CURRENT RESULTS ON NEWTON POLYGONS OF CURVES

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ABSTRACT. In this chapter, I first survey the current state of knowledge about Newton polygons and Ekedahl-Oort types of Jacobians of smooth curves. Then I include a new result, joint with Karemaker, which produces supersingular curves of genus $g$ over $\mathbb{F}_p$ for infinitely many new values of $g$ for all odd primes $p$. I also include a new proof that, for all primes $p$ and all $g \geq 4$, there exists a smooth curve defined over $\mathbb{F}_p$ with genus $g$ and $p$-rank $g - 4$ whose Newton polygon contains the slopes $1/4$ and $3/4$.

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1. INTRODUCTION

There are deep results about stratifications of moduli spaces of abelian varieties in positive characteristic $p$ by invariants such as the Newton polygon or Ekedahl-Oort type. It is not known which Newton polygons and Ekedahl-Oort types occur for Jacobians of smooth curves or, more generally, how the Torelli locus intersects the Newton polygon and Ekedahl-Oort strata in the moduli space $A_g$ of principally polarized abelian varieties of dimension $g$. In this paper, I survey the current state of knowledge about these open problems.

In addition, the paper contains a new result, Corollary 6.3 (joint with Karemaker), in which we demonstrate that there exist supersingular curves of genus $g$ defined over $\mathbb{F}_p$ for infinitely many new values of $g$ for each odd prime $p$.

Another new result is Proposition 7.2; it is an improved inductive method for producing smooth curves with a given Newton polygon by reducing to the $p$-rank 0 case. As an application, in Corollary 7.7, for all primes $p$ and all $g \geq 4$, I give a new proof that there exists a smooth curve over $\overline{\mathbb{F}}_p$ of genus $g$ and $p$-rank $g - 4$ whose Newton polygon has slopes $0, 1/4, 3/4, 1$.

Let $k$ be an algebraically closed field of characteristic $p > 0$. Unless mentioned otherwise, all objects are defined over $k$ and all curves are smooth, projective, and connected.

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1.1. Ordinary and supersingular elliptic curves.

The rest of Section 1 is written informally to give background and motivation. More precise definitions are in Section 2.

Let \( E/k \) be an elliptic curve and let \( \ell \) be prime. The \( \ell \)-torsion group scheme \( E[\ell] \) of \( E \) is the kernel of the multiplication-by-\( \ell \) morphism \( [\ell] : E \to E \). Then

\[
\#E[\ell](k) = \begin{cases} 
\ell^2 & \text{if } \ell \neq p \\
\ell & \text{if } \ell = p, E \text{ ordinary} \\
1 & \text{if } \ell = p, E \text{ supersingular}
\end{cases}
\]

If \( p = 2 \), then \( y^2 + y = x^3 \) is an equation for the unique isomorphism class of supersingular elliptic curve over \( k \). The elliptic curve \( y^2 = x^3 - x \) is supersingular if and only if \( p \equiv 3 \mod 4 \); and \( y^2 = x^3 + 1 \) is supersingular if and only if \( p \equiv 2 \mod 3 \). For odd primes \( p \), Igusa proved that every supersingular elliptic curve is defined over \( F_{p^2} \) and that

\[
E_\lambda : y^2 = x(x-1)(x-\lambda)
\]

is supersingular for exactly \( (p-1)/2 \) choices of \( \lambda \in \mathbb{F}_p \). He used this to count the number of isomorphism classes of supersingular elliptic curves (approximately \( p/12 \)).

The following conditions are equivalent to \( E \) being ordinary: the Newton polygon of \( E \) has slopes 0 and 1; the group scheme \( E[\ell] \) is isomorphic to \( L :\mathbb{Z}/p \oplus \mu_p \).

The following conditions are equivalent to \( E \) being supersingular:

(A) The only \( p \)-torsion point is the identity: \( E[p](k) = \{\text{id}\} \).

(B) The Newton polygon of \( E \) is a line segment of slope 1/2.

(C) The group scheme \( E[p] \) contains \( g \) copies of \( \alpha_p \).

Here is more information about condition (C). The group scheme \( \alpha_p \) is the kernel of Frobenius on \( \mathbb{G}_a \). As a \( k \)-scheme, \( \alpha_p \simeq \text{Spec}(k[x]/x^p) \) with co-multiplication \( m^*(x) = x \otimes 1 + 1 \otimes x \) and co-inverse \( \text{inv}^*(x) = -x \). Then \( E[p] \) is isomorphic to a group scheme \( I_{1,1} \), where \( I_{1,1} \) fits in a non-split exact sequence

\[
0 \to \alpha_p \to I_{1,1} \to \alpha_p \to 0.
\]

The image of \( \alpha_p \) in \((1.1)\) is the kernel of \( F \) (Frobenius) and \( V \) (Verschiebung). In fact, \( I_{1,1} \) is the unique local-local symmetric BT\(_1\) group scheme of rank \( p^2 \).

1.2. Abelian varieties of dimension \( g > 1 \).

Let \( A \) be an abelian variety of dimension \( g \) over \( k \). Let \( A[p] \) be the kernel of the inseparable multiplication-by-\( p \) morphism. The following conditions are all different for \( g \geq 3 \).

A. \( p \)-rank 0 - The only \( p \)-torsion point is the identity: \( A[p](k) = \{\text{id}\} \).

B. supersingular - The Newton polygon of \( A \) is a line segment of slope 1/2.

C. superspecial - The group scheme \( A[p] \) contains \( g \) copies of \( \alpha_p \).
Then \( g \geq 3 \), \( g \geq 2 \), \( C \Rightarrow B \Rightarrow A \), but \( A \nleftrightarrow B \nleftrightarrow C \).

Here is the motivating question.

**Question 1.1.** If \( p \) is prime and \( g \geq 2 \), does there exist a smooth curve \( X/k \) of genus \( g \) whose Jacobian (A) has \( p \)-rank 0; (B) is supersingular, or (C) is superspecial?

We will see that the answer to question (A) is yes for all \( g \) and \( p \); that the answer to question (B) is sometimes yes, but most often is not known; and that the answer to question (C) is no unless \( g \) is small relative to \( p \).

More generally, \( A \) has the following invariants, as defined in Section 2.

A. **\( p \)-rank** - the integer \( f \), with \( 0 \leq f \leq g \), such that \( \#A[p](k) = p^f \).

B. **Newton polygon** - the data of slopes for the \( p \)-divisible group \( A[p^{\infty}] \sim \oplus \lambda G_{c,d} \).

C. **Ekedahl-Oort type** - the isomorphism class of the symmetric BT1 group scheme \( A[p] \).

The \( p \)-rank, Newton polygon, and Ekedahl-Oort type of a curve are defined to be those of its Jacobian. Question 1.1 asks whether the most rare \( p \)-rank, Newton polygon, or Ekedahl-Oort type occurs for the Jacobian of a smooth curve; here is the natural generalization of that question.

**Question 1.2.** If \( p \) is prime and \( g \geq 2 \), which \( p \)-ranks, Newton polygons, and Ekedahl-Oort types occur for the Jacobians of smooth curves \( X/k \) of genus \( g \)?

1.3. **Stratifications.**

Let \( \mathcal{A}_g = \mathcal{A}_g \otimes F_p \) be the moduli space of principally polarized abelian varieties of dimension \( g \). There are stratifications of \( \mathcal{A}_g \) by \( p \)-rank, Newton polygon, and Ekedahl-Oort type. Via the Torelli map, the moduli space \( \mathcal{M}_g = \mathcal{M}_g \otimes F_p \) also has stratifications by these invariants.

Here is a geometric generalization of Question 1.2.

**Question 1.3.** If \( p \) is prime and \( g \geq 2 \), (how) does the open Torelli locus intersect the strata of \( \mathcal{A}_g \) by \( p \)-rank, Newton polygon, or Ekedahl-Oort type?

In this paper, we describe what is currently known about Questions 1.1 - 1.3. Before reading further, it is a useful exercise to think about which situations are most tractable.

1.4. **A reading list.**

Unfortunately, there is not space here to describe major results about the stratifications on \( \mathcal{A}_g \). Here is an incomplete list of valuable references.

A. **\( p \)-rank:**

Oort, *Subvarieties of moduli spaces* [Oor74]

Norman and Oort, *Moduli of abelian varieties* [NO80]

B. **Newton polygon:**
Katz, *Slope filtration of F-crystals*, [Kat79]
de Jong and Oort, *Purity of stratification by Newton polygons* [dJO00]
Chai and Oort, *Monodromy and irreducibility of leaves* [CO11]

**C. Ekedahl-Oort type:**
Kraft, *Kommutative algebraische p-Gruppen* [Kra]
Oort, *A stratification of a moduli space of abelian varieties* [Oor01b]
Moonen and Wedhorn, *Discrete invariants of varieties in pos. char.* [MW04]
Ekedahl and Van der Geer, *Cycle classes of the E-O stratification*... [EvdG09]

Some of the powerful techniques used to study the stratifications on \( A_g \) are not available on \( M_g \), such as deformation tools (Serre-Tate theory and Dieudonné theory) and Hecke operators. The Torelli morphism \( T : \overline{M}_g \to \overline{A}_g \) is a good tool, but it is not flat since the fibers have positive dimension on the boundary \( \partial M_g \), whose points represent singular curves of genus \( g \).

## 2. Notation

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( \sigma \) denote the Frobenius automorphism of \( k \) and its lift to the Witt vectors \( W(k) \). Let \( A \) be a principally polarized abelian variety of dimension \( g \) over \( k \). Here are some facts about \( p \)-divisible groups and \( p \)-torsion group schemes.

### 2.1. The \( p \)-divisible group.

By the Dieudonné-Manin classification [Man63], there is an isogeny of \( p \)-divisible groups

\[
A[p^\infty] \sim \bigoplus_{(c,d)} \mathcal{G}_{c,d}^{m_{c,d}},
\]

where \((c,d)\) ranges over pairs of relatively prime nonnegative integers, and \( \mathcal{G}_{c,d} \) denotes a \( p \)-divisible group of codimension \( c \), dimension \( d \), and thus height \( c + d \). The Dieudonné module \( D_\lambda := \mathbb{D}_*(\mathcal{G}_{c,d}) \) (see 2.3 below) is a free \( W(k) \)-module of rank \( c + d \). Over \( \text{Frac} W(k) \), there is a basis \( x_1, \ldots, x_{c+d} \) for \( D_\lambda \) such that \( F^d x_i = p^c x_i \). The Newton polygon of \( A \) is a combinatorial invariant determined by the numbers \( m_{c,d} \); it can be viewed as the \( p \)-adic Newton polygon of the operator \( F \) on \( \mathbb{D}_*(A[p^\infty]) \). The slopes of the Newton polygon are the multi-set of values of \( \lambda \).

The abelian variety \( A \) is *supersingular* if and only if \( \lambda = 1/2 \) is the only slope of its \( p \)-divisible group \( A[p^\infty] \). Let \( \sigma_g \) denote the supersingular Newton polygon of height \( 2g \). If \( \tilde{I}_{1,1} = \mathcal{G}_{1,1} \) denotes the \( p \)-divisible group of dimension 1 and height 2, then \( A \) is supersingular if and only \( A[p^\infty] \sim \tilde{I}_{1,1}^8 \). There is a partial ordering on Newton polygons, with \( \sigma_g \) being the smallest in this partial ordering.
2.2. The \( p \)-torsion group scheme.

The multiplication-by-\( p \) morphism \([p] : A \to A\) is a finite flat morphism of degree \( p^{2g} \). The \( p \)-torsion group scheme of \( A \) is
\[
A[p] = \text{Ker}[p] = \text{Ker}(\text{Ver} \circ \text{Fr}),
\]
where \( \text{Fr} : A \to A^{(p)} \) denotes the relative Frobenius morphism and \( \text{Ver} : A^{(p)} \to A \) is the Verschiebung morphism. In fact, \( A[p] \) is a symmetric \( \text{BT}_1 \) group scheme as defined in [Oor01b, 2.1, Definition 9.2]; it is killed by \([p]\), with \( \text{Ker}(\text{Fr}) = \text{Im}(\text{Ver}) \) and \( \text{Ker}(\text{Ver}) = \text{Im}(\text{Fr}) \).

The morphism \( F \) comes from the \( p \)-power map on the structure sheaf; it is purely inseparable of degree \( p^g \). The Verschiebung morphism \( V : A^{(p)} \to A \) is the dual of \( F \). A principal polarization on \( A \) induces a principal quasipolarization on \( A[p] \), i.e., an antisymmetric isomorphism \( \psi : A[p] \to A[p]^D \). (This definition must be modified slightly if \( p = 2 \).) Summarizing, \( A[p] \) is a principally quasipolarized (pqp) \( \text{BT}_1 \) group scheme of rank \( p^{2g} \).

Isomorphisms classes of pqp \( \text{BT}_1 \) group schemes over \( k \) (also known as Ekedahl-Oort types) have been completely classified [Oor01b, Theorem 9.4 & 12.3], building on unpublished work of Kraft [Kra] (which did not include polarizations) and of Moonen [Moo01] (for \( p \geq 3 \)). (When \( p = 2 \), there are complications with the polarization which are resolved in [Oor01b, 9.2, 9.5, 12.2].)

2.3. Covariant Dieudonné modules.

The \( p \)-divisible group \( A[p^{\infty}] \) and the \( p \)-torsion group scheme \( A[p] \) can be described using covariant Dieudonné theory; see e.g., [Oor01b, 15.3]. Briefly, let \( \mathbb{E} = \mathbb{E}(k) = W(k)[F, V] \) denote the non-commutative ring generated by semilinear operators \( F \) and \( V \) with relations
\[
FV = VF = p, \quad F\lambda = \lambda^\sigma F, \quad AV = V\lambda^\sigma,
\]
for all \( \lambda \in W(k) \). There is an equivalence of categories \( \mathcal{D}_* \) between \( p \)-divisible groups over \( k \) and \( \mathbb{E} \)-modules which are free of finite rank over \( W(k) \).

Similarly, let \( \mathbb{E} = \mathbb{E} \otimes_{W(k)} k \) be the reduction of the Cartier ring mod \( p \); it is a non-commutative ring \( k[F, V] \) subject to the same constraints as (2.1), except that \( FV = VF = 0 \) in \( \mathbb{E} \). Again, there is an equivalence of categories \( \mathcal{D}_* \) between finite commutative group schemes (of rank \( 2g \)) annihilated by \( p \) and \( \mathbb{E} \)-modules of finite dimension (\( 2g \)) over \( k \). If \( M = \mathcal{D}_*(G) \) is the Dieudonné module over \( k \) of \( G \), then a principal quasipolarization \( \psi : G \to G^D \) induces a nondegenerate symplectic form
\[
\langle \cdot, \cdot \rangle : M \times M \longrightarrow k
\]
on the underlying \( k \)-vector space of \( M \), subject to the additional constraint that, for all \( x \) and \( y \) in \( M \),
\[
\langle Fx, y \rangle = \langle x, Vy \rangle^\sigma.
\]
If $A$ is the Jacobian of a curve $X$, then there is an isomorphism of $\mathbb{E}$-modules between the contravariant Dieudonné module over $k$ of $\text{Jac}(X)[p]$ and the de Rham cohomology group $H^1_{\text{dR}}(X)$ by [Oda69, Section 5]. The canonical principal polarization on $\text{Jac}(X)$ then induces a canonical isomorphism $D_*(\text{Jac}(X)[p]) \simeq H^1_{\text{dR}}(X)$; we will use this identification without further comment.

For elements $A_1, \ldots, A_r \in \mathbb{E}$, let $\mathbb{E}(A_1, \ldots, A_r)$ denote the left ideal $\sum_{i=1}^r \mathbb{E}A_i$ of $\mathbb{E}$ generated by $\{A_i \mid 1 \leq i \leq r\}$.

2.4. The p-rank and a-number.

For a BT$_1$ group scheme $G/k$, the p-rank of $G$ is $f = \dim_k \text{Hom}(\mu_p, G)$ where $\mu_p$ is the kernel of Frobenius on $G_m$. Then $p^f$ is the cardinality of $G(k)$. The a-number of $G$ is

$$a = \dim_k \text{Hom}(\alpha_p, G),$$

where $\alpha_p$ is the kernel of Frobenius on $G_a$. It is well-known that $0 \leq f \leq g$ and $1 \leq a + f \leq g$.

Moreover, since $\mu_p$ and $\alpha_p$ are both simple group schemes, the p-rank and a-number are additive;

$$f(G \oplus H) = f(G) + f(H) \text{ and } a(G \oplus H) = a(G) + a(H).$$

If $\tilde{G}$ is a $p$-divisible group, its p-rank and a-number are those of its $p$-torsion; $f(\tilde{G}) = f(\tilde{G}[p])$ and $a(\tilde{G}) = a(\tilde{G}[p])$. Similarly, if $A$ is an abelian variety, then $f(A) = f(A[p])$ and $a(A) = a(A[p])$.

2.5. The Ekedahl-Oort type.

As in [Oor01b, Sections 5 & 9], the isomorphism type of a pqp BT$_1$ group scheme $G$ over $k$ can be encapsulated into combinatorial data. If $G$ is symmetric with rank $p^{2g}$, then there is a final filtration $N_1 \subset N_2 \subset \cdots \subset N_{2g}$ of $D_*(G)$ as a $k$-vector space which is stable under the action of $V$ and $F^{-1}$ such that $i = \dim(N_i)$ [Oor01b, 5.4].

The Ekedahl-Oort type of $G$ is

$$v = [v_1, \ldots, v_g], \text{ where } v_i = \dim(V(N_i)).$$

The p-rank is $\max\{i \mid v_i = i\}$ and the a-number equals $g - v_g$. There is a restriction $v_i \leq v_{i+1} \leq v_i + 1$ on the Ekedahl-Oort type. There are $2^g$ Ekedahl-Oort types of length $g$ since all sequences satisfying this restriction occur. By [Oor01b, 9.4, 12.3], there are bijections between (i) Ekedahl-Oort types of length $g$; (ii) pqp BT$_1$ group schemes over $k$ of rank $p^{2g}$; and (iii) pqp Dieudonné modules of dimension $2g$ over $k$.

By [EvdG09], the Ekedahl-Oort type can also be described by its Young type $\mu$. Given $v$, for $1 \leq j \leq g$, consider the strictly decreasing sequence

$$\mu_j = \#\{i \mid 1 \leq i \leq g, i - v_i \geq j\}.$$
There is a Young diagram with $\mu_j$ squares in the $j$th row. The Young type of is $\mu = \{\mu_1, \mu_2, \ldots\}$, where one eliminates all $\mu_j$ which are 0. The $p$-rank is $g - \mu_1$ and the $a$-number is $a = \max\{j \mid \mu_j \neq 0\}$. See [Pri08] for examples.

**Example 2.1.** The group scheme $I_{1,1}$. There is a unique BT$_1$ group scheme of rank $p^2$ and $p$-rank 0, which we denote $I_{1,1}$. The structure of $I_{1,1}$ is uniquely determined over $\mathbb{F}_p$ by the exact sequence (1.1). The mod $p$ Dieudonné module of $I_{1,1}$ is

$$M_{1,1} := D_+(I_{1,1}) \simeq \mathbb{E}/\mathbb{E}(F + V).$$

### 3. SMALL GENUS

#### 3.1. The case $g = 2$.

The following table shows the 4 symmetric BT$_1$ group schemes that occur for principally polarized abelian surfaces. They are listed by name, together with their codimension in $\mathcal{A}_2$, $p$-rank $f$, $a$-number $a$, Ekedahl-Oort type $\nu$, Young type $\mu$, Dieudonné module, and Newton polygon. Recall that $L \simeq \mathbb{Z}/p \oplus \mu_p$.

<table>
<thead>
<tr>
<th>Name</th>
<th>cod</th>
<th>$f$</th>
<th>$a$</th>
<th>$\nu$</th>
<th>$\mu$</th>
<th>Dieudonné module</th>
<th>Newton polygon</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^2$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$[1,2]$</td>
<td>$\emptyset$</td>
<td>$D(L)^2$</td>
<td>$(G_{1,0} \oplus G_{0,1})^2$</td>
</tr>
<tr>
<td>$L \oplus I_{1,1}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$[1,1]$</td>
<td>${1}$</td>
<td>$D(L) \oplus D_{1,1}$</td>
<td>$G_{1,0} \oplus G_{0,1} \oplus G_{1,1}$</td>
</tr>
<tr>
<td>$I_{2,1}$</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>$[0,1]$</td>
<td>${2}$</td>
<td>$\mathbb{E}/\mathbb{E}(F^2 + V^2)$</td>
<td>$G_{i,1}$</td>
</tr>
<tr>
<td>$(I_{1,1})^2$</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>$[0,0]$</td>
<td>${2,1}$</td>
<td>$(D_{1,1})^2$</td>
<td>$G_{i,1}^2$</td>
</tr>
</tbody>
</table>

The supersingular objects are those in the last two rows.

The Torelli locus $T^0_2 = T(\mathcal{M}_2)$ is open and dense in $\mathcal{A}_2$. It follows that all 3 Newton polygons and all 4 Ekedahl-Oort types occur for Jacobians of smooth curves of genus 2 over $\overline{\mathbb{F}}_p$, except for one case: there exists a superspecial smooth curve of genus 2 over $\overline{\mathbb{F}}_p$ if and only if $p \geq 5$. This is a special case of [IKO86, Proposition 3.1], in which the authors determine the number of smooth curves $X$ with $\text{Jac}_X[p] \simeq (I_{1,1})^2$.

#### 3.2. The case $g = 3$.

The following table shows the 8 symmetric BT$_1$ group schemes that occur for principally polarized abelian threefolds.
The objects in the last two rows are always supersingular but the situation for \( I_{3,1} \) and \( I_{3,2} \) is more subtle. By [Oor91, Theorem 5.12], if \( A[p] \cong I_{3,1} \), then the Newton polygon is usually \( G_{1,2} + G_{2,1} \) (slopes \( 1/3, 2/3 \)) but it can also be \( G_{3,1} \) (supersingular). This shows that the Ekedahl-Oort stratification does not refine the Newton polygon stratification for \( g \geq 3 \).

The Torelli locus \( T_3^0 = T(\mathcal{M}_3) \) is open and dense in \( A_3 \). It follows that all 5 Newton polygons and all 8 Ekedahl-Oort types occur for Jacobians of smooth curves over \( \mathbb{F}_p \), except when \( p = 2 \) for \( (I_{1,1})^3 \) and possibly when \( p = 2 \) for \( I_{1,1} \oplus I_{2,1} \). For the \( p \)-rank 0 cases, here are some references for the existence of a smooth curve \( X \) of genus 3 over \( \mathbb{F}_p \) such that \( \text{Jac}_X \) has the given \( p \)-torsion group scheme:

1. \( I_{3,1} \), for all \( p \) by [Oor91, Theorem 5.12(1)];
2. \( I_{3,2} \), [Pri09, Lemma 4.8] for \( p \geq 3 \) and [EP13, Example 5.7(3)] for \( p = 2 \);
3. \( I_{1,1} \oplus I_{2,1} \), [Pri09, Lemma 4.8] for \( p \geq 3 \) (using [Oor01b, Proposition 7.3]); the situation for \( p = 2 \) has not been checked;
4. \( (I_{1,1})^3 \), if and only if \( p = 3 \) by [Oor91, Theorem 5.12].

4. RESULTS ABOUT THE \( p \)-RANK OF CURVES

Fix \( p \) prime and \( g \in \mathbb{N} \) and \( 0 \leq f \leq g \).

4.1. A. Smooth curves of given genus and \( p \)-rank.

The moduli space \( \mathcal{M}_g \) of curves of genus \( g \) can be stratified by \( p \)-rank into strata \( \mathcal{M}_g^f \) whose points represent curves of genus \( g \) and \( p \)-rank \( f \). Similarly, one can stratify the moduli space \( \mathcal{H}_g \) of hyperelliptic curves by \( p \)-rank. In most cases, it is not known whether \( \mathcal{M}_g^f \) and \( \mathcal{H}_g^f \) are irreducible.

**Theorem 4.1.** [FvdG04, Theorem 2.3] Every component \( S \) of \( \overline{\mathcal{M}}_g^f \) has dimension \( 2g - 3 + f \) (codim \( g - f \)); in particular, there exists a smooth curve over \( \mathbb{F}_p \) with genus \( g \) and \( p \)-rank \( f \).
**Theorem 4.2.** (p odd) [GP05, Theorem 1], see also [API1, Lemma 3.1], (p = 2) [PZ12, Corollary 1.3] Every component of $\mathcal{H}_g^f$ has dimension $g - 1 + f$ ($\text{codim } g - f$); in particular, there exists a smooth hyp. curve over $\mathbb{F}_p$ with genus $g$ and p-rank $f$.

### 4.2. Brief sketch of the proof of Theorem 4.1

The proof of Theorem 4.1 uses the boundary of $\overline{\mathcal{M}}_g$, whose points represent singular curves.

First, we define the $p$-rank of a singular curve. Suppose $X$ has 2 irreducible components $X_1$ and $X_2$, which are smooth curves, intersecting in an ordinary double point. By [BLR90, Example 8, Page 246], $\text{Jac}(X) \simeq \text{Jac}(X_1) \oplus \text{Jac}(X_1)$. If $f_i$ is the $p$-rank of $X_i$, then the $p$-rank of $X$ is $f_1 + f_2$. It follows that the $p$-rank of a singular curve of compact type is the sum of the $p$-ranks of its components.

(More generally, the $p$-rank of a semi-abelian variety $A$ is $f = \dim_{\mathbb{F}_p} \text{Hom}(\mu_p, A)$. The toric part of $A$ contributes to its $p$-rank. This can be used to determine the $p$-rank of a curve of non-compact type, but we will not need that material in this paper.)

Thus, it is easy to construct a singular curve of genus $g$ with $p$-rank $f$, by constructing a chain of $f$ ordinary and $g - f$ supersingular elliptic curves, joined at ordinary double points. This singular curve can be deformed to a smooth one, but it is not obvious that the $p$-rank stays constant in this deformation. To prove that there is a smooth curve of genus $g$ with $p$-rank $f$, singular curves are still useful, but the argument must be done more carefully.

**Proof.** (Sketch of proof of Theorem 4.1) The dimension of $\overline{\mathcal{M}}_g$ is $3g - 3$. There are singular curves that are ordinary, namely chains of $g$ ordinary elliptic curves. Since $\overline{\mathcal{M}}_g$ is irreducible and the $p$-rank is lower semi-continuous, the generic geometric point of $\overline{\mathcal{M}}_g$ is ordinary, with $p$-rank $g$. Using purity of the Newton polygon stratification [dJO00], $\dim(S) \geq (3g - 3) - (g - f) = 2g - 3 + f$.

Let $\mathcal{M}_{g;1}$ denote the moduli space whose points represent curves $X$ of genus $g$ together with a marked point $x$. For each $1 \leq i \leq g - 1$, there is a clutching morphism

$$
\kappa_{i,g-i} : \overline{\mathcal{M}}_{i;1} \times \overline{\mathcal{M}}_{g-i;1} \to \overline{\mathcal{M}}_g,
$$

defined as follows: if $\eta_1$ (resp. $\eta_2$) is a point representing a curve $X_1$ (resp. $X_2$) of genus $i$ (resp. $g - i$) with marked point $x_1$ (resp. $x_2$), then $\kappa_{i,g-i}(\eta_1, \eta_2)$ represents the stable curve of genus $g$, with components $X_1$ and $X_2$, formed by identifying $x_1$ and $x_2$ in an ordinary double point. The image of $\kappa_{i,g-i}$ is contained in the component $\Delta_i$ of the boundary of $\overline{\mathcal{M}}_g$. The clutching morphism is a closed immersion if $i \neq g - i$ and is always a finite unramified map [Knu83, Corollary 3.9]. By [FvdG04, Lemma 2.5], $S$ intersects $\Delta_i$ for each $1 \leq i \leq g - 1$. Since $\text{codim}(\Delta_i, \overline{\mathcal{M}}_g) = 1$, it follows that $\dim(S) \leq \dim(S \cap \Delta_i) + 1$.

Since the $p$-rank of a singular curve of compact type is the sum of the $p$-ranks of its components, one can restrict the clutching morphism to the $p$-rank strata:

$$
\kappa_i : \overline{\mathcal{M}}_{i;1} \times \overline{\mathcal{M}}_{g-i;1}^f \to \overline{\mathcal{M}}_g^{f_1 + f_2}.
$$
This means that \( \dim(S \cap \Delta_i) \) is bounded above by \( \dim(\overline{M}_{i;1}^{f_1}) + \dim(\overline{M}_{g-i;1}^{f_2}) \). Adding a marked point adds one to the dimension. Now \( \dim(\overline{M}_{i;1}^{f_1}) = 2i - 3 + f_1 + 1 \) and \( \dim(\overline{M}_{g-i;1}^{f_2}) = 2(g - i) - 3 + f_2 + 1 \), working inductively. Thus \( \dim(S \cap \Delta_i) \leq 2g - 4 + f \) and so \( \dim(S) \leq 2g - 3 + f \). \( \square \)

4.3. The \( p \)-rank of curves with automorphisms.

The automorphism group of a curve can place restrictions on its \( p \)-rank. If \( H \subset \text{Aut}(X) \), consider the cover \( \phi_H: X \to Z \), where \( Z = X/H \) is the quotient curve. If \( x \in X \), let \( I_x \) be the inertia group of \( \phi_H \) at \( x \) and let \( e_x = |I_x| \).

4.3.1. Automorphisms of prime-to-\( p \) degree.

First, we consider the case that \( p \nmid |H| \).

Bouw proved that there are non-trivial constraints on the \( p \)-rank of \( X \), arising from the action of the Frobenius operator \( F \) on the \( H \)-module \( H^1(X, \mathcal{O}) \). We describe her result in the case that \( Z \) is the projective line and \( H \cong \mathbb{Z}/d \). In this case, there is a formula for the dimensions of the \( \mathbb{Z}/d \)-eigenspaces \( L_i \) of \( H^1(X, \mathcal{O}) \), for \( 1 \leq i \leq d - 1 \), in terms of the inertia type of \( \phi_H \). Now \( F \) permutes the eigenspaces \( L_i \); it is simple to determine when two eigenspaces are in the same orbit under \( F \). Bouw’s key observation is that the stable rank of \( F \) on \( L_i \) is bounded by the minimum of the stable rank among all the eigenspaces in the orbit of \( L_i \).

This leads to the following result.

**Theorem 4.3.** (See [Bou01, page 300, Theorem 6.1] for precise version) Suppose \( \phi: X \to \mathbb{P}^1 \) is a cyclic degree \( d \) cover. Then the inertia type of \( \phi \) determines a (typically non-trivial) upper bound \( B \) for the \( p \)-rank of \( X \). Under mild conditions, this upper bound \( B \) occurs as the \( p \)-rank of a cover \( \phi \) of this inertia type.

4.3.2. Wildly ramified automorphisms.

The cover \( \phi_H \) is wildly ramified if \( p \) divides the order of the inertia group at a ramified point. There are significant restrictions on the \( p \)-rank and the Newton polygon for wildly ramified covers.

**Theorem 4.4.** (Deuring-Shafarevich formula) [Sub75, Theorem 4.2], see also [Cre84]. Suppose that \( H \) is a \( p \)-group. Then \( f_X - 1 = |H|(f_Z - 1) + \sum_{x \in X}(e_x - 1) \).

Here is a consequence of the Deuring-Shafarevich formula. Suppose \( H \) is a \( p \)-group and \( \phi_H \) is a cover of the projective line, branched just at one point, say \( \infty \). Then there
is a unique point in $\phi_H^{-1}(\infty)$, and $\phi_H$ is totally ramified at that point. Then Theorem 4.4 implies that $X$ has $p$-rank 0.

5. UNLIKELY NEWTON POLYGONS DO OCCUR FOR JACOBIANS

5.1. Unlikely intersections.

In 2005, Oort made the following observation. The moduli space $A_g$ has dimension $g(g + 1)/2$ and and its supersingular locus $A_g[\sigma_g]$ has dimension $\lfloor g^2/4 \rfloor$. The difference $\delta_g := g(g + 1)/2 - \lfloor g^2/4 \rfloor$ is the length of the longest chain of Newton polygons connecting the ordinary Newton polygon $\nu_g$ to the supersingular Newton polygon $\sigma_g$.

Remark 5.1. If $g \geq 9$, then $\delta_g > 3g - 3 = \dim(M_g)$.

Because of Remark 5.1, at least one of the following is true:

1. Either $M_g$ does not admit a perfect stratification by Newton polyon: this means that there are two Newton polygons $\xi_1$ and $\xi_2$ such that $A_g[\xi_1]$ is in the closure of $A_g[\xi_2]$, but $M_g[\xi_1]$ is not in the closure of $M_g[\xi_2]$.

2. Or some Newton polygons do not occur for Jacobians of smooth curves.

At this time, no Newton polygon has been excluded from occurring for a Jacobian for any prime. One test case is when $g = 11$ with Newton polygon $G_{5,6} \oplus G_{6,5}$ (slopes $5/11, 6/11$). When $p = 2$, Blache proved the somewhat surprising result that $G_{5,6} \oplus G_{6,5}$ does occur as the Newton polygon of the Jacobian of a smooth curve.

5.2. Non-existence results for Ekedahl-Oort types.

Theorem 5.2. [Eke87] If $X/\mathbb{F}_p$ is a superspecial curve of genus $g$, then $g \leq p(p - 1)/2$.

The bound in Theorem 5.2 is realized by the Hermitian curve $X_p : y^p + y = x^{p+1}$. In [Re01], there is a generalization of Theorem 5.2 which shows that large $a$-numbers cannot occur for Jacobians of curves when $p$ is small.

When $p = 2$, there is a complete classification of the Ekedahl-Oort types which occur for Jacobians of hyperelliptic curves [EP13]. Briefly, for the curve $y^2 + y = h(x)$, with $\text{div}_\infty(h(x)) = \sum_{i=0}^f c_i P_i$, the multi-set $\{c_i\}$ determines the Ekedahl-Oort type. The number of Ekedahl-Oort types that occur for Jacobians of hyperelliptic curves of genus $g$ is approximately the number of partitions of $g + 1$ when $p = 2$. 
5.3. **Non-existence results for Newton polygons.**

The only non-existence results for Newton polygons are for Jacobians of curves with automorphisms. Specifically, wild ramification places restrictions on the Newton polygon. For example, when \( p = 2 \), there does not exist a hyperelliptic supersingular curve of genus 3 by [Oor91]. This was generalized as follows.

**Theorem 5.3.** [SZ02] If \( p = 2 \) and \( n \geq 2 \), then there does not exist a hyperelliptic supersingular curve of genus \( 2^n - 1 \).

This has been generalized for odd \( p \) in [Bla12]. For the connection between the Newton polygon of Artin-Schreier curves and exponential sums, see also [BF07, BFZ08, Bla11].

5.4. **The first open case.**

There are some open questions even for curves of genus 4.

**Question 5.4.** For all \( p \), does there exist a smooth curve of genus 4 which is supersingular? For all \( p \), does there exist a smooth curve whose Newton polygon is \( (G_{1,2} \oplus G_{2,1}) \oplus G_{1,1} \)?

**Question 5.5.** For all \( p \), does there exist a smooth curve of genus 4 with \( p \)-rank 0 and \( a \)-number \( a \) at least 2? More generally, which of these Young types occur for Jacobians of smooth curves of genus 4 for all \( p \):

\[
\{4,1\}, \{4,2\}, \{4,3\}, \{4,2,1\}, \{4,3,1\}, \{4,3,2\}, \{4,3,2,1\}?
\]

Here is the partial ordering of Young types in dimension \( g = 4 \):

\[
\emptyset \rightarrow \{1\} \rightarrow \{2\} \rightarrow \{3\} \rightarrow \{4\} \\
\downarrow \quad \downarrow \quad \downarrow \\
\{2,1\} \rightarrow \{3,1\} \rightarrow \{4,1\} \\
\downarrow \quad \downarrow \quad \downarrow \\
\{3,2\} \rightarrow \{4,2\} \rightarrow \{4,3\} \\
\downarrow \quad \downarrow \quad \downarrow \\
\{3,2,1\} \rightarrow \{4,2,1\} \rightarrow \{4,3,1\} \\
\downarrow \quad \downarrow \quad \downarrow \\
\{4,3,2\} \rightarrow \{4,3,2,1\}.
\]

6. **Supersingular curves in Artin-Schreier towers**

In this section, we show that the supersingular Newton polygon does occur frequently for Jacobians of smooth curves, even though the supersingular locus has the highest codimension in \( A_g \).
In [vdGvdV95], the authors prove that there exists a supersingular curve of every genus in characteristic 2. This section contains a joint result with Karemaker using their techniques, namely a new existence result for supersingular curves in odd characteristic.

The starting point for these results is the following curve:

\[ Y_0 : y^2 - y = x^3 \text{ when } p = 2 \text{ and } Y_0 : y^p - y = x^2 \text{ when } p \text{ is odd.} \]

When \( p = 2 \), \( Y_0 \) is a supersingular elliptic curve. When \( p \) is odd, \( Y_0 \) is a hyperelliptic curve with genus \((p - 1)/2\) by the Riemann-Hurwitz formula and \( p \)-rank 0 by the Deuring-Shafarevich formula. In fact, \( Y_0 \) is supersingular. The reason for this goes back to the work of Hasse and Davenport [DH35] who showed that the eigenvalues of Frobenius for \( Y_0 \) are Gauss sums and so the characteristic polynomial of Frobenius is a \((p - 1)\)st power of a linear polynomial.

A polynomial \( R(x) \in k[x] \) is additive if 
\[
R(x_1 + x_2) = R(x_1) + R(x_2).
\]
This is equivalent to \( R(x) \) being a linear combination of monomials \( x^d \) where \( d \) is a power of \( p \). Suppose that \( R(x) \) is additive and consider the Artin-Schreier curve

\[ Y : y^p - y = xR(x). \]

Then it turns out that \( \text{Jac}(Y) \) is isogenous to a product of Jacobians of Artin-Schreier curves arising from additive polynomials of smaller degree. Working inductively, this shows that \( \text{Jac}(Y) \) is isogenous to a product of supersingular curves (isomorphic to \( Y_0 \)). This yields the following result.

**Theorem 6.1.** [vdGvdV92, Theorem 13.7], [Bla12, Corollary 3.7(ii)], [BHM+ Proposition 1.8.5] If \( R(x) \in k[x] \) is an additive polynomial of degree \( p^h \), then \( Y : y^p - y = xR(x) \) is supersingular with genus \( p^h(p - 1)/2 \).

More generally, when \( R(x) \in \mathbb{F}_q[x] \), then the \( L \)-polynomial of \( Y \) is determined in [BHM+ Theorem 1.8.4].

The next result follows from Theorem 6.1.

**Theorem 6.2.** [vdGvdV95, Theorem 2.1] If \( p = 2 \) and \( g \in \mathbb{N} \), then there exists a supersingular curve \( Y_g \) of genus \( g \) defined over \( \mathbb{F}_2 \).

In fact, the authors prove that \( Y_g \) can be defined over \( \mathbb{F}_2 \) [vdGvdV95, Theorem 3.5].

### 6.1. New result about supersingular curves.

Here is a generalization of Theorem 6.2 to odd characteristic.

**Corollary 6.3.** [Karemaker/Pries] Let \( p \) be prime. Suppose that \( g = G p(p - 1)^2/2 \) where \( G \) is an integer of the form

\[
G = \sum_{i=1}^{t} p^{s_i}(1 + p + \cdots p^{r_i}), \text{ for some } t \in \mathbb{N}, r_i, s_i \in \mathbb{Z}_{\geq 0} \text{ such that } s_i \geq s_{i-1} + r_{i-1} + 2.
\]

Then there exists a supersingular curve over \( \mathbb{F}_p \) of genus \( g \).

Remark: when \( p = 2 \), then every \( g = G \in \mathbb{N} \) can be written as in (6.1).
Proof. Let \( u_i = (s_i + 1) - \sum_{j=1}^{i-1} (r_j + 1) \) and note \( u_{i+1} \geq u_i + 1 \). Choose a finite dimensional \( \mathbb{F}_p \)-linear subspace \( L_i \) of dimension \( d_i := r_i + 1 \) in the vector subspace of \( k[x] \) of additive polynomials of degree \( p^{u_i} \). Let \( L = \oplus_{i=1}^{t} L_i \).

If \( f \in L \), let \( C_f : y^p - y = xf \). Let \( Y \) be the fiber product of \( C_f \rightarrow \mathbb{P}^1 \) for all \( f \in L - \{0\} \). By [KR89, Theorem B], \( \text{Jac}(Y) \) is isogenous to \( \oplus_{f \neq 0} \text{Jac}(C_f) \). By Theorem 6.1, \( \text{Jac}(C_f) \) is supersingular for each \( f \) and so \( \text{Jac}(Y) \) is supersingular.

The genus of \( Y \) is \( g_Y = \sum_{f \neq 0} g_{C_f} \). Now \( g_{C_f} = p^{u_i}(p - 1)/2 \) for \( f \in L \) which have a non-zero contribution from \( L_{i_j} \), but not from \( L_j \) for \( j > i \). There are \( p^{d_i} - 1 \) non-zero polynomials in \( L_i \). The number of \( f \in L \) which have a non-zero contribution from \( L_{i_j} \), but not from \( L_j \) for \( j > i \) is \( (p^{d_i} - 1) \prod_{j=1}^{i-1} p^{d_j} \). So

\[
g_Y = \sum_{i=1}^{t} (p^{d_i} - 1) \prod_{j=1}^{i-1} p^{d_j} p^{u_i}(p - 1)/2
= \sum_{i=1}^{t} (p - 1) (p^{r_i} + \cdots + 1) p^{r_i + \cdots + 1} p^{u_i - 1} p(p - 1)/2
= \sum_{i=1}^{t} (p^{r_i} + \cdots + 1) p^{r_i} p(p - 1)^2/2 = Gp(p - 1)^2/2.
\]

□

7. Existence Results for Newton Polygons

7.1. Reduction to the case of \( p \)-rank 0.

This section contains an inductive result; using a smooth curve of genus \( g \) with a given Newton polygon \( \nu \), the inductive result allows us to produce a smooth curve with larger genus and \( p \)-rank, whose Newton polygon contains the slopes of \( \nu \). However, in order to apply the inductive result, it is necessary to check some geometric information about the strata of \( \mathcal{M}_g \) with Newton polygon \( \nu \).

The importance of this result is that it allows us to restrict to the case of \( p \)-rank 0 in Question 1.3. This type of inductive process was used in earlier work, but Proposition 7.2 is stronger than [Pri09, Proposition 3.7] and [APT14, Proposition 5.4]. Here is some notation needed for the result.

Notation 7.1. Let \( \nu \) be a Newton polygon (resp. Ekedahl-Oort type) occurring for abelian varieties of dimension \( g \). Let \( \mathcal{A}_g[\nu] \) be the stratum in \( \mathcal{A}_g \) whose points represent abelian varieties of type \( \nu \). Let \( c_{\nu} = \text{codim}(\mathcal{A}_g[\nu], \mathcal{A}_g) \). For \( e \in \mathbb{N} \), let \( \nu^e \) be the Newton polygon
and note that \( \dim S \) let \( \partial S \) be empty. Let \( \eta \) occur and \( \codim S \) be a generic point of the stable curve of genus \( g \). Then, for all \( e \in \mathbb{N} \), there exists a component \( S_e \) of \( M_{g} + e \) such that \( \codim(S_e, M_{g} + e) = c_v \).

**Proof.** The proof is by induction on \( e \), with the case \( e = 0 \) being trivial. By replacing \( v^{+(e-1)} \) by \( v \), \( S_{e-1} \) by \( S \), and \( g + (e-1) \) by \( g \), it suffices to consider the case \( e = 1 \). Consider the clutching morphism

\[
\kappa_{1,g} : M_{1,1} \times M_{g,1} \to \overline{M}_{g+1},
\]

defined as follows: if \( \eta_1 \) is a point representing an elliptic curve \( E \) with origin \( o \), and \( \eta_2 \) is a point representing a curve \( X \) of genus \( g \) with marked point \( x \), then \( \kappa_{1,g} (\eta_1, \eta_2) \) represents the stable curve of genus \( g + 1 \), with components \( E \) and \( X \), formed by identifying \( o \) and \( x \) in an ordinary double point. The image of \( \kappa_{1,g} \) is contained in the component \( \Delta_1 \) of the boundary of \( \overline{M}_{g+1} \).

By hypothesis, \( \dim(S) = \dim(M_k) - c_v = 3g - 3 - c_v \). Consider the forgetful morphism \( \phi : M_{g,1} \to M_k \). The fiber of \( \phi \) over \( \eta_2 \) is isomorphic to \( X \) and is thus irreducible. Let \( S_1 = \phi^*(S) \); it is irreducible and \( \dim(S_1) = \dim(S) + 1 \). Let \( T = \kappa_{1,g-1}(M_{1,1} \times S_1) \) and note that \( \dim(T) = \dim(S) + 2 \).

The points of \( T \) represent singular curves of genus \( g + 1 \) having \( (G_{0,1} \oplus G_{1,0}) \oplus v \) as Newton polygon (resp. \( L \oplus v \) as Ekedahl-Oort type). This shows that \( \overline{M}_{g+1}[v^{+1}] \) is non-empty. Let \( S_1 \) be a component of \( \overline{M}_{g+1}[v^{+1}] \) containing \( T \). Since \( \codim(\Delta_1, \overline{M}_{g+1}) = 1 \), it follows that either (i) \( \dim(S_1) = \dim(T) \), in which case \( W \) is contained in the boundary of \( M_{g+1} \) or (ii) \( \dim(S_1) = \dim(T) + 1 \).

The next claim is that case (i) does not occur. By definition, \( c_v = \codim(A_g[v], A_g) \). An exercise with Newton polygons demonstrates that \( c_v = \codim(A_g[v^{+1}], A_g) \). The generic point of \( M_{g+1} \) is ordinary so, by purity \( [d]O00 \), \( \codim(S_1, M_{g+1}) \leq c_v \). This means that \( \dim(S_1) \geq 3(g + 1) - 3 - c_v = \dim(S) + 3 \). This shows that only case (ii) can occur and \( \codim(S_1, M_{g+1}) = c_v \).

Furthermore, in case (ii), \( S_1 \) is not contained in \( \Delta_1 \). Since the points of \( S \) represent smooth curves, the generic point of \( S_1 \) is not contained in any other boundary component and is thus in \( M_{g+1} \). Thus the generic point of \( S_1 \) represents a smooth curve. \( \square \)

**Remark 7.3.** If the codimension of \( S \) in \( M_g \) is smaller than expected, then the same proof shows that the codimension of \( S_e \) in \( M_{g+e} \) is also smaller than expected:

\[
\text{if } \codim(S, M_g) = c_v - \epsilon, \text{ then } \codim(S_e, M_{g+e}) = c_v - \epsilon.
\]

In Section 7.3 we give an application of Proposition 7.2.
7.2. Some information about central leaves and isogeny leaves.

Let $\xi$ be a symmetric Newton polygon of height $2g$. Consider the closure $W_\xi$ of the stratum $A_g[\xi]$ of $A_g$ whose points represent principally polarized abelian varieties with Newton polygon $\xi$.

At a point $x$ representing a principally polarized abelian variety $A$ with Newton polygon $\xi$, one can define the central leaf $C(x)$ and an isogeny leaf $I(x)$ of $W_\xi$ at $x$. By [Oor04, Theorem 5.3], $W_\xi$ has an ‘almost’ product structure by $C(x)$ and $I(x)$. Informally, $C(x)$ is the subset of $W_\xi$ whose points represent abelian varieties $A'$ such that there is an isomorphism of $p$-divisible groups $A'[p^\infty] \simeq A[p^\infty]$. To define $I(x)$, first define $H_\alpha(x)$ to be the set of points of $A_g$ connected to $x$ by iterated $\alpha_p$-isogenies (over extension fields). Then $I(x)$ is the union of all irreducible components of $H_\alpha(x)$ that contain $x$.

The dimensions of the central leaf and the isogeny leaf at $\eta_A$ depend only on the slopes of $\xi$. As in [Oor00, 3.3] or [Oor01a, 1.9], define
\[
\text{sdim}(\xi) = \#\Delta(\xi),
\]
where
\[
\Delta(\xi) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leq g, \ (x, y) \text{ on or above } \xi\}.
\]

By [Oor01a, Theorem 4.1], the dimension $W_\xi$ is
\[
\dim(W_\xi) = \text{sdim}(\xi).
\]

For example: when $\xi = G_{3,1} \oplus G_{1,3}$ (slopes $1/4$ and $3/4$), then $\text{sdim}(\xi) = 6$; when $\xi = G_{2,3} \oplus G_{3,2}$ (slopes $2/5$ and $3/5$), then $\text{sdim}(\xi) = 7$.

Let $c(x) = \dim(C(x))$ and $i(x) = \dim(I(x))$. By [Oor04, Corollary 5.8], the ‘almost’ product structure implies that
\[
c(x) + i(x) = \text{sdim}(\xi).
\]

The formula for $c(x)$ is more complicated, but depends only on $\xi$. It relies on counting lattice points $(x, y)$ in a region determined by the slopes $\lambda$ and $1 - \lambda$ of $\xi$. We provide the formula in a special case. Suppose $\xi = G_{m,n} + G_{n,m}$ where $\gcd(m, n) = 1$ and $m > n$. Then
\[
c(\xi) = \#\{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq x \leq g, \ \frac{n}{m+n}x \leq y < \frac{m}{m+n}x\}.
\]

For example: when $\xi = G_{3,1} \oplus G_{1,3}$, then $c(\xi) = 5$ and so $i(\xi) = 1$; when $\xi = G_{3,2} + G_{2,3}$, then $c(\xi) = 3$ and so $i(\xi) = 4$.

7.3. Application.

In [AP14] Lemma 5.3(a)], for all primes $p$, the authors prove that there exists a smooth curve defined over $\overline{\mathbb{F}}_p$ with genus 4 whose Newton polygon has slopes $1/4, 3/4$. That proof uses the Newton polygon stratification of the moduli space of principally polarized abelian fourfolds equipped with an action by $\mathbb{Z}[\xi_3]$ of signature $(3, 1)$. In this section, we use a recent result [ST Proposition 2.1] about abelian fourfolds to give a new proof of [AP14] Lemma 5.3(a)].

**Definition 7.4.** Let $\nu_4^0 = G_{3,1} \oplus G_{1,3}$ (slopes $1/4$ and $3/4$).
Remark 7.5. If $A$ has Newton polygon type $\nu_4^0$, then its Ekedahl-Oort type is $I_{4,1}$ (defined as the unique symmetric BT$_1$ group scheme of rank 8, p-rank 0, and $a$-number 1). The group scheme $I_{4,1}$ can also be described by its final filtration $[0,1,2,3]$, Dieudonné module $E/E(F^4 + V^4)$, or Young type $\{4\}$. See [Pri08, Lemma 3.1] for details.

Proposition 7.6. Let $A$ be a 4-dimensional principally polarized abelian variety over $\mathbb{F}_p$ with Newton polygon $\nu_4^0$ (slopes $1/4$, $3/4$). Then $A$ is isogenous to a Jacobian of a smooth curve. In particular, there exists a smooth curve $X_4$ of genus 4 whose Jacobian has Newton polygon $\nu_4^0$.

Proof. By [ST, Proposition 2.1], $A$ is isogenous to the Jacobian of a curve $X$ of compact type. If $X$ were reducible, then $\text{Jac}(X)$ would decompose as $\text{Jac}(X_1) \oplus \text{Jac}(X_2)$. This would contradict the fact that $\nu_4^0$ is indecomposable as a symmetric object. So $X$ is irreducible, and thus smooth. \qed

Recall that $M_{g,f}^f$ denotes the p-rank $f$ strata of the moduli space of curves of genus $g$. For most pairs $(g,f)$, it is not known whether $M_{g,f}^f$ is irreducible.

Corollary 7.7. Let $p$ be prime. If $g \geq 4$, then the Newton polygon $(G_{0,1} \oplus G_{1,0})^{g-4} \oplus \nu_4^0$ occurs at the generic point of at least one irreducible component of $M_{g-4}^g$.

Proof. This follows from Propositions 7.2 and 7.6. \qed

Remark 7.8. Corollary 7.7 was proven earlier in [AP14, Corollary 5.6].

Remark 7.9. Let $\nu_5' = G_{3,2} \oplus G_{2,3}$ (slopes $2/5$ and $3/5$). Let $A$ be a 5-dimensional principally polarized abelian variety over $\mathbb{F}_p$ with Newton polygon $\nu_5'$. One would like to use the strategy of [ST, Proposition 2.1] to prove that $A$ is isogenous to the Jacobian of a smooth curve, and thus that there exists a smooth curve $X_5'$ of genus 5 whose Jacobian has Newton polygon $\nu_5'$.

For this strategy, let $T_5$ denote the closure of the image of the Torelli morphism $\tau : M_5 \to A_5$ in $A_5$. Now $\text{dim}(A_5) = 15$ and $\text{dim}(M_5) = 12$, so $T_5$ has codimension 3 in $A_5$. Let $x$ be the point of $A_5$ representing $A$. By Section 7.2 the dimension of the isogeny leaf $I(x)$ at $x$ is $i(\xi) = 4$. Also $I(x)$ is proper by [Oor04, Proposition 4.11]. If $I(x)$ intersects $T_5$, then it would imply that $A$ is isogenous to the Jacobian of a smooth curve $X_5'$.

Question 7.10. In Remark 7.9, does the isogeny leaf $I(x)$ of dimension 4 intersect the Torelli locus $T_5$ of codimension 3 in $A_5$?

8. OPEN PROBLEMS

We end with a few open problems about the p-rank 0 strata of $M_g$. The p-rank 0 stratum of $A_3$ is irreducible for $g \geq 3$.

Question 8.1. For $g \geq 4$, is the p-rank 0 stratum of $M_g$ irreducible?

The Newton polygon $G_{1,8-1} \oplus G_{8-1,1}$ is generic in every component in $A_8^0$. 
Question 8.2. If $g \geq 5$, does there exist a smooth curve of genus $g$ with $p$-rank 0 whose Newton polygon is $G_{1,g−1} \oplus G_{g−1,1}$?

In other words, Question 8.2 asks whether the slopes $1/g$ and $(g−1)/g$ occur for the Newton polygon of the Jacobian of a smooth curve of genus $g$. (The answer is yes for $g = 4$, as seen in Proposition 7.6(1).)

Analogously, consider the $p$-torsion group scheme $I_{g,1}$, defined as the unique symmetric BT$_1$ group scheme of rank $p^2g$, $p$-rank 0, and $a$-number 1. It can be described combinatorially by the Ekedahl-Oort type $[0, 1, 2, \ldots, g−1]$, the Dieudonné module $E/E(F^8 + V^8)$, or the Young type $\{g\}$.

Question 8.3. If $g \geq 5$, does there exist a smooth curve of genus $g$ with $p$-rank 0 whose Ekedahl-Oort type is $I_{g,1}$?

Proposition 7.6(1) implies that the $g = 4$ analogue of Questions 8.2 and 8.3 has a positive answer.

REFERENCES


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