Where parallel lines meet
...a geometric love story

Renzo Cavalieri

Colorado State University

MATH DAY 2009

Q: *How many points of intersection do two distinct lines in the plane have?*
A: well it depends...

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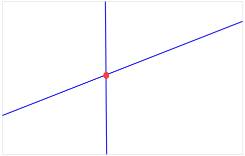
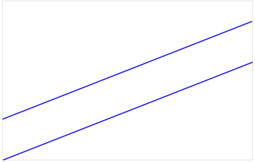


plane

if you are unlucky 0



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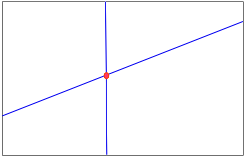

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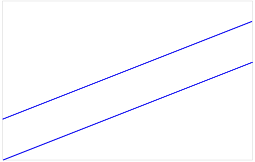

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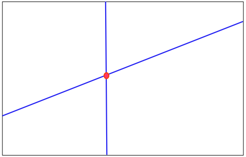

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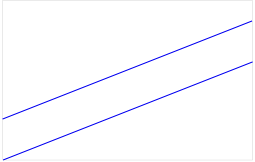

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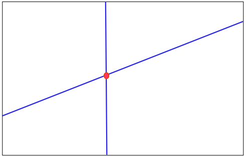

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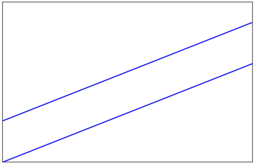

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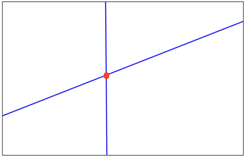

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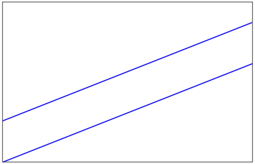

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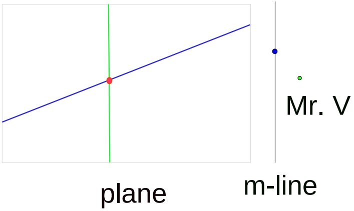
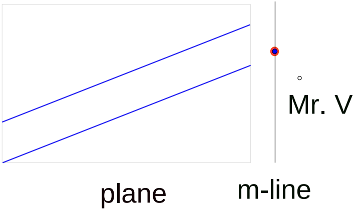
A deus ex machina solution

Let us beef up the plane by adding a line (*m*-line) and a point (Mr. V), and declare that any line of slope *m* contains the point *m* in the *m*-line, and that any vertical line contains Mr. V. We call this new creature the amazing Projective Plane. The *m*-line and Mr. V together are called the line at infinity.

Now the answer to the previous question is always 1.





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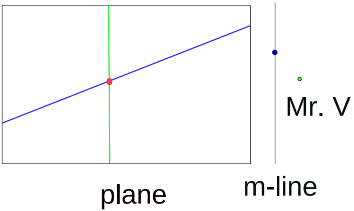

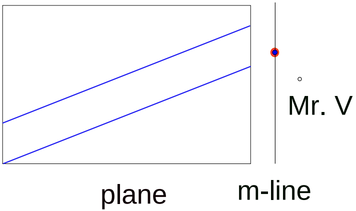

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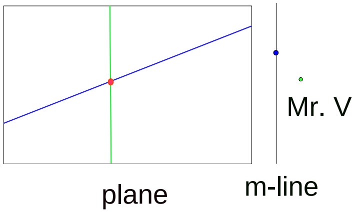

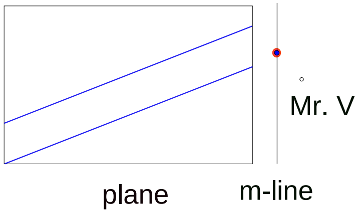

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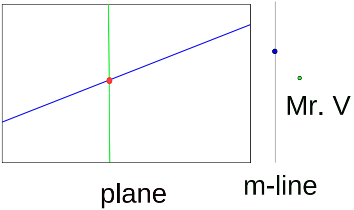

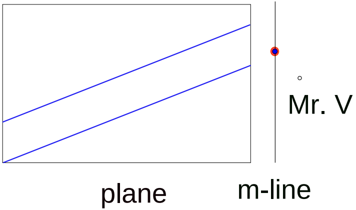

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Back to the Renaissance ($\sim$ 1500's)

Raffaello Sanzio
Sposalizio della Vergine
(1504)
Perspective

Imagine you are a painter. You observe the three-dimensional world through your eyes and then reproduce it on a two-dimensional canvas.

Mathematically, you are drawing a straight line between the point you are observing and the eye, and marking the point of intersection with the canvas.

In this picture the plane is the canvas and the projective plane is the set of rays of light going to the painter's eye. The line at infinity is the set of rays of light that are parallel to the canvas.

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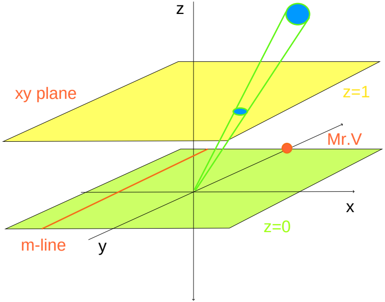
A more mathematical formulation



The points of the projective plane correspond to lines through the origin in three-dimensional space.

$$\mathbb{P}^2 = \frac{\mathbb{R}^3 \setminus (0,0,0)}{(X,Y,Z) \sim (\lambda X, \lambda Y, \lambda Z)}$$

Equivalence classes of triples are denoted:

$$(X : Y : Z)$$

and called homogeneous coordinates.

The usual *xy* plane can be identified with the plane $Z = 1$.
The *m*-line with $X = 1, Z = 0$.
Mr. V is the point $(0 : 1 : 0)$.

Intersecting parallel lines in $\mathbb{P}^2$.

Given two lines in the usual *xy* plane, we can:

1. put them in $\mathbb{P}^2$ by homogenizing their equation.
2. intersect them by solving a homogeneous linear system.

Example:

$$y - 3x + 2 = 0 \quad \text{and} \quad 2y - 6x + 9 = 0$$

$$Y - 3X + 2Z = 0 \quad \text{and} \quad 2Y - 6X + 9Z = 0$$

gives the solution:

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corresponding to the point (on the line at infinity) $(1 : 3 : 0)$.

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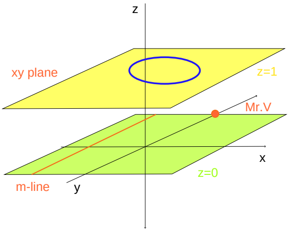
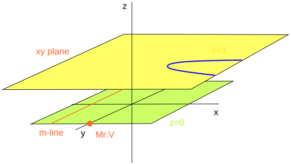
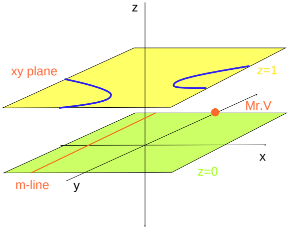
There are three types of conics in the plane:



ELLIPSE



PARABOLA

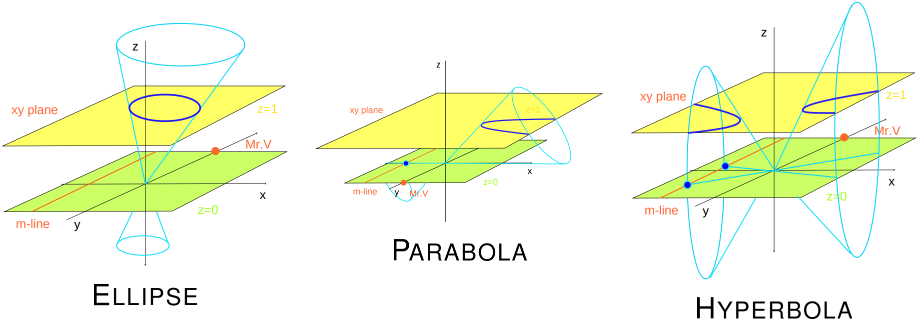


HYPERBOLA

...but in the Projective Plane...

Conics

...they are all sections of a cone!



...according to how the cone is positioned with respect to the canvas, it intersects differently the line at infinity!

How to recognize a plane conic?

Simple!

1. Projectivize!
2. Intersect with the line at infinity!

Example:

$$34x^2 + 17xy + 3x - 323y - 888 = 0$$

1. $$34X^2 + 17XY + 3XZ - 323YZ - 888Z^2 = 0$$
2. intersect with $Z = 0$:

$$34X^2 + 17XY = 0$$

This equation has two solutions. Our conic intersects the line at infinity in the points $(0 : 1 : 0)$ and $(1 : -2 : 0)$ and therefore it is a hyperbola!

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