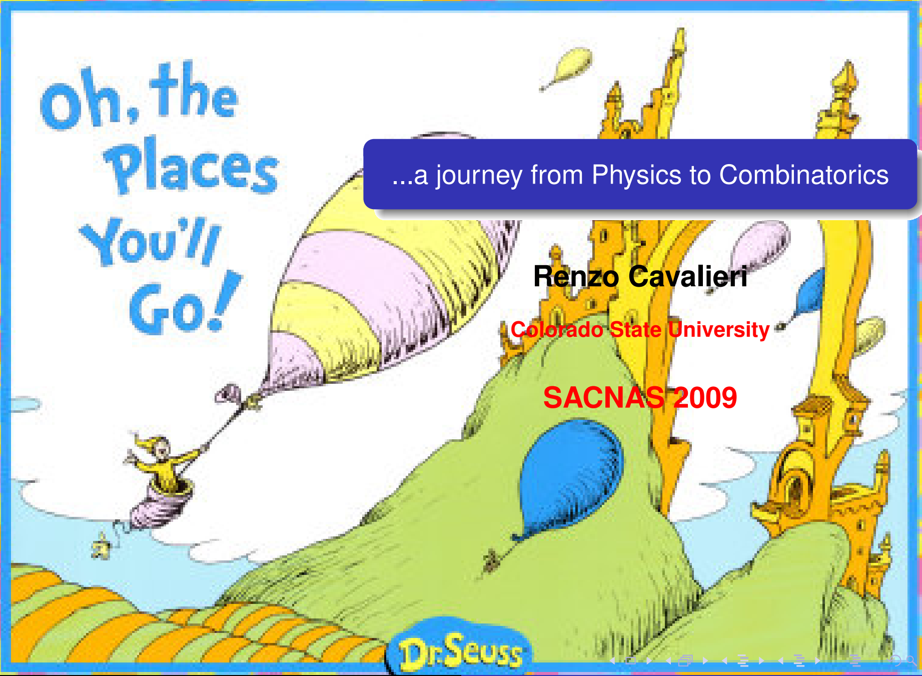

...a journey from Physics to Combinatorics

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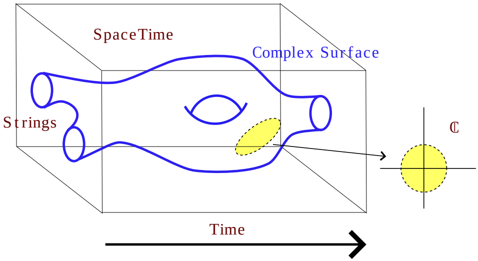
Physics

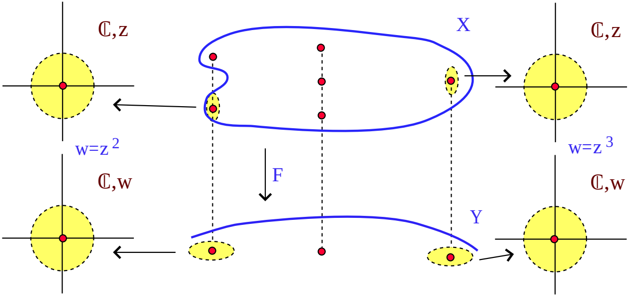
Physicists tell us that matter consists of tiny little strings that wiggle their way through Space Time. In doing so they trace surfaces, which represent the evolution of a physical system. These surfaces come with a complex structure.



It seems reasonable that physicists want to know as much as possible about analytic functions between surfaces.

Geometry

Geometers have been studying such maps for over a century, and know they are ramified covers: away from a finite number of points (branch points), there are exactly *d* preimages.

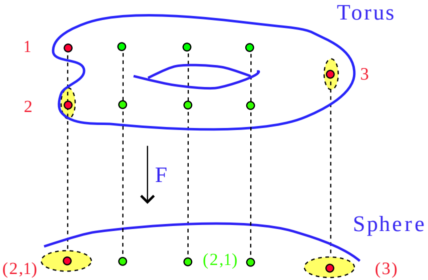


Above a branch point, the local expression of the function is $z \mapsto z^n$, and the collection of the *n*'s above a branch is called the ramification profile.

The Quest

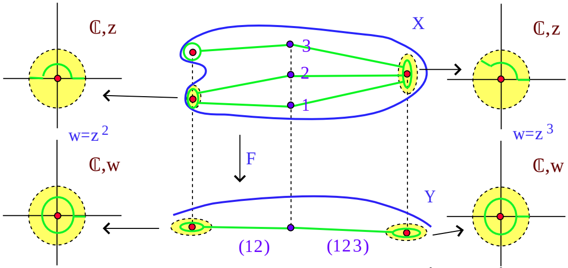
QUESTION: how many covers of degree d from a genus g surface to a sphere, with specified ramification profile over two points and generic ramification over r other points?

$$H_1^3((3),(2,1))$$



Such number is called a Hurwitz number, denoted $H_g^r(\alpha,\beta)$.

Algebra gives us a way to compute Hurwitz number via the following construction:



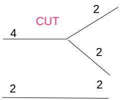
In order to count covers, we instead count $(r+2)$-tuples $\sigma_0, \tau_1, \ldots, \tau_r, \sigma_\infty$ of permutations of d points such that:

1. $(\sigma_0, \sigma_\infty)$ have cycle type (α, β);
2. τ's are simple transpositions;
3. $\sigma_0 \tau_1 \ldots \tau_r \sigma_\infty = Id$
4. $\langle \sigma_0, \tau_1, \ldots, \tau_r, \sigma_\infty \rangle$ acts transitively on the d points

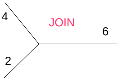
Combinatorics

The cut and join equations tell us how a permutation can change when you compose it with a simple transposition:

$$\sigma = (1234)(56) \quad \tau_1 = (13) \quad \Rightarrow \tau_1 \sigma = (12)(34)(56)$$

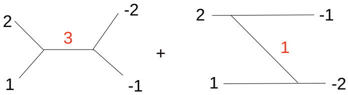


$$\sigma = (1234)(56) \quad \tau_1 = (35) \quad \Rightarrow \tau_1 \sigma = (123564)$$



This gives us a way to compute Hurwitz number by instead counting weighted trivalent graphs.

$$H_0^2((2,1),(2,1)) = \quad 4 \quad =$$



Sketch of further coolness...