

Wild cyclic-by-tame extensions

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Abstract

Suppose G is a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ where p is prime and m is relatively prime to p . Suppose K is a complete local field of characteristic $p > 0$ with algebraically closed residue field. The main result states necessary and sufficient conditions on the ramification filtrations that occur for wildly ramified G -Galois extensions of K . In addition, we prove that there exists a parameter space for G -Galois extensions of K with given ramification filtration, and we calculate its dimension in terms of the ramification filtration. We provide explicit equations for wild cyclic extensions of K of degree p^3 .

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1. Introduction

This paper is about wildly ramified Galois extensions of a complete local field $k((t))$ where k is an algebraically closed field of characteristic $p > 0$. We prove that the lower jumps of the ramification filtration of a Galois extension of $k((t))$ with group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ are all congruent modulo m , Proposition 4.2. We also prove that one can dominate a given Galois extension having group $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$ by a Galois extension having group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with control over the last jump in the ramification filtration, Proposition 5.1. Together with well-known results about ramification filtrations of Galois extensions with group \mathbb{Z}/p^n [11], this yields (see Theorem 5.2):

Theorem 1.1. *Let G be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ where $p \nmid m$. Let $\sigma \in G$ have order p^n and let $m' = |\text{Cent}_G(\sigma)|/p^n$. A sequence*

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$u_1 \leq \dots \leq u_n$ of rational numbers occurs as the set of positive breaks in the upper numbering of the ramification filtration of a G -Galois extension of $k((t))$ if and only if:

- (a) $u_i \in \frac{1}{m}\mathbb{N}$ for $1 \leq i \leq n$;
- (b) $\gcd(m, mu_1) = m'$;
- (c) $p \nmid mu_1$ and, for $1 < i \leq n$, either $u_i = pu_{i-1}$ or both $u_i > pu_{i-1}$ and $p \nmid mu_i$;
- (d) and $mu_i \equiv mu_1 \pmod{m}$ for $1 \leq i \leq n$.

In the first author's doctoral thesis, Theorem 1.1 yields restrictions on the stable reduction of certain branched covers of the projective line.

Our other main result, Theorem 5.6, states that, given a group G and a ramification filtration η satisfying conditions (a)-(d) as in Theorem 1.1, there exists a parameter space \mathcal{M}_η whose k -points are in natural bijection with isomorphism classes of G -Galois extensions of $k((t))$ having ramification filtration η . We calculate the dimension of \mathcal{M}_η in terms of the upper jumps of η .

Here is the paper's outline: in Section 2 we introduce the framework of study, including ramification filtrations and field theory; Section 3 contains several structural descriptions of cyclic p -group extensions; in Section 4, we prove results about tame actions on cyclic extensions; and the main results on ramification filtrations and parameter spaces for G -Galois extensions appear in Section 5.

Our original motivation for this topic was to find explicit equations for \mathbb{Z}/p^3 -Galois extensions of $k((t))$, see Section 6. Such equations are useful and are difficult to find in the literature. For example, in [5, II, Lemma 5.1], the authors use equations for \mathbb{Z}/p^2 -Galois extensions in order to prove a case of Oort's Conjecture, namely, that every \mathbb{Z}/p^2 -Galois extension of $k((t))$ lifts to characteristic 0 [5, Thm. 2].

Similar results for elementary abelian p -group extensions are in [2].

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2. Framework of study

This section contains background on extensions of complete local fields and ramification filtrations and introduces the situation studied in this paper, in which the Galois group is a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$.

2.1. Extensions of complete local fields

Let k be an algebraically closed field of characteristic $p > 0$. We fix a compatible system of roots of unity of k . In particular, this fixes an isomorphism $\mathbb{Z}/p \simeq \mathbb{F}_p$ and fixes a primitive m th root of unity ζ in k . Let R be an equal

characteristic complete discrete valuation ring with residue field k and fraction field K . Then $R \simeq k[[t]]$ and $K \simeq k((t))$ for some uniformizing parameter t .

Suppose L/K is a separable Galois field extension with group G . Let S be the integral closure of R in L . Then S/R is a Galois extension of rings with group G which is totally ramified over the prime ideal (t) .

This type of field extension arises in the following context. Suppose $\phi: Y \rightarrow X$ is a Galois cover of smooth k -curves. Suppose $y \in Y$ is a ramified point with inertia group G . Consider the complete local rings $S = \hat{\mathcal{O}}_{Y,y}$ and $R = \hat{\mathcal{O}}_{X,\phi(y)}$. Then S/R is a Galois extension of rings with group G which is totally ramified over the unique valuation of R as described in the preceding paragraph.

For a Galois extension L/K as above, the group G is a semi-direct product of the form $P \rtimes \mathbb{Z}/m$ where P is a p -group and $p \nmid m$ [12, IV, Cor. 4]. Throughout the paper, we assume that the subgroup P is cyclic.

2.2. Subgroups of a semi-direct product

Suppose G is a semi-direct product of the form $P \rtimes \mathbb{Z}/m$ where $P \simeq \mathbb{Z}/p^n$ and $p \nmid m$. Let σ be a chosen generator of P . Let c be a chosen element of order m in G and let $M = \langle c \rangle$. Let $m' = |\text{Cent}_G(\sigma)|/p^n$. In other words, $m' = \#\{g \in M \mid g\sigma g^{-1} = \sigma\}$.

For $0 \leq i \leq n$, the element $\sigma_i := \sigma^{p^i}$ has order p^{n-i} and $H_i := \langle \sigma_i \rangle$ is the unique subgroup of order p^{n-i} in G . Then $\{\text{id}\} = H_n \subset H_{n-1} \subset \cdots \subset H_0 = P$.

The semi-direct product is determined by the conjugation action of M on P . Since $c\sigma c^{-1}$ also generates P , then $c\sigma c^{-1} = \sigma^{\alpha'}$ for some integer α' such that $1 \leq \alpha' < p^n$ and $p \nmid \alpha'$. The action of c stabilizes H_i . Let $J_i := (H_{i-1}/H_i) \rtimes M$.

Lemma 2.1. (i) *The value of α' does not depend on the choice of generator of P ;*

(ii) *The value of α' depends on the choice of generator of M as follows; if $c_0 = c^\beta$ for some integer β , then $\alpha'_0 \equiv (\alpha')^\beta \pmod{p^n}$.*

Proof. (i) If $\tau = \sigma^\gamma$, then $c\tau c^{-1} = (c\sigma c^{-1})^\gamma = (\sigma^{\alpha'})^\gamma = \tau^{\alpha'}$.

(ii) By induction, $c^i \sigma c^{-i} = \sigma^{(\alpha')^i}$. Thus $c_0 \sigma c_0^{-1} = \sigma^{\alpha'_0}$. □

Lemma 2.2. *The groups J_i are canonically isomorphic for $1 \leq i \leq n$.*

Proof. The groups J_i are semi-direct products of the form $\mathbb{Z}/p \rtimes \mathbb{Z}/m$. Thus it suffices to show that the action of c on the equivalence class of σ_{i-1} modulo $\langle \sigma_i \rangle$ is the same for $1 \leq i \leq n$. Note that $c\sigma^p c^{-1} = (\sigma^p)^{\alpha'}$. Thus $c\sigma_i c^{-1} = \sigma_i^{\alpha'}$. □

The residue of α' modulo p can be identified with an element $\alpha \in (\mathbb{Z}/p)^*$ and thus with an element $\alpha \in \mathbb{F}_p^*$. Also m/m' is the order of α in \mathbb{F}_p^* .

2.3. Towers of fields

Suppose L/K is a separable Galois extension whose group G is of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ with $p \nmid m$. We fix an identification of $\text{Aut}(L/K)$ with G and indicate this by writing that L/K is a G -Galois extension.

Consider the fixed fields $L_i = L^{H_i}$ and $K_i = L^{H_i \rtimes M}$ for $0 \leq i \leq n$. So, $L_n = L$ and $K_0 = K$. Let v_i be the natural valuation on L_i . Let Θ_i be the integral closure of R in L_i . Then L/L_i is an H_i -Galois extension and L_i/L_0 is a P/H_i -Galois extension. Also L_i/K_{i-1} is a J_i -Galois extension. This yields a tower of fields:

$$\begin{array}{ccccccc}
 L_0 & \xrightarrow{\mathbb{Z}/p} & L_1 & \xrightarrow{\mathbb{Z}/p} & \dots & \xrightarrow{\mathbb{Z}/p} & L_{n-1} & \xrightarrow{\mathbb{Z}/p} & L \\
 \uparrow \mathbb{Z}/m & & \uparrow & & & & \uparrow & & \uparrow \\
 K_0 & \xrightarrow{\quad} & K_1 & \xrightarrow{\quad} & \dots & \xrightarrow{\quad} & K_{n-1} & \xrightarrow{\quad} & K_n
 \end{array}$$

By Kummer theory, there exists $x \in L_0$ such that $L_0 \simeq K[x]/(x^m - 1/t)$. After choosing $c \in G$ such that $c(x) = \zeta x$, one can determine the values of α' and α for the extension L/K .

2.4. Ramification filtrations

Here is a brief review of the theory of ramification filtrations from [12, IV]. Consider the natural valuation $v = v_n$ on L and a uniformizing parameter $\pi \in L$. For $r \in \mathbb{N}$, let I_r be the r th ramification group in the lower numbering for the extension L/K . In other words, I_r is the normal subgroup of all $g \in G$ such that $v(g(\pi) - \pi) \geq r + 1$.

The ramification filtration is important because it determines the degree δ of the different of S/R . Namely, by [12, IV, Prop. 4], $\delta = \sum_{r \geq 0} (|I_r| - 1)$. If $\phi : Y \rightarrow X$ is a cover of smooth projective connected k -curves, the genus of Y can be found using the Riemann-Hurwitz formula [6, IV, Cor. 2.4] and this formula relies on the degree of the different at each ramification point of ϕ .

Let $g \in G$ with $g \neq 1$. The *lower jump* for g is the non-negative integer j so that $v(g(\pi) - \pi) = j + 1$. Then $g \in I_j$ and $g \notin I_{j+1}$. By [12, IV, Prop. 11], $p \nmid j$ for any positive lower jump j . If $|P| = p^n$, then there are n positive indices $j_1 \leq \dots \leq j_n$ at which there is a break in the ramification filtration in the lower numbering, which are called the *lower jumps* of L/K .

There is also a ramification filtration I^ℓ in the upper numbering. The *upper jumps* of L/K are the positive breaks $u_1 \leq \dots \leq u_n$ in the ramification filtration in the upper numbering. The lower numbering is stable for subextensions [12, IV, Prop. 2] and the upper numbering is stable for quotients [12, IV, Prop. 14]. Using Herbrand's formula [12, IV, §3], one can translate between the two ramification filtrations: letting $j_0 = u_0 = 0$, then $u_i - u_{i-1} = (j_i - j_{i-1})/p^{i-1}m$ for $1 \leq i \leq n$.

3. Wild cyclic extensions

In this section, we describe the equations and ramification filtration of the \mathbb{Z}/p^n -Galois subextension L/L_0 . The material in this section is mostly known, but it is all necessary for later results in the paper.

3.1. Cyclic towers of Artin-Schreier extensions

Lemma 3.1. *The i th lower jump j_i of L/K equals the lower jump of L_i/L_{i-1} .*

Proof. The i th lower jump j_i of L/K is the lower jump of the automorphism σ_{i-1} . This is the same as the lower jump of σ_{i-1} for the extension L/L_{i-1} by [12, IV, Prop. 2]. Since this is the smallest lower jump for the extension L/L_{i-1} , it also equals the upper jump of σ_{i-1} for L/L_{i-1} . By [12, IV, Prop. 14], this is then the same as the upper jump, and thus the lower jump, of L_i/L_{i-1} . \square

3.2. Witt Vectors and p -power cyclic extensions

We recall some Witt vector theory. Let \wp be the operation $\text{Fr} - \text{Id}$ on Witt vectors, where Fr denotes Frobenius. An element a of a field F of characteristic p is a \wp th power in F if the polynomial $z^p - z - a$ has a root in F .

By [7, p. 331, Ex. 50], every Galois extension of $L_0 \cong k((x^{-1}))$ with group \mathbb{Z}/p^n has Witt vector equations

$$(y_1^p, \dots, y_n^p) = (y_1, \dots, y_n) +' (x_1, \dots, x_n). \quad (1)$$

where $x_i \in L_0$ for $1 \leq i \leq n$ such that x_1 is not a \wp th power in L_0 and where $+'$ denotes addition of Witt vectors: Moreover, there is a generator τ of \mathbb{Z}/p^n such that the action of τ on Witt vectors is

$$\tau(y_1, \dots, y_n) = (y_1, \dots, y_n) +' (1, 0, \dots, 0). \quad (2)$$

Modifying (x_1, \dots, x_n) by an element $w \in W^n(L_0)$, where W^n is the n th truncation of the Witt vectors, changes the isomorphism class of the extension precisely when $w \notin \wp(W^n(L_0))$. Thus, since k is algebraically closed, one can choose (x_1, \dots, x_n) to be in *standard form*, i.e., $x_i \in k[x]$ and either $x_i = 0$ or x_i has no exponent divisible by p .

To make (1) more explicit, for $0 \leq i \leq n-1$, let $W_i = \sum_{d=0}^i p^d X_{d+1}^{p^{i-d}}$ be the i th Witt polynomial, [12, II, §6]. Define $S_i \in \mathbb{Z}[X_1, \dots, X_{i+1}, Y_1, \dots, Y_{i+1}]$ to be the unique formal polynomial such that

$$\begin{aligned} W_i(X_1, \dots, X_{i+1}) + W_i(Y_1, \dots, Y_{i+1}) = \\ W_i(S_0(X_1, Y_1), S_1(X_1, X_2, Y_1, Y_2), \dots, S_i(X_1, \dots, X_{i+1}, Y_1, \dots, Y_{i+1})). \end{aligned}$$

The indexing of these variables is shifted by one from that of [12, II, §6] in order to be more consistent with notation in this paper. By [12, II, Thm. 6], the S_i are well defined and have integer coefficients.

Lemma 3.2. In $\mathbb{Z}[X_1, \dots, X_i, Y_1, \dots, Y_i]$,

$$S_{i-1}(X_1, \dots, X_i, Y_1, \dots, Y_i) = X_i + Y_i + \sum_{d=1}^{i-1} \frac{1}{p^{i-d}} (X_d^{p^{i-d}} + Y_d^{p^{i-d}} - S_{d-1}^{p^{i-d}})$$

and the degree of every monomial of S_{i-1} is congruent to one modulo $p-1$.

Proof. The equation follows from $\sum_{d=0}^{i-1} p^d S_d^{p^{i-1-d}} = \sum_{d=0}^{i-1} p^d (X_{d+1}^{p^{i-1-d}} + Y_{d+1}^{p^{i-1-d}})$ (see [11, Footnote 4]) and the statement about degrees from induction. \square

For $1 \leq i \leq n$, let $\bar{S}_{i-1} \in \mathbb{F}_p[X_1, \dots, X_i, Y_1, \dots, Y_i]$ be the reduction of S_{i-1} modulo p and let $f_i(Y_1, \dots, Y_{i-1}, X_1, \dots, X_i) = \bar{S}_{i-1} - Y_i$. Then $f_i = X_i + g_i$ where $g_i \in \mathbb{F}_p[X_1, \dots, X_{i-1}, Y_1, \dots, Y_{i-1}]$ is a polynomial whose terms each have degree congruent to one modulo $p-1$. The meaning of (1) is that a Galois extension with group \mathbb{Z}/p^n has equations $y_i^p - y_i = f_i(y_1, \dots, y_{i-1}, x_1, \dots, x_n)$.

Lemma 3.3. Let L/L_0 be a \mathbb{Z}/p^n -Galois extension and σ a generator of \mathbb{Z}/p^n . There exist $x_i \in L_0$ and $y_i \in L$ for $1 \leq i \leq n$ such that L/L_0 is isomorphic to the $\langle \sigma \rangle$ -Galois extension with Witt vector equations and Galois action

$$(y_1^p, \dots, y_n^p) = (y_1, \dots, y_n) + '(x_1, \dots, x_n)$$

$$\sigma(y_1, \dots, y_n) = (y_1, \dots, y_n) + '(1, 0, \dots, 0).$$

Furthermore, there is a unique choice for (x_1, \dots, x_n) in standard form.

Proof. There exist $x_i \in L_0$ and $y_i \in L$ and a generator τ of \mathbb{Z}/p^n such that L/L_0 has Witt vector equations (1) and Galois action (2). Now $\sigma = \tau^b$ for some $b \in (\mathbb{Z}/p^n)^*$. Then $\sigma(y_1, \dots, y_n) = (y_1, \dots, y_n) + 'b(1, 0, \dots, 0)$. Since b is invertible in $\mathbb{Z}/p^n \cong W^n(\mathbb{Z}/p) \subset W^n(L_0)$, one can replace (y_1, \dots, y_n) and (x_1, \dots, x_n) with the Witt vectors $\frac{1}{b}(y_1, \dots, y_n)$ and $\frac{1}{b}(x_1, \dots, x_n)$. Since Fr is a ring homomorphism [7, p. 331, Ex. 48], the extension L/L_0 still has Witt vector equations (1) and now $\sigma(y_1, \dots, y_n) = (y_1, \dots, y_n) + '(1, 0, \dots, 0)$.

By a generalization of [8, Lemma 2.1.5], there is a unique choice of (x_1, \dots, x_n) in standard form compatible with the restriction on the Galois action. \square

3.3. Ramification filtrations for cyclic p -group extensions

The ramification filtration of a \mathbb{Z}/p^n -Galois extension is completely determined by either its lower or upper jumps, which in turn can be determined by the Witt vector equation.

Lemma 3.4. Let L/L_0 be a \mathbb{Z}/p^n -Galois extension with Witt vector (x_1, \dots, x_n) in standard form. Let $u = \max\{-p^{n-i}v_0(x_i)\}_{i=1}^n$. Then u is the last upper jump of L/L_0 .

Proof. This follows from [4, Thm. 1.1]; see also [14, Prop. 4.2(1)]. \square

We retrieve the following classical result.

Lemma 3.5. *A sequence of positive integers $w_1 \leq \dots \leq w_n$ occurs as the set of upper jumps of a \mathbb{Z}/p^n -Galois extension of L_0 if and only if $p \nmid w_1$ and, for $1 < i \leq n$, either $w_i = pw_{i-1}$ or both $w_i > pw_{i-1}$ and $p \nmid w_i$.*

Proof. The result, originally found in [11], follows from Lemma 3.4; see also [9, Lemma 19]. \square

The following lemma will be used to compare the upper jumps of the G -Galois extension L/K and the \mathbb{Z}/p^n -Galois extension L/L_0 .

Lemma 3.6. *Suppose L/K has upper jumps $u_1 \leq \dots \leq u_n$. Then L/L_0 has upper jumps $w_1 \leq \dots \leq w_n$ where $w_i = mu_i$ for $1 \leq i \leq n$.*

Proof. By [12, IV, Prop. 2], the lower jumps of L/L_0 equal the lower jumps $j_1 \leq \dots \leq j_n$ of L/K . Herbrand's formula [12, IV, §3] implies that $u_i - u_{i-1} = (j_i - j_{i-1})/p^{i-1}m$ and that $w_i - w_{i-1} = (j_i - j_{i-1})/p^{i-1}$ for $1 \leq i \leq n$. \square

4. Cyclic-by-tame extensions

Suppose L/K is a separable G -Galois field extension as in Sections 2.2-3.1. In this section, we find necessary conditions on the ramification filtrations and equations arising from the \mathbb{Z}/m -Galois action on L .

4.1. The case of Galois extensions with group $\mathbb{Z}/p \rtimes \mathbb{Z}/m$

Lemma 4.1. *Consider the J_1 -Galois extension L_1/K with equations $x^m = 1/t$ and $y_1^p - y_1 = x_1$ and Galois action $c(x) = \zeta x$ and $\sigma(y_1) = y_1 + 1$.*

- (i) *The lower jump j of L_1/L_0 satisfies $m' = \gcd(m, j)$.*
- (ii) *Also $m|j(p-1)$. In particular, $j \equiv jp^r \pmod{m}$ for any $r \in \mathbb{N}$.*
- (iii) *Also $c(y_1) = \alpha^{-1}y_1 = \zeta^j y_1$.*

Proof. (i) This follows from [12, IV, Prop. 9], see also [8, Lemma 1.4.1(iv)].

- (ii) The conjugation action of \mathbb{Z}/m on \mathbb{Z}/p gives a homomorphism $\nu : \mathbb{Z}/m \rightarrow \text{Aut}(\mathbb{Z}/p)$. By definition, $\text{Im}(\nu)$ has order m/m' and $\text{Ker}(\nu) = \langle c^{m/m'} \rangle$. Thus $m|m'(p-1)$. By part (i), $m' = \gcd(m, j)$, so $m|j(p-1)$.

- iii) [8, Lemma 1.4.1(ii)-(iii)]. \square

4.2. A congruence condition on the ramification filtration

Proposition 4.2. (i) *The lower jumps in the ramification filtration of the P -Galois extension L/L_0 are all congruent modulo m .*

- (ii) *The upper jumps in the ramification filtration of the P -Galois extension L/L_0 are all congruent modulo m .*

Proof. (i) The i th lower jump of L/L_0 is j_i by [12, IV, Prop. 2]. Let π be a uniformizer of Θ_n and let $u = c(\pi)/\pi \in \Theta_n^*$. Then u equals $\theta_0(c) \in k^*$ in the notation of [12, IV, Prop. 7]. The order of u is m by [12, IV, Prop. 7]. By the proof of Lemma 2.2, $c\sigma_{i-1}c^{-1} = \sigma_{i-1}^{\alpha_i}$ for $1 \leq i \leq n$. Since σ_{i-1} generates $H_{i-1}/H_i = I_{j_i}/I_{j_i+1}$, [12, IV, Prop. 9] shows that $\theta_{j_i}(\sigma_{i-1}^{\alpha_i}) = u^{j_i}\theta_{j_i}(\sigma_{i-1})$ for $1 \leq i \leq n$. Thus $u^{j_i} = \alpha \in k^*$ for $1 \leq i \leq n$ and so $j_1 \equiv \dots \equiv j_n \pmod{m}$.

(ii) Let $w_1 \leq \dots \leq w_n$ be the upper jumps of the P -Galois extension L/L_0 . Since P is abelian, the Hasse-Arf Theorem implies that $w_i \in \mathbb{N}$. By Herbrand's formula, $w_i - w_{i-1} = (j_i - j_{i-1})/p^{i-1}$. Thus $w_i - w_{i-1} \equiv 0 \pmod{m}$ by part (i). □

Class field theory approach: If k is instead a finite field, here is a different proof of Proposition 4.2 which uses class field theory.

Second proof of Proposition 4.2. The G -Galois extension L/K dominates the $\langle c \rangle$ -Galois extension L_0/K where $L_0 \simeq k((x^{-1}))$, $x^m = 1/t$, and $c(x) = \zeta x$. Let L/L_0 be the P -Galois subextension, which has upper jumps $w_1 \leq \dots \leq w_n$ where $w_i = mu_i$ by Lemma 3.6. Thus the upper ramification group I^ℓ of L/L_0 equals H_i if $w_i < \ell \leq w_{i+1}$.

Let $Q = (x^{-1})$ be the maximal ideal of $k[[x^{-1}]]$. Consider the unit groups $U^d = 1 + Q^d$ of $k[[x^{-1}]]$ [12, IV.2]. By [12, IV, Prop. 6], U^d/U^{d+1} is canonically isomorphic to Q^d/Q^{d+1} . Now, Q^d carries a natural $\langle c \rangle$ -module structure where $c((x^{-1})^d) = \zeta_m^{-d}(x^{-1})^d$. Thus U^d/U^{d+1} carries a natural structure as a $\langle c \rangle$ -module, and this structure depends on the congruence class of d modulo m .

By [12, XV.2, Cor. 3 & pg. 229], there is a reciprocity isomorphism $\omega : L_0^*/NL^* \rightarrow P$ and thus there are isomorphisms $\omega_n : U^d/(U^{d+1}NU_L^{\psi(d)}) \rightarrow I^d/I^{d+1}$. Here $N : L \rightarrow L_0$ is the norm map and ψ is Herbrand's function. In particular, taking $d = w_i$, then $U^{w_i}/(U^{w_i+1}NU_L^{\psi(w_i)}) = H_{i-1}/H_i$.

Now H_{i-1}/H_i has a $\langle c \rangle$ -module structure and this $\langle c \rangle$ -module structure is independent of i by Lemma 2.2. After pulling back by ω , this implies that the $\langle c \rangle$ -module structure of $U^{w_i}/(U^{w_i+1}NU_L^{\psi(w_i)})$ and thus of U^{w_i} is independent of i . Thus $\zeta_m^{-w_i}$ is independent of i and so $w_i \equiv w_1 \pmod{m}$.

The lower jumps are also congruent modulo m by Herbrand's formula. □

At this point, one can prove that the conditions in Theorem 1.1 are necessary; we will postpone this until Section 5.2.

4.3. Actions and isomorphisms

This section contains two results that will be needed in Section 5.

Proposition 4.3. *Suppose $L_0 \simeq K[x]/(x^m - 1/t)$ and $c(x) = \zeta x$. Suppose L/L_0 is a P -Galois extension with Witt vector equations (1), Galois action (2), and first lower jump j such that $\zeta^j = \alpha^{-1}$. Then L/K is a G -Galois extension if and only if $c(x_i) = \zeta^j x_i$ and $c(y_i) = \zeta^j y_i$ for $1 \leq i \leq n$.*

Proof. Suppose L/K is a G -Galois extension. Then L_1/K is a J_1 -Galois extension. By Lemma 4.1(iii), $c(y_1)/y_1 = \alpha^{-1} = \zeta^j$. Since $y_1^p - y_1 = x_1$, this implies that $c(x_1) = \zeta^j x_1$. As an inductive hypothesis, suppose that $c(x_i) = \zeta^j x_i$ and $c(y_i) = \zeta^j y_i$ for $1 \leq i \leq n-1$.

Now L_n/K_{n-1} is a J_n -Galois extension of local fields and J_n and J_1 are canonically isomorphic by Lemma 2.2. In other words, the value of α for $\text{Aut}(L_n/K_{n-1})$ is the same as for $\text{Aut}(L_1/K)$. By Kummer Theory, there exists a uniformizer π_{n-1} of L_{n-1} such that c acts on π_{n-1} via multiplication by some γ in μ_m . Then L_n/K_{n-1} satisfies the hypotheses of Lemma 4.1, with $1/\pi_{n-1}$, y_n , j_n , and γ^{-1} replacing x , y_1 , j , and ζ respectively. Applying Lemma 4.1(iii) to L_n/K_{n-1} implies that $c(y_n)/y_n = \gamma^{-j_n} = \alpha^{-1} = \zeta^j$.

The equation for L_n/L_{n-1} is $y_n^p - y_n = x_n + g_n$ where the terms of the polynomial $g_n \in \mathbb{F}_p[x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}]$ each have degree congruent to one modulo $p-1$. By the inductive hypothesis and Lemma 4.1(ii), c scales g_n by ζ^j . Thus c scales both $y_n^p - y_n - x_n$ and y_n by ζ^j , which implies $c(x_n) = \zeta^j x_n$.

Conversely, suppose $c(x_i) = \zeta^j x_i$ and $c(y_i) = \zeta^j y_i$ for $1 \leq i \leq n$. The proof that L/K is G -Galois proceeds by induction on n ; the case $n=1$ can be computed explicitly, see e.g. [8, Lemma 1.4.1]. As an inductive hypothesis, suppose that L_{n-1}/K is a G/H_{n-1} -Galois extension. To finish, it suffices to show that the action of c extends to an automorphism of L_n , i.e., that c stabilizes the equation $y_n^p - y_n = f_n$ for L_n/L_{n-1} . By Lemmas 3.2 and 4.1(ii), the action of c scales every term of this equation by ζ^j . \square

Lemma 4.4. *Suppose L/K is a G -Galois extension as in Section 2.3.*

- (i) *There is a Witt vector (x_1, \dots, x_n) in standard form for the subextension L/L_0 and it is uniquely determined up to multiplication by $\mu_{m/m'}$.*
- (ii) *There are $\varphi(m)/\varphi(m/m')$ different non-isomorphic G -Galois structures on the field extension L/K such that the action of σ on L is as in (2).*

Proof. For part (i), by Lemma 3.3, for fixed x , there is a uniquely determined Witt vector (x_1, \dots, x_n) in standard form for the subextension L/L_0 . Now x is determined up to multiplication by ζ^d , for $d \in \mathbb{Z}$. By Proposition 4.3, every monomial in x_i has degree congruent to $j \pmod{m}$. Replacing x with $\zeta^d x$ scales x_i by ζ^{dj} . The values of ζ^{dj} range over $\mu_{m/m'}$ by Lemma 4.1(i).

For part (ii), a G -Galois structure on L/K satisfying the requirement for σ is determined by an isomorphism $\iota : G \rightarrow \text{Aut}(L/K)$ such that $\iota(\sigma)(y_1, \dots, y_n) = (y_1, \dots, y_n) + (1, 0, \dots, 0)$. If $h \in \text{Aut}(L/K)$, then the map $h : L \rightarrow L$ yields an isomorphism of G -Galois extensions $L/K \rightarrow L/K$, the first with structure morphism ι and the second with structure morphism $h\iota h^{-1}$. Thus, modifying ι by an inner automorphism yields an isomorphic G -Galois structure on L/K . So the number of isomorphism classes of G -Galois structures with this requirement on σ is given by the number of elements of $\text{Aut}(G)$ fixing σ , divided by the number of $\text{Inn}(G)$ fixing σ .

An automorphism γ of G which fixes σ is determined by $\gamma(c)$. Also $\gamma(c)$ must have order m and have the same conjugation action as c on σ , as determined by

Lemma 2.1(ii). When G is abelian, then $\alpha' = 1$ and there are $\varphi(m)$ choices for $\gamma(c)$. This yields the count $\varphi(m)/\varphi(m/m')$ since $m' = m$ and since $\text{Inn}(G)$ is trivial. If G is non-abelian, then the image of $\gamma(c)$ in M must have order m and be congruent to c modulo $\langle c^{m/m'} \rangle = \ker(\nu)$. There are $p^n \varphi(m)/\varphi(m/m')$ choices for $\gamma(c)$. This yields the desired count, since there are p^n inner automorphisms of G which fix σ , namely conjugation by powers of σ . \square

5. Main results

Let G be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$. This section contains three results: first we prove that one can dominate a given Galois extension having group $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$ by a Galois extension having group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with control over the last upper jump; second, we give necessary and sufficient conditions for the ramification filtration of a G -Galois extension; third, we define a parameter space for G -Galois extensions of K with given ramification filtration η and calculate its dimension in terms of the upper jumps.

5.1. A wild embedding problem

We prove that one can embed a given Galois extension having group $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$ by a Galois extension having group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, with control over the last upper jump. See [3, 24.42] for an earlier version of this result, in which $m = 1$ and there is no control over the upper jump. Recall that G/H_{n-1} is a semi-direct product of the form $\mathbb{Z}/p^{n-1} \rtimes \mathbb{Z}/m$.

Proposition 5.1. *Suppose L_{n-1}/K is a G/H_{n-1} -Galois extension with upper jumps $u_1 \leq \dots \leq u_{n-1}$. Let $u_n \in \frac{1}{m}\mathbb{N}$ be such that either $u_n = pu_{n-1}$ or both $u_n > pu_{n-1}$ and $p \nmid mu_n$. Suppose also that $mu_n \equiv mu_1 \pmod{m}$. Then there exists a G -Galois extension L_n/K with upper jumps $u_1 \leq \dots \leq u_n$ that dominates L_{n-1}/K .*

Proof. Without loss of generality, one can suppose $L_0 \simeq K[x]/(x^m - 1/t)$ and $c(x) = \zeta x$. The \mathbb{Z}/p^{n-1} -Galois extension L_{n-1}/L_0 has upper jumps $mu_1 \leq \dots \leq mu_{n-1}$ by Lemma 3.6. By Section 3.2, L_{n-1}/L_0 is given by a Witt vector equation $(y_1^p, \dots, y_{n-1}^p) = (y_1, \dots, y_{n-1}) + (x_1, \dots, x_{n-1})$ for some $x_i \in L_0$, such that x_1 is not a p th power in L_0 . Furthermore, one can choose (x_1, \dots, x_{n-1}) to be in standard form. In particular, if $x_i \neq 0$, then $p \nmid v_0(x_i)$.

By Proposition 4.3, if $1 \leq i \leq n-1$, then $c(x_i) = \zeta^j x_i$ and $c(y_i) = \zeta^j y_i$ where $j = mu_1$. By Lemma 3.4, $mu_{n-1} = \max\{-p^{n-i}v_0(x_i)\}_{i=1}^{n-1}$.

If $u_n \neq pu_{n-1}$, let $x_n = x^{mu_n}$. In this case, $-v_0(x_n) = mu_n$. If $u_n = pu_{n-1}$, let $x_n = 0$. In this case, $-v_0(x_n) = -\infty < pmu_{n-1}$. In both cases, (x_1, \dots, x_n) is a Witt vector in standard form. Then the Witt vector equation $(y_1^p, \dots, y_n^p) = (y_1, \dots, y_n) + (x_1, \dots, x_n)$ yields a P -Galois extension L_n/L_0 dominating L_{n-1}/L_0 , with upper jumps $mu_1 \leq \dots \leq mu_n$ by Lemma 3.4 (i.e., [4, Thm. 1.1]).

By the definition of x_n , then $c(x_n) = \zeta^j x_n$. Let $c(y_n) = \zeta^j y_n$. By Proposition 4.3, L_n/K is a G -Galois extension dominating L_{n-1}/K , and it has upper jumps $u_1 \leq \dots \leq u_n$ by Lemma 3.6. \square

5.2. Conditions on the ramification filtration

The ramification filtration of a Galois extension with group G of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ is completely determined by either its lower or upper jumps. Here are the statement and proof of Theorem 1.1, giving necessary and sufficient conditions on the ramification filtrations of G -Galois extensions of K .

Theorem 5.2. *Let G be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ where $p \nmid m$. Let $\sigma \in G$ have order p^n and let $m' = |\text{Cent}_G(\sigma)|/p^n$. A sequence $u_1 \leq \dots \leq u_n$ of rational numbers occurs as the set of positive breaks in the upper numbering of the ramification filtration of a G -Galois extension of $k((t))$ if and only if:*

- (a) $u_i \in \frac{1}{m}\mathbb{N}$ for $1 \leq i \leq n$;
- (b) $\gcd(m, mu_1) = m'$;
- (c) $p \nmid mu_1$ and, for $1 < i \leq n$, either $u_i = pu_{i-1}$ or both $u_i > pu_{i-1}$ and $p \nmid mu_i$;
- (d) and $mu_i \equiv mu_1 \pmod{m}$ for $1 \leq i \leq n$.

Proof. Conditions (a)-(d) are necessary: let $u_1 \leq \dots \leq u_n$ be the set of upper jumps of a G -Galois extension of $k((t))$. The upper jumps of the \mathbb{Z}/p^n -subextension L/L_0 are $w_1 \leq \dots \leq w_n$ where $w_i = mu_i$ by Lemma 3.6. Condition (a) follows since $w_i \in \mathbb{N}$ by the Hasse-Arf Theorem. Condition (b) follows from Lemma 4.1(i). Condition (c) is due to [11], see Lemma 3.5. Condition (d) follows from Proposition 4.2(ii).

Conditions (a)-(d) are sufficient: recall that G has generators σ (of order p^n) and c (of order m) and $c\sigma c^{-1} = \sigma^{\alpha'}$ for some integer α' such that $1 \leq \alpha' < p^n$ and $p \nmid \alpha'$. Let $\alpha \in \mathbb{F}_p^* \simeq (\mathbb{Z}/p)^*$ be such that $\alpha \equiv \alpha' \pmod{p}$. Let $j = mu_1$. By condition (b), ζ^j has order m/m' in k^* . Likewise, α^{-1} has order m/m' in k^* . Thus there exists an integer β such that $\zeta^{\beta j} = \alpha^{-1}$.

Consider the $\langle c \rangle$ -Galois extension L_0/K with equation $x^m = 1/t$ and Galois action $c(x) = \zeta^\beta x$. Let $x_1 \in x^j k[[x^{-m}]]^*$. Consider the \mathbb{Z}/p -Galois extension L_1/L with equation $y_1^p - y_1 = x_1$ and Galois action $\sigma(y_1) = y_1 + 1$. By [8, Lemma 1.4.1], L_1/K is a J_1 -Galois extension. It has lower jump j and thus upper jump u_1 . By conditions (a), (c), (d), and Proposition 5.1, there exists a G -Galois extension L/K dominating L_1/K with upper jumps $u_1 \leq \dots \leq u_n$. \square

Corollary 5.3. *Let G be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ where $p \nmid m$. Suppose η is a ramification filtration of G satisfying conditions (a)-(d). Let f be the order of p modulo m/m' and let $q = p^f$. Then there exists a G -Galois extension L/K with ramification filtration η which is defined over \mathbb{F}_q .*

Proof. It suffices to produce a G -Galois extension L/K whose equations and Galois action have coefficients in \mathbb{F}_q . Note that ζ^{j_1} has order m/m' in k^* . By the definition of f , the field \mathbb{F}_{p^f} contains the (m/m') th roots of unity, and thus contains ζ^{j_1} . The case $n = 1$ follows by direct computation with the equation

$y_1^p - y_1 = x_1^{mu_1}$, see [8, Lemma 1.4.1]. The result then proceeds by induction on n . For the inductive step, one produces an equation for the extension L/L_{n-1} using Proposition 5.1. In the proof of that result, recall that $x_n \in \mathbb{F}_p[x]$ by definition. Thus the equation has coefficients in \mathbb{F}_p by Lemma 3.2. The Galois action is defined over \mathbb{F}_q by (2) and Proposition 4.3. \square

5.3. Parameter space for G -Galois extensions

Given a sequence $u_1 \leq \dots \leq u_n$ satisfying conditions (a)-(d), let η be the ramification filtration of G having upper jumps $u_1 \leq \dots \leq u_n$. By Theorem 5.2, there exists a G -Galois extension of $k((t))$ with ramification filtration η . We prove there is a scheme \mathcal{M}_η such that there is a natural bijection between the k -points of \mathcal{M}_η and isomorphism classes of G -Galois extensions of $k((t))$ with ramification filtration η . We calculate the dimension of \mathcal{M}_η in terms of the sequence $u_1 \leq \dots \leq u_n$.

Notation 5.4. Given positive integers w and m , let

$$\epsilon_p(w, m) = \#\{e \in \mathbb{Z} \mid 1 \leq e \leq w, e \equiv w \pmod{m}, p \nmid e\}.$$

Lemma 5.5. *Let $\delta_p(w, m) = 1$ if $w \equiv ap \pmod{m}$ for some $1 \leq a \leq r$, where r is the remainder when $\lfloor w/p \rfloor$ is divided by m , and $\delta_p(w, m) = 0$ otherwise. Then $\epsilon_p(w, m) = \lceil w/m \rceil - \lfloor w/mp \rfloor - \delta_p(w, m)$.*

Proof. The number of integers e such that $1 \leq e \leq w$ and $e \equiv w \pmod{m}$ is $\lceil w/m \rceil$. To count the number of these which are divisible by p , consider the set $A = \{p, 2p, \dots, \lfloor w/p \rfloor p\}$. Then A contains at least $\lfloor \lfloor w/p \rfloor / m \rfloor = \lfloor w/mp \rfloor$ elements e such that $e \equiv w \pmod{m}$. Let r be the remainder when $\lfloor w/p \rfloor$ is divided by m . Then A contains one additional element $e \equiv w \pmod{m}$ if and only if an element of $\{p, 2p, \dots, rp\}$ is congruent to w modulo m . The formula holds since $\delta_p(w, m) = 1$ precisely in this case. \square

Given a positive integer N , the root of unity $\zeta_{m/m'}$ acts on the affine variety \mathbb{A}^N via multiplication on each coordinate. Let $\mathbb{A}^N / \mu_{m/m'}$ denote the quotient.

Theorem 5.6. *Let G be a semi-direct product of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$ where $p \nmid m$. Let $u_1 \leq \dots \leq u_n$ be a sequence satisfying conditions (a)-(d) and η be the ramification filtration of G with upper jumps $u_1 \leq \dots \leq u_n$. Let $N_\eta = \sum_{i=1}^n \epsilon_p(mu_i, m)$. Then there is an open subscheme $U_\eta \subset \mathbb{A}^{N_\eta} / \mu_{m/m'}$ and a finite étale map $\pi : \mathcal{M}_\eta \rightarrow U_\eta$ of degree $\varphi(m)/\varphi(m/m')$ such that the k -points of \mathcal{M}_η are in natural bijection with isomorphism classes of G -Galois extensions of $k((t))$ with ramification filtration η .*

It is clear that $\dim(\mathcal{M}_\eta) = N_\eta$ depends only on p, m, u_1, \dots, u_n .

Proof. By Lemma 4.4, it suffices to show that the collection of Witt vectors (x_1, \dots, x_n) in standard form, which, as in Proposition 4.3, yield G -Galois extensions L/K with ramification invariants $u_1 \leq \dots \leq u_n$, is in natural bijection with the k -points of an open subscheme of \mathbb{A}^{N_η} .

The proof is by induction on n . For the case $n = 1$, Lemma 3.4 shows that $x_1 \in k[x]$ must have degree mu_1 . By Proposition 4.3, the extension L_1/K is J_1 -Galois if and only if $c(x_1) = \zeta^{mu_1} x_1$, in other words, if and only if all exponents of x_1 are congruent to mu_1 modulo m . Since x_1 is in standard form, it has no exponents with degree divisible by p . Thus the number of possible exponents is $\epsilon = \epsilon_p(mu_1, m)$. Since the leading coefficient of x_1 is nonzero, the choice of x_n is equivalent to the choice of a k -point in an open subscheme of \mathbb{A}^ϵ . (See also [8, Proposition 2.2.6]).

Now, suppose that (x_1, \dots, x_{n-1}) is a Witt vector in standard form, which yields a G/H_{n-1} -Galois extension L_{n-1}/K with upper jumps $u_1 \leq \dots \leq u_{n-1}$. Let $\epsilon = \epsilon_p(mu_n, m)$. It suffices to show that Witt vectors (x_1, \dots, x_n) in standard form which yield an extension L/K dominating L_{n-1}/K with upper jumps $u_1 \leq \dots \leq u_n$ are in natural bijection with the k -points of an open subscheme $\tilde{U}_n \subset \mathbb{A}^\epsilon$.

The Witt vector (x_1, \dots, x_n) for the extension L/K is determined by the choice of $x_n \in k[x]$ in standard form. By Proposition 4.3, the extension L/K is G -Galois if and only if $c(x_n) = \zeta^{mu_1} x_n$, in other words, if and only if all exponents of x_n are congruent to mu_1 modulo m . Recall that $mu_1 \equiv mu_n \pmod m$ by Proposition 4.2.

By Lemma 3.4, the extension L/K has upper jump u_n if and only if $\deg(x_n) = -v_0(x_n) \leq mu_n$, where equality must hold if $u_n > pu_{n-1}$. Thus, an exponent e appearing in x_n satisfies $0 \leq e \leq mu_n$, and $e \equiv mu_n \pmod m$, and $p \nmid e$. The number of these exponents is $\epsilon = \epsilon_p(mu_n, m)$. The leading coefficient of x_n must be non-zero when $u_n > pu_{n-1}$. The choice of x_n is thus equivalent to the choice of a k -point in an open subscheme of \mathbb{A}^ϵ . \square

Remark 5.7. Consider the contravariant functor F_η from the category of schemes to sets, which associates to a scheme B the set of G -Galois extensions of $\mathcal{O}_B((t))$ whose geometric fibres have ramification filtration η . The scheme \mathcal{M}_η does not represent F_η on the category of k -schemes because there are non-constant G -Galois covers defined over a base scheme B , which become constant after pullback by a finite morphism $B' \rightarrow B$. The scheme \mathcal{M}_η is a fine moduli space for F_η on a category where such morphisms are trivialized; see [8, Thm. 2.2.10] for the case $n = 1$.

Remark 5.8. In [1, Prop. 4.1.1], the authors calculate the dimension of the tangent space of the versal deformation space of a \mathbb{Z}/p^n -Galois extension in terms of its ramification filtration. Theorem 5.6 is less technical than their result and it is not clear how to compare them directly.

6. Equations for \mathbb{Z}/p^3 -Galois extensions

It is well-known that the methods of Section 3.2 can be used to find equations for \mathbb{Z}/p^n -extensions [10], but the equations themselves are difficult to find in the literature. Here are formulae for the general \mathbb{Z}/p^3 -Galois extension of K .

Example 6.1. Suppose L/K is a \mathbb{Z}/p^3 -Galois extension of $K \cong k((t))$. Then there exist $x_1, x_2, x_3 \in K$ so that L/K is isomorphic to the following extension:

$$\begin{aligned} y_1^p - y_1 &= x_1; \\ y_2^p - y_2 &= \frac{x_1^p + y_1^p - (x_1 + y_1)^p}{p} + x_2; \\ y_3^p - y_3 &= \frac{x_1^{p^2} + y_1^{p^2} - (x_1 + y_1)^{p^2}}{p^2} + \frac{x_2^p + y_2^p - (x_2 + y_2 + \frac{x_1^p + y_1^p - (x_1 + y_1)^p}{p})^p}{p} + x_3. \end{aligned}$$

A generator σ of the Galois group can be chosen so that its action is given by:

$$\begin{aligned} \sigma(y_1) &= y_1 + 1; \\ \sigma(y_2) &= y_2 + \frac{y_1^p + 1 - (y_1 + 1)^p}{p}; \\ \sigma(y_3) &= y_3 + \frac{y_1^{p^2} + 1 - (y_1 + 1)^{p^2}}{p^2} + \frac{y_2^p - (y_2 + \frac{y_1^p + 1 - (y_1 + 1)^p}{p})^p}{p}. \end{aligned}$$

The integral coefficients in Example 6.1 can be considered to be in $\mathbb{F}_p \subset k$.

Proof. For the equations, it suffices to recursively compute $f_i = \bar{S}_{i-1} - y_i$ for $1 \leq i \leq 3$, starting with $S_0(x_1, y_1) = x_1 + y_1$ and $S_1(x_1, x_2, y_1, y_2) = x_2 + y_2 + (x_1^p + y_1^p - (x_1 + y_1)^p)/p$. The Galois action is given by $\sigma(y_i) = y_i + \tilde{f}_i$, where $\tilde{f}_i = f_i(y_1, \dots, y_{i-1}, 1, 0, \dots, 0)$. To see this, note that $y_i^p = y_i + f_i$ and (1) imply that $(y_1 + f_1, \dots, y_n + f_n) = (y_1, \dots, y_n) + '(x_1, \dots, x_n)$. Substituting $(1, 0, \dots, 0)$ for (x_1, \dots, x_n) yields $(y_1 + f_1, \dots, y_n + f_n) = (y_1, \dots, y_n) + '(1, 0, \dots, 0)$, which equals $\sigma(y_1, \dots, y_n)$ by Lemma 3.3. \square

Example 6.2. When $p = 2$ and $x = t^{-j}$, here are equations for a $\mathbb{Z}/8$ -Galois extension of $k((t))$, which is defined over \mathbb{F}_2 and has upper jumps $j, 2j$, and $4j$:

$$y^2 - y = x; \quad z^2 - z = xy; \quad w^2 - w = x^3y + y^3x + xyz.$$

The Galois action is given by $y \mapsto y + 1$, $z \mapsto z + y$, and $w \mapsto w + y^3 + y + yz$.

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