CURRENT RESULTS ON NEWTON POLYGONS OF CURVES

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ABSTRACT. In this chapter, I first survey the current state of knowledge about Newton polygons and Ekedahl-Oort types of Jacobians of smooth curves. Then I include a new result, joint with Karemaker, which produces supersingular curves of genus g over $\overline{\mathbb{F}}_p$ for infinitely many new values of g for all odd primes p. I also include a new proof that, for all primes p and all $g \geq 4$, there exists a smooth curve defined over $\overline{\mathbb{F}}_p$ with genus g and p-rank $g-4$ whose Newton polygon contains the slopes $1/4$ and $3/4$.

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1. INTRODUCTION

There are deep results about stratifications of moduli spaces of abelian varieties in positive characteristic p by invariants such as the Newton polygon or Ekedahl Oort type. It is not known which Newton polygons and Ekedahl-Oort types occur for Jacobians of smooth curves or, more generally, how the Torelli locus intersects the Newton polygon and Ekedahl-Oort strata in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. In this paper, I survey the current state of knowledge about these open problems.

In addition, the paper contains a new result, Corollary 6.3 (joint with Karemaker), in which we demonstrate that there exist supersingular curves of genus g defined over $\overline{\mathbb{F}}_p$ for infinitely many new values of g for each odd prime p.

Another new result is Proposition 7.2; it is an improved inductive method for producing smooth curves with a given Newton polygon by reducing to the p-rank 0 case. As an application, in Corollary 7.7, for all primes p and all $g \geq 4$, I give a new proof that there exists a smooth curve over $\overline{\mathbb{F}}_p$ of genus g and p-rank $g-4$ whose Newton polygon has slopes $0, 1/4, 3/4, 1$.

Let k be an algebraically closed field of characteristic $p > 0$. Unless mentioned otherwise, all objects are defined over k and all curves are smooth, projective, and connected.

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1.1. Ordinary and supersingular elliptic curves.

The rest of Section 1 is written informally to give background and motivation. More precise definitions are in Section 2.

Let E/k be an elliptic curve and let ℓ be prime. The ℓ-torsion group scheme $E[\ell]$ of E is the kernel of the multiplication-by-ℓ morphism $[\ell] : E \to E$. Then

$$\#E[\ell](k) = \begin{cases} \ell^2 & \text{if } \ell \neq p \\ \ell & \text{if } \ell = p, E \text{ ordinary} \\ 1 & \text{if } \ell = p, E \text{ supersingular} \end{cases}.$$

If $p = 2$, then $y^2 + y = x^3$ is an equation for the unique isomorphism class of supersingular elliptic curve over k. The elliptic curve $y^2 = x^3 - x$ is supersingular if and only if $p \equiv 3 \bmod 4$; and $y^2 = x^3 + 1$ is supersingular if and only if $p \equiv 2 \bmod 3$. For odd primes p, Igusa proved that every supersingular elliptic curve is defined over $\mathbb{F}_{p^2}$ and that

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

is supersingular for exactly $(p-1)/2$ choices of $\lambda \in \overline{\mathbb{F}}_p$. He used this to count the number of isomorphism classes of supersingular elliptic curves (approximately $p/12$).

The following conditions are equivalent to E being ordinary: the Newton polygon of E has slopes 0 and 1; the group scheme $E[p]$ is isomorphic to $L := \mathbb{Z}/p \oplus \mu_p$.

The following conditions are equivalent to E being supersingular:

(A) The only p-torsion point is the identity: $E[p](k) = \{\mathrm{id}\}$.

(B) The Newton polygon of E is a line segment of slope $1/2$.

(C) The group scheme $E[p]$ contains 1 copy of α_p.

Here is more information about condition (C). The group scheme α_p is the kernel of Frobenius on $\mathbb{G}_a$. As a k-scheme, $\alpha_p \simeq \mathrm{Spec}(k[x]/x^p)$ with co-multiplication $m^*(x) = x \otimes 1 + 1 \otimes x$ and co-inverse $\mathrm{inv}^*(x) = -x$. Then $E[p]$ is isomorphic to a group scheme $I_{1,1}$, where $I_{1,1}$ fits in a non-split exact sequence

$$0 \to \alpha_p \to I_{1,1} \to \alpha_p \to 0. \tag{1.1}$$

The image of α_p in (1.1) is the kernel of F (Frobenius) and V (Verschiebung). In fact, $I_{1,1}$ is the unique local-local symmetric BT_1 group scheme of rank p^2.

1.2. Abelian varieties of dimension $g > 1$.

Let A be an abelian variety of dimension g over k. Let $A[p]$ be the kernel of the inseparable multiplication-by-p morphism. The following conditions are all different for $g \geq 3$.

A. p-rank 0 - The only p-torsion point is the identity: $A[p](k) = \{\mathrm{id}\}$.

B. supersingular - The Newton polygon of A is a line segment of slope $1/2$.

C. superspecial - The group scheme $A[p]$ contains g copies of α_p.

Then

$$C \Rightarrow B \Rightarrow A, \text{ but } A \overset{g\geq 3}{\not\Rightarrow} B \overset{g\geq 2}{\not\Rightarrow} C.$$

Here is the motivating question.

Question 1.1. If p is prime and $g \geq 2$, does there exist a smooth curve X/k of genus g whose Jacobian (A) has p-rank 0; (B) is supersingular, or (C) is superspecial?

We will see that the answer to question (A) is yes for all g and p; that the answer to question (B) is sometimes yes, but most often is not known; and that the answer to question (C) is no unless g is small relative to p.

More generally, A has the following invariants, as defined in Section 2.

A. ***p*-rank** - the integer f, with $0 \leq f \leq g$, such that $\#A[p](k) = p^f$.

B. Newton polygon - the data of slopes for the p-divisible group $A[p^\infty] \sim \oplus_\lambda G_{c,d}^{m_\lambda}$.

C. Ekedahl-Oort type -the isomorphism class of the symmetric BT_1 group scheme $A[p]$.

The p-rank, Newton polygon, and Ekedahl-Oort type of a curve are defined to be those of its Jacobian. Question 1.1 asks whether the most rare p-rank, Newton polygon, or Ekedahl-Oort type occurs for the Jacobian of a smooth curve; here is the natural generalization of that question.

Question 1.2. If p is prime and $g \geq 2$, which p-ranks, Newton polygons, and Ekedahl-Oort types occur for the Jacobians of smooth curves X/k of genus g?

1.3. **Stratifications.**

Let $\mathcal{A}_g = \mathcal{A}_g \otimes \mathbb{F}_p$ be the moduli space of principally polarized abelian varieties of dimension g. There are stratifications of $\mathcal{A}_g$ by p-rank, Newton polygon, and Ekedahl-Oort type. Via the Torelli map, the moduli space $\mathcal{M}_g = \mathcal{M}_g \otimes \mathbb{F}_p$ also has stratifications by these invariants.

Here is a geometric generalization of Question 1.2.

Question 1.3. If p is prime and $g \geq 2$, (how) does the open Torelli locus intersect the strata of $\mathcal{A}_g$ by p-rank, Newton polygon, or Ekedahl-Oort type?

In this paper, we describe what is currently known about Questions 1.1 - 1.3. Before reading further, it is a useful exercise to think about which situations are most tractable.

1.4. **A reading list.**

Unfortunately, there is not space here to describe major results about the stratifications on $\mathcal{A}_g$. Here is an incomplete list of valuable references.

A. ***p*-rank:**

Oort, *Subvarieties of moduli spaces* [Oor74]

Norman and Oort, *Moduli of abelian varieties* [NO80]

B. Newton polygon:

Katz, *Slope filtration of F-crystals*, [Kat79]
de Jong and Oort, *Purity of stratification by Newton polygons* [dJO00]
Chai and Oort, *Monodromy and irreducibility of leaves* [CO11]

C. Ekedahl-Oort type:
Kraft, *Kommutative algebraische p-Gruppen* [Kra]
Oort, *A stratification of a moduli space of abelian varieties* [Oor01b]
Moonen and Wedhorn, *Discrete invariants of varieties in pos. char.* [MW04]
Ekedahl and Van der Geer, *Cycle classes of the E-O stratification...* [EvdG09]

Some of the powerful techniques used to study the stratifications on $\mathcal{A}_g$ are not available on $\mathcal{M}_g$, such as deformation tools (Serre-Tate theory and Dieudonné theory) and Hecke operators. The Torelli morphism $T : \overline{\mathcal{M}}_g \to \tilde{\mathcal{A}}_g$ is a good tool, but it is not flat since the fibers have positive dimension on the boundary $\partial\mathcal{M}_g$, whose points represent singular curves of genus g.

2. NOTATION

Let k be an algebraically closed field of characteristic $p > 0$. Let σ denote the Frobenius automorphism of k and its lift to the Witt vectors $W(k)$. Let A be a principally polarized abelian variety of dimension g over k. Here are some facts about p-divisible groups and p-torsion group schemes.

2.1. The p-divisible group.

By the Dieudonné-Manin classification [Man63], there is an isogeny of p-divisible groups

$$A[p^\infty] \sim \oplus_{\lambda=\frac{d}{c+d}} \widetilde{G}_{c,d}^{m_\lambda},$$

where (c,d) ranges over pairs of relatively prime nonnegative integers, and $\widetilde{G}_{c,d}$ denotes a p-divisible group of codimension c, dimension d, and thus height $c+d$. The Dieudonné module $\widetilde{D}_\lambda := \mathbb{D}_*(\widetilde{G}_{c,d})$ (see 2.3 below) is a free $W(k)$-module of rank $c+d$. Over $\operatorname{Frac} W(k)$, there is a basis $x_1, \ldots, x_{c+d}$ for $\widetilde{D}_\lambda$ such that $F^d x_i = p^c x_i$. The Newton polygon of A is a combinatorial invariant determined by the numbers m_λ; it can be viewed as the p-adic Newton polygon of the operator F on $\mathbb{D}_*(A[p^\infty])$. The slopes of the Newton polygon are the multi-set of values of λ.

The abelian variety A is *supersingular* if and only if $\lambda = 1/2$ is the only slope of its p-divisible group $A[p^\infty]$. Let σ_g denote the supersingular Newton polygon of height $2g$. If $\widetilde{I}_{1,1} = \widetilde{G}_{1,1}$ denotes the p-divisible group of dimension 1 and height 2, then A is supersingular if and only $A[p^\infty] \sim \widetilde{I}_{1,1}^g$. There is a partial ordering on Newton polygons, with σ_g being the smallest in this partial ordering.

2.2. The p-torsion group scheme.

The multiplication-by-p morphism $[p] : A \to A$ is a finite flat morphism of degree p^{2g}. The *p-torsion group scheme* of A is

$$A[p] = \mathrm{Ker}[p] = \mathrm{Ker}(\mathrm{Ver} \circ \mathrm{Fr}),$$

where $\mathrm{Fr} : A \to A^{(p)}$ denotes the relative Frobenius morphism and $\mathrm{Ver} : A^{(p)} \to A$ is the Verschiebung morphism. In fact, $A[p]$ is a symmetric BT_1 group scheme as defined in [Oor01b, 2.1, Definition 9.2]; it is killed by $[p]$, with $\mathrm{Ker}(\mathrm{Fr}) = \mathrm{Im}(\mathrm{Ver})$ and $\mathrm{Ker}(\mathrm{Ver}) = \mathrm{Im}(\mathrm{Fr})$.

The morphism F comes from the p-power map on the structure sheaf; it is purely inseparable of degree p^g. The Verschiebung morphism $V : A^{(p)} \to A$ is the dual of $F_{A^{\mathrm{dual}}}$. The principal polarization on A induces a principal quasipolarization on $A[p]$, i.e., an anti-symmetric isomorphism $\psi : A[p] \to A[p]^D$. (This definition must be modified slightly if $p = 2$.) Summarizing, $A[p]$ is a principally quasipolarized (pqp) BT_1 group scheme of rank p^{2g}.

Isomorphisms classes of pqp BT_1 group schemes over k (also known as Ekedahl-Oort types) have been completely classified [Oor01b, Theorem 9.4 & 12.3], building on unpublished work of Kraft [Kra] (which did not include polarizations) and of Moonen [Moo01] (for $p \geq 3$). (When $p = 2$, there are complications with the polarization which are resolved in [Oor01b, 9.2, 9.5, 12.2].)

2.3. Covariant Dieudonné modules.

The p-divisible group $A[p^\infty]$ and the p-torsion group scheme $A[p]$ can be described using covariant Dieudonné theory; see e.g., [Oor01b, 15.3]. Briefly, let $\widetilde{\mathbb{E}} = \widetilde{\mathbb{E}}(k) = W(k)[F, V]$ denote the non-commutative ring generated by semilinear operators F and V with relations

$$FV = VF = p,\ F\lambda = \lambda^\sigma F,\ \lambda V = V\lambda^\sigma, \tag{2.1}$$

for all $\lambda \in W(k)$. There is an equivalence of categories $\mathbb{D}_*$ between p-divisible groups over k and $\widetilde{\mathbb{E}}$-modules which are free of finite rank over $W(k)$.

Similarly, let $\mathbb{E} = \widetilde{\mathbb{E}} \otimes_{W(k)} k$ be the reduction of the Cartier ring mod p; it is a non-commutative ring $k[F, V]$ subject to the same constraints as (2.1), except that $FV = VF = 0$ in $\mathbb{E}$. Again, there is an equivalence of categories $\mathbb{D}_*$ between finite commutative group schemes (of rank $2g$) annihilated by p and $\mathbb{E}$-modules of finite dimension ($2g$) over k. If $M = \mathbb{D}_*(G)$ is the Dieudonné module over k of G, then a principal quasipolarization $\psi : G \to G^D$ induces a a nondegenerate symplectic form

$$\langle \cdot, \cdot \rangle : M \times M \longrightarrow k \tag{2.2}$$

on the underlying k-vector space of M, subject to the additional constraint that, for all x and y in M,

$$\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma. \tag{2.3}$$

If A is the Jacobian of a curve X, then there is an isomorphism of $\mathbb{E}$-modules between the *contravariant* Dieudonné module over k of $\mathrm{Jac}(X)[p]$ and the de Rham cohomology group $H^1_{\mathrm{dR}}(X)$ by [Oda69, Section 5]. The canonical principal polarization on $\mathrm{Jac}(X)$ then induces a canonical isomorphism $\mathbb{D}_*(\mathrm{Jac}(X)[p]) \simeq H^1_{\mathrm{dR}}(X)$; we will use this identification without further comment.

For elements $A_1, \ldots, A_r \in \mathbb{E}$, let $\mathbb{E}(A_1, \ldots, A_r)$ denote the left ideal $\sum_{i=1}^r \mathbb{E} A_i$ of $\mathbb{E}$ generated by $\{A_i \mid 1 \leq i \leq r\}$.

2.4. The p-rank and a-number.

For a BT_1 group scheme G/k, the *p-rank* of G is $f = \dim_{\mathbb{F}_p}\mathrm{Hom}(\mu_p, G)$ where μ_p is the kernel of Frobenius on $\mathbb{G}_m$. Then p^f is the cardinality of $G(k)$. The *a-number* of G is

$$a = \dim_k \mathrm{Hom}(\alpha_p, G),$$

where α_p is the kernel of Frobenius on $\mathbb{G}_a$. It is well-known that $0 \leq f \leq g$ and $1 \leq a + f \leq g$.

Moreover, since μ_p and α_p are both simple group schemes, the p-rank and a-number are additive;

$$f(G \oplus H) = f(G) + f(H) \text{ and } a(G \oplus H) = a(G) + a(H). \tag{2.4}$$

If $\widetilde{G}$ is a p-divisible group, its p-rank and a-number are those of its p-torsion; $f(\widetilde{G}) = f(\widetilde{G}[p])$ and $a(\widetilde{G}) = a(\widetilde{G}[p])$. Similarly, if A is an abelian variety, then $f(A) = f(A[p])$ and $a(A) = a(A[p])$.

2.5. The Ekedahl-Oort type.

As in [Oor01b, Sections 5 & 9], the isomorphism type of a pqp BT_1 group scheme G over k can be encapsulated into combinatorial data. If G is symmetric with rank p^{2g}, then there is a *final filtration* $N_1 \subset N_2 \subset \cdots \subset N_{2g}$ of $\mathbb{D}_*(G)$ as a k-vector space which is stable under the action of V and F^{-1} such that $i = \dim(N_i)$ [Oor01b, 5.4].

The *Ekedahl-Oort type* of G is

$$\nu = [\nu_1, \ldots, \nu_g], \text{ where } \nu_i = \dim(V(N_i)).$$

The p-rank is $\max\{i \mid \nu_i = i\}$ and the a-number equals $g - \nu_g$. There is a restriction $\nu_i \leq \nu_{i+1} \leq \nu_i + 1$ on the Ekedahl-Oort type. There are 2^g Ekedahl-Oort types of length g since all sequences satisfying this restriction occur. By [Oor01b, 9.4, 12.3], there are bijections between (i) Ekedahl-Oort types of length g; (ii) pqp BT_1 group schemes over k of rank p^{2g}; and (iii) pqp Dieudonné modules of dimension $2g$ over k.

By [EvdG09], the Ekedahl-Oort type can also be described by its Young type μ. Given ν, for $1 \leq j \leq g$, consider the strictly decreasing sequence

$$\mu_j = \#\{i \mid 1 \leq i \leq g,\ i - \nu_i \geq j\}.$$

There is a Young diagram with μ_j squares in the jth row. The *Young type* of is $\mu = \{\mu_1, \mu_2, ...\}$, where one eliminates all μ_j which are 0. The p-rank is $g - \mu_1$ and the a-number is $a = \max\{j \mid \mu_j \neq 0\}$. See [Pri08] for examples.

Example 2.1. *The group scheme $I_{1,1}$.* There is a unique BT_1 group scheme of rank p^2 and p-rank 0, which we denote $I_{1,1}$. The structure of $I_{1,1}$ is uniquely determined over $\overline{\mathbb{F}}_p$ by the exact sequence (1.1). The mod p Dieudonné module of $I_{1,1}$ is

$$M_{1,1} := \mathbb{D}_*(I_{1,1}) \simeq \mathbb{E}/\mathbb{E}(F+V).$$

3. Small genus

3.1. The case $g = 2$.

The following table shows the 4 symmetric BT_1 group schemes that occur for principally polarized abelian surfaces. They are listed by name, together with their codimension in $\mathcal{A}_2$, p-rank f, a-number a, Ekedahl-Oort type ν, Young type μ, Dieudonné module, and Newton polygon. Recall that $L \simeq \mathbb{Z}/p \oplus \mu_p$.

Name	cod	f	a	ν	μ	Dieudonné module	Newton polygon
L^2	0	2	0	$[1,2]$	$\emptyset$	$D(L)^2$	$(G_{1,0} \oplus G_{0,1})^2$
$L \oplus I_{1,1}$	1	1	1	$[1,1]$	$\{1\}$	$D(L) \oplus D_{1,1}$	$G_{1,0} \oplus G_{0,1} \oplus G_{1,1}$
$I_{2,1}$	2	0	1	$[0,1]$	$\{2\}$	$\mathbb{E}/\mathbb{E}(F^2+V^2)$	$G_{1,1}^2$
$(I_{1,1})^2$	3	0	2	$[0,0]$	$\{2,1\}$	$(D_{1,1})^2$	$G_{1,1}^2$

The supersingular objects are those in the last two rows.

The Torelli locus $\mathcal{T}_2^0 = T(\mathcal{M}_2)$ is open and dense in $\mathcal{A}_2$. It follows that all 3 Newton polygons and all 4 Ekedahl-Oort types occur for Jacobians of smooth curves of genus 2 over $\overline{\mathbb{F}}_p$, except for one case: there exists a superspecial smooth curve of genus 2 over $\overline{\mathbb{F}}_p$ if and only if $p \geq 5$. This is a special case of [IKO86, Proposition 3.1], in which the authors determine the number of smooth curves X with $\mathrm{Jac}_X[p] \simeq (I_{1,1})^2$.

3.2. The case $g = 3$.

The following table shows the 8 symmetric BT_1 group schemes that occur for principally polarized abelian threefolds.

Name	cod	f	a	ν	μ	Dieudonné module
L^3	0	3	0	$[1,2,3]$	$\emptyset$	$D(L)^3$
$L^2 \oplus I_{1,1}$	1	2	1	$[1,2,2]$	$\{1\}$	$D(L)^2 \oplus D_{1,1}$
$L \oplus I_{2,1}$	2	1	1	$[1,1,2]$	$\{2\}$	$D(L) \oplus \mathbb{E}/\mathbb{E}(F^2+V^2)$
$L \oplus (I_{1,1})^2$	3	1	2	$[1,1,1]$	$\{2,1\}$	$D(L) \oplus (D_{1,1})^2$
$I_{3,1}$	3	0	1	$[0,1,2]$	$\{3\}$	$\mathbb{E}/\mathbb{E}(F^3+V^3)$
$I_{3,2}$	4	0	2	$[0,1,1]$	$\{3,1\}$	$(F^2+V) \oplus (V^2+F)$
$I_{1,1} \oplus I_{2,1}$	5	0	2	$[0,0,1]$	$\{3,2\}$	$D_{1,1} \oplus \mathbb{E}/\mathbb{E}(F^2+V^2)$
$(I_{1,1})^3$	6	0	3	$[0,0,0]$	$\{3,2,1\}$	$(D_{1,1})^3$

The objects in the last two rows are always supersingular but the situation for $I_{3,1}$ and $I_{3,2}$ is more subtle. By [Oor91, Theorem 5.12], if $A[p] \simeq I_{3,1}$, then the Newton polygon is usually $G_{1,2} + G_{2,1}$ (slopes $1/3, 2/3$) but it can also be $G_{1,1}^3$ (supersingular). This shows that the Ekedahl-Oort stratification does not refine the Newton polygon stratification for $g \geq 3$.

The Torelli locus $\mathcal{T}_3^0 = T(\mathcal{M}_3)$ is open and dense in $\mathcal{A}_3$. It follows that all 5 Newton polygons and all 8 Ekedahl-Oort types occur for Jacobians of smooth curves over $\overline{\mathbb{F}}_p$, except when $p = 2$ for $(I_{1,1})^3$ and possibly when $p = 2$ for $I_{1,1} \oplus I_{2,1}$. For the p-rank 0 cases, here are some references for the existence of a smooth curve X of genus 3 over $\overline{\mathbb{F}}_p$ such that Jac_X has the given p-torsion group scheme:

(1) $I_{3,1}$, for all p by [Oor91, Theorem 5.12(1)];
(2) $I_{3,2}$, [Pri09, Lemma 4.8] for $p \geq 3$ and [EP13, Example 5.7(3)] for $p = 2$;
(3) $I_{1,1} \oplus I_{2,1}$, [Pri09, Lemma 4.8] for $p \geq 3$ (using [Oor01b, Proposition 7.3]); the situation for $p = 2$ has not been checked;
(4) $(I_{1,1})^3$, if and only if $p = 3$ by [Oor91, Theorem 5.12].

4. Results about the p-rank of curves

Fix p prime and $g \in \mathbb{N}$ and $0 \leq f \leq g$.

4.1. A. Smooth curves of given genus and p-rank.

The moduli space $\mathcal{M}_g$ of curves of genus g can be stratified by p-rank into strata $\mathcal{M}_g^f$ whose points represent curves of genus g and p-rank f. Similarly, one can stratify the moduli space $\mathcal{H}_g$ of hyperelliptic curves by p-rank. In most cases, it is not known whether $\mathcal{M}_g^f$ and $\mathcal{H}_g^f$ are irreducible.

Theorem 4.1. [FvdG04, Theorem 2.3] *Every component S of $\overline{\mathcal{M}}_g^f$ has dimension $2g - 3 + f$ (codim $g - f$); in particular, there exists a smooth curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.*

Theorem 4.2. *(p odd)* [GP05, Theorem 1], *see also* [AP11, Lemma 3.1], *($p = 2$)* [PZ12, Corollary 1.3] *Every component of $\mathcal{H}_g^f$ has dimension $g-1+f$ (codim $g-f$); in particular, there exists a smooth hyp. curve over $\overline{\mathbb{F}}_p$ with genus g and p-rank f.*

4.2. **Brief sketch of the proof of Theorem 4.1.**

The proof of Theorem 4.1 uses the boundary of $\overline{\mathcal{M}}_g$, whose points represent singular curves.

First, we define the p-rank of a singular curve. Suppose X has 2 irreducible components X_1 and X_2, which are smooth curves, intersecting in an ordinary double point. By [BLR90, Example 8, Page 246], $\mathrm{Jac}(X) \simeq \mathrm{Jac}(X_1) \oplus \mathrm{Jac}(X_1)$. If f_i is the p-rank of X_i, then the p-rank of X is $f_1 + f_2$. It follows that the p-rank of a singular curve of compact type is the sum of the p-ranks of its components.

(More generally, the p-rank of a semi-abelian variety A is $f = \dim_{\mathbb{F}_p}\mathrm{Hom}(\mu_p, A)$. The toric part of A contributes to its p-rank. This can be used to determine the p-rank of a curve of non-compact type, but we will not need that material in this paper.)

Thus, it is easy to construct a *singular* curve of genus g with p-rank f, by constructing a chain of f ordinary and $g-f$ supersingular elliptic curves, joined at ordinary double points. This singular curve can be deformed to a smooth one, but it is not obvious that the p-rank stays constant in this deformation. To prove that there is a *smooth* curve of genus g with p-rank f, singular curves are still useful, but the argument must be done more carefully.

Proof. (Sketch of proof of Theorem 4.1) The dimension of $\overline{\mathcal{M}}_g$ is $3g-3$. There are singular curves that are ordinary, namely chains of g ordinary elliptic curves. Since $\overline{\mathcal{M}}_g$ is irreducible and the p-rank is lower semi-continuous, the generic geometric point of $\overline{\mathcal{M}}_g$ is ordinary, with p-rank g. Using purity of the Newton polygon stratification [dJO00], $\dim(S) \geq (3g-3) - (g-f) = 2g-3+f$.

Let $\mathcal{M}_{g;1}$ denote the moduli space whose points represent curves X of genus g together with a marked point x. For each $1 \leq i \leq g-1$, there is a clutching morphism

$$\kappa_{i,g-i} : \overline{\mathcal{M}}_{i;1} \times \overline{\mathcal{M}}_{g-i;1} \to \overline{\mathcal{M}}_g,$$

defined as follows: if η_1 (resp. η_2) is a point representing a curve X_1 (resp. X_2) of genus i (resp. $g-i$) with marked point x_1 (resp. x_2), then $\kappa_{i,g-i}(\eta_1, \eta_2)$ represents the stable curve of genus g, with components X_1 and X_2, formed by identifying x_1 and x_2 in an ordinary double point. The image of $\kappa_{i,g-i}$ is contained in the component Δ_i of the boundary of $\mathcal{M}_g$. The clutching morphism is a closed immersion if $i \neq g-i$ and is always a finite unramified map [Knu83, Corollary 3.9]. By [FvdG04, Lemma 2.5], S intersects Δ_i for each $1 \leq i \leq g-1$. Since $\mathrm{codim}(\Delta_i, \overline{\mathcal{M}}_g) = 1$, it follows that $\dim(S) \leq \dim(S \cap \Delta_i) + 1$.

Since the p-rank of a singular curve of compact type is the sum of the p-ranks of its components, one can restrict the clutching morphism to the p-rank strata:

$$\kappa_i : \overline{\mathcal{M}}_{i;1}^{f_1} \times \overline{\mathcal{M}}_{g-i;1}^{f_2} \to \overline{\mathcal{M}}_g^{f_1+f_2}.$$

This means that $\dim(S \cap \Delta_i)$ is bounded above by $\dim(\overline{\mathcal{M}}^{f_1}_{i;1}) + \dim(\overline{\mathcal{M}}^{f_2}_{g-i;1})$. Adding a marked point adds one to the dimension. Now $\dim(\overline{\mathcal{M}}^{f_1}_{i;1}) = 2i - 3 + f_1 + 1$ and $\dim(\overline{\mathcal{M}}^{f_2}_{g-i;1}) = 2(g-i) - 3 + f_2 + 1$, working inductively. Thus $\dim(S \cap \Delta_i) \leq 2g - 4 + f$ and so $\dim(S) \leq 2g - 3 + f$. □

4.3. **The p-rank of curves with automorphisms.**

The automorphism group of a curve can place restrictions on its p-rank. If $H \subset \mathrm{Aut}(X)$, consider the cover $\phi_H : X \to Z$, where $Z = X/H$ is the quotient curve. If $x \in X$, let I_x be the inertia group of ϕ_H at x and let $e_x = |I_x|$.

4.3.1. *Automorphisms of prime-to-p degree.*

First, we consider the case that $p \nmid |H|$.

Bouw proved that there are non-trivial constraints on the p-rank of X, arising from the action of the Frobenius operator F on the H-module $H^1(X, \mathcal{O})$. We describe her result in the case that Z is the projective line and $H \simeq \mathbb{Z}/d$. In this case, there is a formula for the dimensions of the $\mathbb{Z}/d$-eigenspaces L_i of $H^1(X, \mathcal{O})$, for $1 \leq i \leq d-1$, in terms of the inertia type of ϕ_H. Now F permutes the eigenspaces L_i; it is simple to determine when two eigenspaces are in the same orbit under F. Bouw's key observation is that the stable rank of F on L_i is bounded by the minimum of the stable rank among all the eigenspaces in the orbit of L_i.

This leads to the following result.

Theorem 4.3. *(See* [Bou01, page 300, Theorem 6.1] *for precise version) Suppose $\phi : X \to \mathbb{P}^1$ is a cyclic degree d cover. Then the inertia type of ϕ determines a (typically non-trivial) upper bound B for the p-rank of X. Under mild conditions, this upper bound B occurs as the p-rank of a cover ϕ of this inertia type.*

4.3.2. *Wildly ramified automorphisms.*

The cover ϕ_H is wildly ramified if p divides the order of the inertia group at a ramified point. There are significant restrictions on the p-rank and the Newton polygon for wildly ramified covers.

Theorem 4.4. *(Deuring-Shafarevich formula)* [Sub75, Theorem 4.2], *see also* [Cre84]. *Suppose that H is a p-group. Then $f_X - 1 = |H|(f_Z - 1) + \sum_{x \in X}(e_x - 1)$.*

Here is a consequence of the Deuring-Shafarevich formula. Suppose H is a p-group and ϕ_H is a cover of the projective line, branched just at one point, say ∞. Then there is a unique point in $\phi_H^{-1}(\infty)$, and ϕ_H is totally ramified at that point. Then Theorem 4.4 implies that X has p-rank 0.

5. Unlikely Newton polygons do occur for Jacobians

5.1. **Unlikely intersections.**

In 2005, Oort made the following observation. The moduli space $\mathcal{A}_g$ has dimension $g(g+1)/2$ and and its supersingular locus $\mathcal{A}_g[\sigma_g]$ has dimension $\lfloor g^2/4 \rfloor$. The difference $\delta_g := g(g+1)/2 - \lfloor g^2/4 \rfloor$ is the length of the longest chain of Newton polygons connecting the ordinary Newton polygon ν_g to the supersingular Newton polygon σ_g.

Remark 5.1. If $g \geq 9$, then $\delta_g > 3g-3 = \dim(\mathcal{M}_g)$.

Because of Remark 5.1, at least one of the following is true:

(1) Either $\mathcal{M}_g$ does not admit a perfect stratification by Newton polyon: this means that there are two Newton polygons ξ_1 and ξ_2 such that $\mathcal{A}_g[\xi_1]$ is in the closure of $\mathcal{A}_g[\xi_2]$, but $\mathcal{M}_g[\xi_1]$ is not in the closure of $\mathcal{M}_g[\xi_2]$.
(2) Or some Newton polygons do not occur for Jacobians of smooth curves.

At this time, no Newton polygon has been excluded from occurring for a Jacobian for any prime. One test case is when $g = 11$ with Newton polygon $G_{5,6} \oplus G_{6,5}$ (slopes $5/11, 6/11$). When $p = 2$, Blache proved the somewhat surprising result that $G_{5,6} \oplus G_{6,5}$ does occur as the Newton polygon of the Jacobian of a smooth curve.

5.2. **Non-existence results for Ekedahl-Oort types.**

Theorem 5.2. [Eke87] *If $X/\overline{\mathbb{F}}_p$ is a superspecial curve of genus g, then $g \leq p(p-1)/2$.*

The bound in Theorem 5.2 is realized by the Hermitian curve $X_p : y^p + y = x^{p+1}$. In [Re01], there is a generalization of Theorem 5.2 which shows that large a-numbers cannot occur for Jacobians of curves when p is small.

When $p = 2$, there is a complete classification of the Ekedahl-Oort types which occur for Jacobians of hyperelliptic curves [EP13]. Briefly, for the curve $y^2 + y = h(x)$, with $\mathrm{div}_\infty(h(x)) = \sum_{i=0}^{f} c_i P_i$, the multi-set $\{c_i\}$ determines the Ekedahl-Oort type. The number of Ekedahl-Oort types that occur for Jacobians of hyperelliptic curves of genus g is approximately the number of partitions of $g+1$ when $p = 2$.

5.3. **Non-existence results for Newton polygons.**

The only non-existence results for Newton polygons are for Jacobians of curves with automorphisms. Specifically, wild ramification places restrictions on the Newton polygon. For example, when $p = 2$, there does not exist a hyperelliptic supersingular curve of genus 3 by [Oor91]. This was generalized as follows.

Theorem 5.3. [SZ02] *If $p = 2$ and $n \geq 2$, then there does not exist a hyperelliptic supersingular curve of genus $2^n - 1$.*

This has been generalized for odd p in [Bla12]. For the connection between the Newton polygon of Artin-Schreier curves and exponential sums, see also [BF07, BFZ08, Bla11].

5.4. **The first open case.**

There are some open questions even for curves of genus 4.

Question 5.4. For all p, does there exist a smooth curve of genus 4 which is supersingular? For all p, does there exist a smooth curve whose Newton polygon is $(G_{1,2} \oplus G_{2,1}) \oplus G_{1,1}$?

Question 5.5. For all p, does there exist a smooth curve of genus 4 with p-rank 0 and a-number a at least 2? More generally, which of these Young types occur for Jacobians of smooth curves of genus 4 for all p:

$$\{4,1\},\ \{4,2\},\ \{4,3\}, \{4,2,1\},\ \{4,3,1\}, \{4,3,2\},\ \{4,3,2,1\}?$$

Here is the partial ordering of Young types in dimension $g = 4$:

$$\begin{tikzcd}
\emptyset \arrow[r] & \{1\} \arrow[r] & \{2\} \arrow[r] \arrow[d] & \{3\} \arrow[r] \arrow[d] & \{4\} \arrow[d] & & \\
& & \{2,1\} \arrow[r] & \{3,1\} \arrow[r] \arrow[d] & \{4,1\} \arrow[d] & & \\
& & & \{3,2\} \arrow[r] \arrow[d] & \{4,2\} \arrow[r] \arrow[d] & \{4,3\} \arrow[d] & \\
& & & \{3,2,1\} \arrow[r] & \{4,2,1\} \arrow[r] & \{4,3,1\} \arrow[d] & \\
& & & & & \{4,3,2\} \arrow[r] & \{4,3,2,1\}.
\end{tikzcd}$$

6. Supersingular curves in Artin-Schreier towers

In this section, we show that the supersingular Newton polygon does occur frequently for Jacobians of smooth curves, even though the supersingular locus has the highest codimension in $\mathcal{A}_g$.

In [vdGvdV95], the authors prove that there exists a supersingular curve of every genus in characteristic 2. This section contains a joint result with Karemaker using their techniques, namely a new existence result for supersingular curves in odd characteristic.

The starting point for these results is the following curve:

$$Y_0 : y^2 - y = x^3 \text{ when } p = 2 \text{ and } Y_0 : y^p - y = x^2 \text{ when } p \text{ is odd.}$$

When $p = 2$, Y_0 is a supersingular elliptic curve. When p is odd, Y_0 is a hyperelliptic curve with genus $(p-1)/2$ by the Riemann-Hurwitz formula and p-rank 0 by the Deuring-Shafarevich formula. In fact, Y_0 is supersingular. The reason for this goes back to the work of Hasse and Davenport [DH35] who showed that the eigenvalues of Frobenius for Y_0 are Gauss sums and so the characteristic polynomial of Frobenius is a $(p-1)$st power of a linear polynomial.

A polynomial $R(x) \in k[x]$ is additive if $R(x_1 + x_2) = R(x_1) + R(x_2)$. This is equivalent to $R(x)$ being a linear combination of monomials x^d where d is a power of p. Suppose that $R(x)$ is additive and consider the Artin-Schreier curve

$$Y : y^p - y = xR(x).$$

Then it turns out that $\mathrm{Jac}(Y)$ is isogenous to a product of Jacobians of Artin-Schreier curves arising from additive polynomials of smaller degree. Working inductively, this shows that $\mathrm{Jac}(Y)$ is isogenous to a product of supersingular curves (isomorphic to Y_0). This yields the following result.

Theorem 6.1. [vdGvdV92, Theorem 13.7], [Bla12, Corollary 3.7(ii)], [BHM$^+$, Proposition 1.8.5] *If $R(x) \in k[x]$ is an additive polynomial of degree p^h, then $Y : y^p - y = xR(x)$ is supersingular with genus $p^h(p-1)/2$.*

More generally, when $R(x) \in \mathbb{F}_q[x]$, then the L-polynomial of Y is determined in [BHM$^+$, Theorem 1.8.4].

The next result follows from Theorem 6.1.

Theorem 6.2. [vdGvdV95, Theorem 2.1] *If $p = 2$ and $g \in \mathbb{N}$, then there exists a supersingular curve Y_g of genus g defined over $\overline{\mathbb{F}}_2$.*

In fact, the authors prove that Y_g can be defined over $\mathbb{F}_2$ [vdGvdV95, Theorem 3.5].

6.1. **New result about supersingular curves.**

Here is a generalization of Theorem 6.2 to odd characteristic.

Corollary 6.3. *[Karemaker/Pries] Let p be prime. Suppose that $g = Gp(p-1)^2/2$ where G is an integer of the form*

(6.1)
$$G = \sum_{i=1}^{t} p^{s_i}(1 + p + \cdots p^{r_i}), \text{ for some } t \in \mathbb{N},\ r_i, s_i \in \mathbb{Z}^{\geq 0} \text{ such that } s_i \geq s_{i-1} + r_{i-1} + 2.$$

Then there exists a supersingular curve over $\overline{\mathbb{F}}_p$ of genus g.

Remark: when $p = 2$, then every $g = G \in \mathbb{N}$ can be written as in (6.1).

Proof. Let $u_i = (s_i+1) - \sum_{j=1}^{i-1}(r_j+1)$ and note $u_{i+1} \geq u_i + 1$. Choose a finite dimensional $\mathbb{F}_p$-linear subspace L_i of dimension $d_i := r_i + 1$ in the vector subspace of $k[x]$ of additive polynomials of degree p^{u_i}. Let $\mathbf{L} = \oplus_{i=1}^t L_i$.

If $f \in \mathbf{L}$, let $C_f : y^p - y = xf$. Let Y be the fiber product of $C_f \to \mathbb{P}^1$ for all $f \in \mathbf{L} - \{0\}$. By [KR89, Theorem B], $\mathrm{Jac}(Y)$ is isogenous to $\oplus_{f\neq 0}\mathrm{Jac}(C_f)$. By Theorem 6.1, $\mathrm{Jac}(C_f)$ is supersingular for each f and so $\mathrm{Jac}(Y)$ is supersingular.

The genus of Y is $g_Y = \sum_{f\neq 0} g_{C_f}$. Now $g_{C_f} = p^{u_i}(p-1)/2$ for $f \in \mathbf{L}$ which have a non-zero contribution from L_i, but not from L_j for $j > i$. There are $p^{d_i} - 1$ non-zero polynomials in L_i. The number of $f \in \mathbf{L}$ which have a non-zero contribution from L_i, but not from L_j for $j > i$ is $(p^{d_i} - 1)\prod_{j=1}^{i-1} p^{d_j}$. So

$$\begin{aligned} g_Y &= \sum_{i=1}^t (p^{d_i}-1)(\prod_{j=1}^{i-1} p^{d_j}) p^{u_i}(p-1)/2 \\ &= \sum_{i=1}^t (p-1)(p^{r_i} + \cdots + 1) p^{\sum_{j=1}^{i-1}(r_j+1)} p^{u_i - 1} p(p-1)/2 \\ &= \sum_{i=1}^t (p^{r_i} + \cdots + 1) p^{s_i} p(p-1)^2/2 = Gp(p-1)^2/2. \end{aligned}$$

□

7. Existence results for Newton polygons

7.1. Reduction to the case of p-rank 0.

This section contains an inductive result; using a smooth curve of genus g with a given Newton polygon ν, the inductive result allows us to produce a smooth curve with larger genus and p-rank, whose Newton polygon contains the slopes of ν. However, in order to apply the inductive result, it is necessary to check some geometric information about the strata of $\mathcal{M}_g$ with Newton polygon ν.

The importance of this result is that it allows us to restrict to the case of p-rank 0 in Question 1.3. This type of inductive process was used in earlier work, but Proposition 7.2 is stronger than [Pri09, Proposition 3.7] and [AP14, Proposition 5.4]. Here is some notation needed for the result.

Notation 7.1. Let ν be a Newton polygon (resp. Ekedahl-Oort type) occurring for abelian varieties of dimension g. Let $\mathcal{A}_g[\nu]$ be the stratum in $\mathcal{A}_g$ whose points represent abelian varieties of type ν. Let $c_\nu = \mathrm{codim}(\mathcal{A}_g[\nu], \mathcal{A}_g)$. For $e \in \mathbb{N}$, let ν^{+e} be the Newton polygon

(resp. Ekedahl-Oort type)

$$\nu^{+e} := (G_{0,1} \oplus G_{1,0})^e \oplus \nu, \ (\text{resp. } \nu^{+e} := L^e \oplus \nu).$$

In other words, ν^{+e} is the Newton polygon in dimension $g + e$, having the slopes of ν together with slopes 0 and 1 with multiplicity e. When working with Ekedahl-Oort types, ν^{+e} is the symmetric BT_1 group scheme which decomposes into the sum of ν with e copies of the p-torsion group scheme of an ordinary elliptic curve.

Proposition 7.2. *With notation as in 7.1, suppose that there exists an irreducible component S of $\mathcal{M}_g[\nu]$ such that* $\mathrm{codim}(S, \mathcal{M}_g) = c_\nu$. *Then, for all $e \in \mathbb{N}$, there exists a component S_e of $\mathcal{M}_{g+e}[\nu^{+e}]$ such that* $\mathrm{codim}(S_e, \mathcal{M}_{g+e}) = c_\nu$.

Proof. The proof is by induction on e, with the case $e = 0$ being trivial. By replacing $\nu^{+(e-1)}$ by ν, S_{e-1} by S, and $g + (e-1)$ by g, it suffices to consider the case $e = 1$. Consider the clutching morphism

$$\kappa_{1,g} : \mathcal{M}_{1;1} \times \mathcal{M}_{g;1} \to \overline{\mathcal{M}}_{g+1},$$

defined as follows: if η_1 is a point representing an elliptic curve E with origin o, and η_2 is a point representing a curve X of genus g with marked point x, then $\kappa_{1,g}(\eta_1, \eta_2)$ represents the stable curve of genus $g+1$, with components E and X, formed by identifying o and x in an ordinary double point. The image of $\kappa_{1,g}$ is contained in the component Δ_1 of the boundary of $\mathcal{M}_{g+1}$.

By hypothesis, $\dim(S) = \dim(\mathcal{M}_g) - c_\nu = 3g - 3 - c_\nu$. Consider the forgetful morphism $\phi : \mathcal{M}_{g;1} \to \mathcal{M}_g$. The fiber of ϕ over η_2 is isomorphic to X and is thus irreducible. Let $S_1 = \phi^*(S)$; it is irreducible and $\dim(S_1) = \dim(S) + 1$. Let $T = \kappa_{1,g-1}(\mathcal{M}_{1;1} \times S_1)$ and note that $\dim(T) = \dim(S) + 2$.

The points of T represent *singular* curves of genus $g+1$ having $(G_{0,1} \oplus G_{1,0}) \oplus \nu$ as Newton polygon (resp. $L \oplus \nu$ as Ekedahl-Oort type). This shows that $\overline{\mathcal{M}}_{g+1}[\nu^{+1}]$ is non-empty. Let S_1 be a component of $\overline{\mathcal{M}}_{g+1}[\nu^{+1}]$ containing T. Since $\mathrm{codim}(\Delta_1, \overline{\mathcal{M}}_{g+1}) = 1$, it follows that either (i) $\dim(S_1) = \dim(T)$, in which case W is contained in the boundary of $\mathcal{M}_{g+1}$ or (ii) $\dim(S_1) = \dim(T) + 1$.

The next claim is that case (i) does not occur. By definition, $c_\nu = \mathrm{codim}(\mathcal{A}_g[\nu], \mathcal{A}_g)$. An exercise with Newton polygons demonstrates that $c_\nu = \mathrm{codim}(\mathcal{A}_{g+1}[\nu^{+1}], \mathcal{A}_{g+1})$. The generic point of $\mathcal{M}_{g+1}$ is ordinary so, by purity [dJO00], $\mathrm{codim}(S_1, \mathcal{M}_{g+1}) \leq c_\nu$. This means that $\dim(S_1) \geq 3(g+1) - 3 - c_\nu = \dim(S) + 3$. This shows that only case (ii) can occur and $\mathrm{codim}(S_1, \mathcal{M}_{g+1}) = c_\nu$.

Furthermore, in case (ii), S_1 is not contained in $\Delta_1[\overline{\mathcal{M}}_{g+1}]$. Since the points of S represent smooth curves, the generic point of S_1 is not contained in any other boundary component and is thus in $\mathcal{M}_{g+1}$. Thus the generic point of S_1 represents a smooth curve. □

Remark 7.3. If the codimension of S in $\mathcal{M}_g$ is smaller than expected, then the same proof shows that the codimension of S_e in $\mathcal{M}_{g+e}$ is also smaller than expected:

$$\text{if } \mathrm{codim}(S, \mathcal{M}_g) = c_\nu - \epsilon, \text{ then } \mathrm{codim}(S_e, \mathcal{M}_{g+e}) = c_\nu - \epsilon.$$

In Section 7.3, we give an application of Proposition 7.2.

7.2. **Some information about central leaves and isogeny leaves.**

Let ξ be a symmetric Newton polygon of height $2g$. Consider the closure W_ξ of the stratum $\mathcal{A}_g[\xi]$ of $\mathcal{A}_g$ whose points represent principally polarized abelian varieties with Newton polygon ξ.

At a point x representing a principally polarized abelian variety A with Newton polygon ξ, one can define the central leaf $C(x)$ and an isogeny leaf $I(x)$ of W_ξ at x. By [Oor04, Theorem 5.3], W_ξ has an 'almost' product structure by $C(x)$ and $I(x)$. Informally, $C(x)$ is the subset of W_ξ whose points represent abelian varieties A' such that there is an isomorphism of p-divisible groups $A'[p^\infty] \simeq A[p^\infty]$. To define $I(x)$, first define $H_\alpha(x)$ to be the set of points of $\mathcal{A}_g$ connected to x by iterated α_p-isogenies (over extension fields). Then $I(x)$ is the union of all irreducible components of $H_\alpha(x)$ that contain x.

The dimensions of the central leaf and the isogeny leaf at η_A depend only on the slopes of ξ. As in [Oor00, 3.3] or [Oor01a, 1.9], define

$$\mathrm{sdim}(\xi) = \#\Delta(\xi),$$

where

$$\Delta(\xi) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leq g,\ (x,y) \text{ on or above } \xi\}.$$

By [Oor01a, Theorem 4.1], the dimension W_ξ is

$$\dim(W_\xi) = \mathrm{sdim}(\xi).$$

For example: when $\xi = G_{3,1} \oplus G_{1,3}$ (slopes $1/4$ and $3/4$), then $\mathrm{sdim}(\xi) = 6$; when $\xi = G_{2,3} \oplus G_{3,2}$ (slopes $2/5$ and $3/5$), then $\mathrm{sdim}(\xi) = 7$.

Let $c(x) = \dim(C(x))$ and $i(x) = \dim(I(x))$. By [Oor04, Corollary 5.8], the 'almost' product structure implies that

$$c(x) + i(x) = \mathrm{sdim}(\xi).$$

The formula for $c(x)$ is more complicated, but depends only on ξ. It relies on counting lattice points (x,y) in a region determined by the slopes λ and $1-\lambda$ of ξ. We provide the formula in a special case. Suppose $\xi = G_{m,n} + G_{n,m}$ where $\gcd(m,n) = 1$ and $m > n$. Then

$$c(\xi) = \#\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq x \leq g, \frac{n}{m+n}x \leq y < \frac{m}{m+n}x\}.$$

For example: when $\xi = G_{3,1} \oplus G_{1,3}$, then $c(\xi) = 5$ and so $i(\xi) = 1$; when $\xi = G_{3,2} + G_{2,3}$, then $c(\xi) = 3$ and so $i(\xi) = 4$.

7.3. **Application.**

In [AP14, Lemma 5.3(a)], for all primes p, the authors prove that there exists a smooth curve defined over $\overline{\mathbb{F}}_p$ with genus 4 whose Newton polygon has slopes $1/4, 3/4$. That proof uses the Newton polygon stratification of the moduli space of principally polarized abelian fourfolds equipped with an action by $\mathbb{Z}[\zeta_3]$ of signature $(3,1)$. In this section, we use a recent result [ST, Proposition 2.1] about abelian fourfolds to give a new proof of [AP14, Lemma 5.3(a)].

Definition 7.4. Let $\nu_4^0 = G_{3,1} \oplus G_{1,3}$ (slopes $1/4$ and $3/4$).

Remark 7.5. If A has Newton polygon type ν_4^0, then its Ekedahl-Oort type is $I_{4,1}$ (defined as the unique symmetric BT_1 group scheme of rank 8, p-rank 0, and a-number 1). The group scheme $I_{4,1}$ can also be described by its final filtration $[0,1,2,3]$, Dieudonné module $\mathbb{E}/\mathbb{E}(F^4+V^4)$, or Young type $\{4\}$. See [Pri08, Lemma 3.1] for details.

Proposition 7.6. *Let A be a 4-dimensional principally polarized abelian variety over $\overline{\mathbb{F}}_p$ with Newton polygon ν_4^0 (slopes $1/4$, $3/4$). Then A is isogenous to a Jacobian of a smooth curve. In particular, there exists a smooth curve X_4 of genus 4 whose Jacobian has Newton polygon ν_4^0.*

Proof. By [ST, Proposition 2.1], A is isogenous to the Jacobian of a curve X of compact type. If X were reducible, then $\mathrm{Jac}(X)$ would decompose as $\mathrm{Jac}(X_1) \oplus \mathrm{Jac}(X_2)$. This would contradict the fact that ν_4^0 is indecomposable as a symmetric object. So X is irreducible, and thus smooth. □

Recall that $\mathcal{M}_g^f$ denotes the p-rank f strata of the moduli space of curves of genus g. For most pairs (g,f), it is not known whether $\mathcal{M}_g^f$ is irreducible.

Corollary 7.7. *Let p be prime. If $g \geq 4$, then the Newton polygon $(G_{0,1} \oplus G_{1,0})^{g-4} \oplus \nu_4^0$ occurs at the generic point of at least one irreducible component of $\mathcal{M}_g^{g-4}$.*

Proof. This follows from Propositions 7.2 and 7.6. □

Remark 7.8. Corollary 7.7 was proven earlier in [AP14, Corollary 5.6].

Remark 7.9. Let $\nu_5' = G_{3,2} \oplus G_{2,3}$ (slopes $2/5$ and $3/5$). Let A be a 5-dimensional principally polarized abelian variety over $\overline{\mathbb{F}}_p$ with Newton polygon ν_5'. One would like to use the strategy of [ST, Proposition 2.1] to prove that A is isogenous to the Jacobian of a smooth curve, and thus that there exists a smooth curve X_5' of genus 5 whose Jacobian has Newton polygon ν_5'.

For this strategy, let T_5 denote the closure of the image of the Torelli morphism $\tau : \mathcal{M}_5 \to \mathcal{A}_5$ in $\mathcal{A}_5$. Now $\dim(\mathcal{A}_5) = 15$ and $\dim(\mathcal{M}_5) = 12$, so T_5 has codimension 3 in $\mathcal{A}_5$. Let x be the point of $\mathcal{A}_5$ representing A. By Section 7.2, the dimension of the isogeny leaf $I(x)$ at x is $i(\xi) = 4$. Also $I(x)$ is proper by [Oor04, Proposition 4.11]. If $I(x)$ intersects T_5, then it would imply that A is isogenous to the Jacobian of a smooth curve X_5'.

Question 7.10. In Remark 7.9, does the isogeny leaf $I(x)$ of dimension 4 intersect the Torelli locus T_5 of codimension 3 in $\mathcal{A}_5$?

8. Open problems

We end with a few open problems about the p-rank 0 strata of $\mathcal{M}_g$. The p-rank 0 stratum of $\mathcal{A}_g$ is irreducible for $g \geq 3$.

Question 8.1. For $g \geq 4$, is the p-rank 0 stratum of $\mathcal{M}_g$ irreducible?

The Newton polygon $G_{1,g-1} \oplus G_{g-1,1}$ is generic in every component in $\mathcal{A}_g^0$.

Question 8.2. If $g \geq 5$, does there exist a smooth curve of genus g with p-rank 0 whose Newton polygon is $G_{1,g-1} \oplus G_{g-1,1}$?

In other words, Question 8.2 asks whether the slopes $1/g$ and $(g-1)/g$ occur for the Newton polygon of the Jacobian of a smooth curve of genus g. (The answer is yes for $g = 4$, as seen in Proposition 7.6(1).)

Analogously, consider the p-torsion group scheme $I_{g,1}$, defined as the unique symmetric BT_1 group scheme of rank p^{2g}, p-rank 0, and a-number 1. It can be described combinatorially by the Ekedahl-Oort type $[0, 1, 2, \ldots, g-1]$, the Dieudonné module $\mathbb{E}/\mathbb{E}(F^g + V^g)$, or the Young type $\{g\}$.

Question 8.3. If $g \geq 5$, does there exist a smooth curve of genus g with p-rank 0 whose Ekedahl-Oort type is $I_{g,1}$?

Proposition 7.6(1) implies that the $g = 4$ analogue of Questions 8.2 and 8.3 has a positive answer.

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