP \rangle^{\mathbb{P}^2}_{0,d_1} + \langle \ell\ell *** \ell \rangle^{\mathbb{P}^2}_{0,d_1} \langle \ell *** PP \rangle^{\mathbb{P}^2}_{0,d_1} +$$

$$+ \langle \ell\ell *** P \rangle^{\mathbb{P}^2}_{0,d_1} \langle 1 *** PP \rangle^{\mathbb{P}^2}_{0,d_1} \Big)$$

right hand side:

$$\sum_{d_1+d_2=d} \Big(\langle \ell P *** 1 \rangle^{\mathbb{P}^2}_{0,d_1} \langle P *** \ell P \rangle^{\mathbb{P}^2}_{0,d_1} + \langle \ell P *** \ell \rangle^{\mathbb{P}^2}_{0,d_1} \langle \ell *** \ell P \rangle^{\mathbb{P}^2}_{0,d_1} +$$

$$+ \langle \ell P *** P \rangle^{\mathbb{P}^2}_{0,d_1} \langle 1 *** \ell P \rangle^{\mathbb{P}^2}_{0,d_1} \Big)$$

Here, we put $***$ to mean that one needs to distribute the remaining marks in all possible ways.

This looks like a huge combinatorial mess, but in fact it is not that bad, because a lot of the terms vanish. In fact, it is much more convenient to tackle the question by analyzing what are the terms that do not vanish!

First observe that of all the terms that contain a 1, there is only one that is non-zero, and it contributes precisely N_d. What are left are the terms with no 1. Notice that we can pull out the ℓ's with the divisor axiom. Now, for those guys not to vanish the only possibility is that the number of points on both sides be the "right one" (i.e. $3d_i - 1$ on each side). At the end of the day, and I am more than glad to leave the actual derivation as a good exercise, one gets the recursive equation:

$$N_d = \sum_{d_1+d_2=d, d_1, d_2>0} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right]$$

Finally, by inputting $N_1 = 1$, we obtain $N_2 = 1$, $N_3 = 12$, $N_4 = 620$, $N_5 = 87304$ ...