GENERIC NEWTON POLYGONS FOR CURVES OF GIVEN p-RANK

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ABSTRACT. We survey results and open questions about the p-ranks and Newton polygons of Jacobians of curves in positive characteristic p. We prove some geometric results about the p-rank stratification of the moduli space of (hyperelliptic) curves. For example, if $0 \le f \le g - 1$, we prove that every component of the p-rank $f + 1$ stratum of $\mathcal{M}_g$ contains a component of the p-rank f stratum in its closure. We prove that the p-rank f stratum of $\overline{\mathcal{M}}_g$ is connected. For all primes p and all $g \ge 4$, we demonstrate the existence of a Jacobian of a smooth curve, defined over $\overline{\mathbb{F}}_p$, whose Newton polygon has slopes $\{0, \frac{1}{4}, \frac{3}{4}, 1\}$. We include partial results about the generic Newton polygons of curves of given genus g and p-rank f.

1. INTRODUCTION

Suppose C is a smooth projective curve of genus g defined over a finite field $\mathbb{F}_q$ of characteristic p. Then its zeta function has the form $Z_{C/\mathbb{F}_q}(T) = \frac{L_{C/\mathbb{F}_q}(T)}{(1-T)(1-qT)}$ for some polynomial $L_{C/\mathbb{F}_q}(T) \in \mathbb{Z}[T]$. The Newton polygon ν of C is that of $L_{C/\mathbb{F}_q}(T)$; it is a lower convex polygon in $\mathbb{R}^2$ with endpoints $(0,0)$ and $(2g, g)$. Its slopes encode important information about C and its Jacobian.

Given a curve $C/\mathbb{F}_q$ of genus g, there are methods to compute its Newton polygon. After some experiments, it becomes clear that the typical Newton polygon has slopes only 0 and 1. For small g and p, the other possible Newton polygons do occur, but rarely, leading us to the following question.

Question 1.1. *Does every Newton polygon of height $2g$ (satisfying the obvious necessary conditions) occur as the Newton polygon of a smooth curve defined over a finite field of characteristic p for each prime p?*

The answer to this question is unknown, although one now knows that every integer f such that $0 \le f \le g$ occurs as the length of the line segment of slope 0 for the Newton polygon of a curve in each characteristic p [12]. As an example, we consider the first open case, when $g = 4$ and ν has slopes $\frac{1}{4}$ and $\frac{3}{4}$. We confirm in Lemma 5.3 that this Newton polygon occurs for a curve of genus 4 for each prime p using a unitary Shimura variety of type $U(3,1)$.

The main idea in this paper is that the occurrence of a certain Newton polygon for a curve of small genus can be used to prove the occurrence of new Newton polygons for smooth curves for every larger genus. As an application, we prove in Corollary 5.6 that the Newton polygon ν_g^{g-4} having $g - 4$ slopes of 0 and 1 and four slopes of $\frac{1}{4}$ and $\frac{3}{4}$ occurs as the Newton polygon of a smooth curve of genus g for all primes p and all $g \ge 4$.

The key condition above is that the curve must be smooth, because it is easy to produce singular curves with decomposable Newton polygons by clutching together curves of smaller genus. In order to deduce results about Newton polygons of smooth curves from results about Newton polygons of singular curves, we rely on geometric methods from [2]. It turns out that one of the best techniques to determine the existence of a curve whose Jacobian has specified behavior is to

study the geometry of the corresponding loci in $\mathcal{M}_g$, the moduli space of smooth proper curves of genus g.

More precisely, the p-rank f and Newton polygon are invariants of the p-divisible group of a principally polarized abelian variety. The stratification of the moduli space $\mathcal{A}_g$ by these invariants is well-understood, in large part because of work of Chai and Oort. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. The Torelli map $\tau : \mathcal{M}_g \hookrightarrow \mathcal{A}_g$, which sends a curve to its Jacobian, allows us to define the analogous stratifications on $\mathcal{M}_g$. For dimension reasons, this gives a lot of information when $1 \leq g \leq 3$ and very little information when $g \geq 4$. For example, in most cases it is not known whether the p-rank f stratum $\mathcal{M}_g^f$ is irreducible.

In Section 2, we review the fundamental definitions and properties of the p-rank and Newton polygon. In Section 3, we review the p-rank and Newton polygon stratifications of $\mathcal{A}_g$. Since degeneration is one of the few techniques for studying stratifications in $\mathcal{M}_g$, in Section 4.1 we recall the Deligne-Mumford compactification of $\mathcal{M}_g$, and explain how it interacts with the p-rank stratification.

In Section 4.2, we review a theorem that we proved about the boundary of the p-rank strata $\mathcal{M}_g^f$ of $\mathcal{M}_g$ in [2]. Using this, we prove that $\overline{\mathcal{M}}_g^f$ is connected for all $g \geq 2$ and $0 \leq f \leq g$ (Corollary 4.5). For $f \geq 1$, we also prove that every component of $\mathcal{M}_g^f$ contains a component of $\mathcal{M}_g^{f-1}$ in its closure (Corollary 4.4).

In Section 5, we consider the finer stratification of $\mathcal{M}_g$ by Newton polygon. We consider a Newton polygon ν_g^f which is the most generic Newton polygon of an abelian variety of dimension g and p-rank f. The expectation is that the generic point of every component of $\mathcal{M}_g^f$ represents a curve with Newton polygon ν_g^f. We prove that this expectation holds in the first non-trivial case when $f = g - 3$ in Corollary 5.5 and prove a slightly weaker statement when $f = g - 4$ in Corollary 5.6.

The discrete invariants associated to these stratifications seem to influence arithmetic attributes of curves over finite fields, such as automorphism groups and maximality. One should note, however, that this relationship is somewhat subtle. On one hand, many exceptional curves turn out to be *supersingular*, meaning that the Newton polygon is a line segment of slope $\frac{1}{2}$. For example, it is not hard to prove that a curve which achieves the Hasse-Weil bound over a finite field must be supersingular. On the other hand, the p-rank stratification is in some ways "transverse" to other interesting loci in $\mathcal{M}_g$, illustrated by the fact that a randomly chosen Jacobian of genus g and p-rank f behaves like a randomly selected principally polarized abelian variety of dimension g. In Section 4.4 and Section 5 we discuss open questions and conjectures on these topics.

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2. STRUCTURES IN POSITIVE CHARACTERISTIC

Consider a principally polarized abelian variety X of dimension g defined over a field K of characteristic $p > 0$. If $N \geq 2$ is relatively prime to p, then the N-torsion group scheme $X[N]$ is étale, and $X[N](\overline{K}) \cong (\mathbb{Z}/N)^{\oplus 2g}$ depends only on the dimension of X. In contrast, $X[p]$ is *never* reduced, and there is a range of possibilities for the geometric isomorphism class of $X[p]_{\overline{K}}$ and, *a fortiori*, the p-divisible group $X[p^\infty] := \lim_{\to n} X[p^n]$. In this section, we review some attributes of $X[p]$ and $X[p^\infty]$, with special emphasis on the case where X is the Jacobian of a curve over a finite field.

2.1. **The p-rank.** The p-rank of X is the rank of the "physical" p-torsion of X. More precisely, it is the integer f such that

$$(2.1) \qquad\qquad\qquad X[p](\overline{K}) \cong (\mathbb{Z}/p)^{\oplus f}.$$

We will see below (2.2.3) that $0 \leq f \leq g$. The abelian variety X is said to be ordinary if its p-rank is maximal, i.e., $f = g$.

Specifying a K-point of $X[p]$ is equivalent to specifying a homomorphism $X[p] \to (\mathbb{Z}/p)$ of group schemes over K, and thus one may also define f by

$$f = \dim_{\mathbb{F}_p} \operatorname{Hom}_{\overline{K}}(X[p], (\mathbb{Z}/p)).$$

Now, $X[p]$ is a self-dual group scheme, and the dual of $(\mathbb{Z}/p)$ is the non-reduced group scheme $\boldsymbol{\mu}_p$, the kernel of Frobenius on the multiplicative group $\mathbb{G}_m$. Consequently, it is equivalent to define the p-rank of X as

$$f = \dim_{\mathbb{F}_p} \operatorname{Hom}_{\overline{K}}(\boldsymbol{\mu}_p, X[p]).$$

(This last formulation is convenient for defining the p-rank of semiabelian varieties and semistable curves.)

If X is the Jacobian of a smooth, projective curve C, then the p-rank equals the maximum rank of a p-group which occurs as the Galois group of an unramified cover of C [20, Corollary 4.18].

2.2. Newton polygons.

2.2.1. *Newton polygon of a curve over a finite field.* Let $C/\mathbb{F}_q$ be a smooth, projective curve of genus g. Then its zeta function

$$Z_{C/\mathbb{F}_q}(T) = \exp\left(\sum_{k \geq 1} \#C(\mathbb{F}_{q^k}) T^k / k\right)$$

is a rational function of the form

$$Z_{C/\mathbb{F}_q} = \frac{L_{C/\mathbb{F}_q}(T)}{(1-T)(1-qT)}$$

where $L_{C/\mathbb{F}_q}(T) \in \mathbb{Z}[T]$ is a polynomial of degree $2g$. The L-polynomial factors over $\overline{\mathbb{Q}}$ as

$$L_{C/\mathbb{F}_q}(T) = \prod_{1 \leq j \leq 2g} (1 - \alpha_j T)$$

where the roots can be ordered so that

(2.2) $$\alpha_j \alpha_{g+j} = q \text{ for each } 1 \leq j \leq g.$$

Each α_j has archimedean size $\sqrt{q}$; for each $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, one has $|\iota(\alpha_j)| = \sqrt{q}$. In contrast, there is a range of possibilities for the p-adic valuations of the α_j. The Newton polygon of C (or of its Jacobian X) is a combinatorial device which encodes these valuations.

Let $\mathbb{K}$ be a field with a discrete valuation v, and let $h(T) = \sum a_i T^i \in \mathbb{K}[T]$ be a polynomial. The Newton polygon of $h(T)$ is defined in the following way.

In the plane, graph the points $(i, v(a_i))$, and form its lower convex hull. This object is called the Newton polygon of h. Equivalently, it suffices to track the multiplicity $e(\lambda)$ with which each slope λ occurs in the diagram. Thus, we will often record a Newton polygon as the function

$$\mathbb{Q} \longrightarrow \mathbb{Z}_{\geq 0}$$

$$\lambda \longmapsto e(\lambda)$$

which, to each λ, assigns the length of the projection of the "slope λ" part of the Newton polygon onto its first coordinate. This function encodes the valuation of the roots of h. More precisely, it is not hard to check that the number of $\alpha \in \overline{\mathbb{K}}$ such that $v(\alpha) = -\lambda$ and $h(\alpha) = 0$ is $e(\lambda)$.
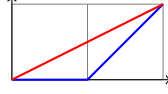
FIGURE 2.1. Newton polygons for elliptic curves.

Write $\lambda = a_\lambda / b_\lambda$ with $\gcd(a_\lambda, b_\lambda) = 1$. Since $h(T)$ is defined over $\mathbb{K}$, there is an integrality constraint

(2.3) $$e(\lambda)\lambda \in \mathbb{Z} \text{ for each } \lambda \in \mathbb{Q},$$

which implies that the line segments of the Newton polygon break at points with integral coordinates. Also,

(2.4) $$\sum_\lambda e(\lambda)/b_\lambda = \deg h.$$

We will often work with the equivalent data

$$m(\lambda) := e(\lambda)/b_\lambda.$$

Now equip $\mathbb{Q}$ with the p-adic valuation, normalized so that $v(q) = 1$. The Newton polygon of $C/\mathbb{F}_q$ is that of $L_{C/\mathbb{F}_q}(T)$. The choice of p-adic valuation means that the Newton polygon of C is unchanged by finite extension of the base field. Moreover, the relation (2.2) implies that

(2.5) $$e(\lambda) = 0 \text{ if } \lambda \notin \mathbb{Q} \cap [0,1]$$

(2.6) $$e(\lambda) = e(1-\lambda).$$

A Newton polygon satisfying (2.3), (2.5) and (2.6) will be called an admissible symmetric Newton polygon of height $\sum_\lambda e(\lambda)/b_\lambda$.

2.2.2. *Examples.* Let $E/\mathbb{F}_q$ be an elliptic curve. There is an integer a such that $|a| \le 2\sqrt{q}$ such that

$$\#E(\mathbb{F}_q) = 1 - a + q.$$

Then

$$Z_{E/\mathbb{F}_q}(T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}.$$

Suppose $\gcd(a, p) = 1$. (This is the generic case.) Then the Newton polygon of E is the lower convex hull of the points

$$\{(0,1), (1,0), (2,1)\},$$

and the slopes of E are $\{0, 1\}$; we have

$$m(\lambda) = \begin{cases} 1 & \lambda \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}.$$

Such an elliptic curve is called ordinary.

Suppose to the contrary that $\gcd(a, p) = p$. Then the Newton polygon of E is the lower convex hull of points

$$\{(0,1), (1, \ge \tfrac{1}{2}), (2,1)\},$$

and the only slope of E is $\{1/2\}$; $m(1/2) = 1$, and all other multiplicities are zero. Such an elliptic curve is called supersingular. (See Figure 2.1.)

We will use the next example (where $g = 4$) in the proof of Lemma 5.3.

Example 2.1. There exists a (hyperelliptic) curve of genus 4 defined over $\mathbb{F}_3$ whose Newton polygon has slopes $1/4$ and $3/4$.

Proof. Using Sage, we calculated a list of possibilities for monic polynomials $f(x) \in \mathbb{F}_3[x]$ of degree nine such that the hyperelliptic curve $C : y^2 = f(x)$ has 3-rank 0. One such possibility is

$$f(x) = x^9 + x^7 + x^6 + 2x^5 + x^4 + 2x^3 + x^2 + x.$$

For the prime 3, Sage cannot directly compute the L-polynomial of C. Instead, we compute the degree 4 truncation of the zeta function of C to be:

$$127T^4 + 40T^3 + 13T^2 + 4T + 1.$$

The zeta function of C has the form $Z_{C/\mathbb{F}_3}(T) = L_{C/\mathbb{F}_3}(T)/(1 - T)(1 - 3T)$ where

$$L_{C/\mathbb{F}_3}(T) = 1 + aT + bT^2 + cT^3 + dT^4 + 3cT^5 + 9bT^6 + 27aT^7 + 81T^8$$

for some coefficients a, b, c, d. By taking the degree four Taylor polynomial of $Z_{C/\mathbb{F}_3}(T)$, we solve $a = b = c = 0$ and $d = 6$. Then the slopes of the Newton polygon of $L_{C/\mathbb{F}_3}(T) = 81T^8 + 6T^4 + 1$ are 1/4 and 3/4. □

2.2.3. *Newton polygon of an abelian variety over an arbitrary field.* There is an equivalent notion of Newton polygon which also makes sense for an arbitrary abelian variety over an arbitrary field of positive characteristic. For each $\lambda \in \mathbb{Q}_{\geq 0}$, write $\lambda = a_\lambda/b_\lambda = a_\lambda/(a_\lambda + c_\lambda)$ with a_λ and b_λ relatively prime. Manin defines a certain p-divisible group $G_\lambda = G_{a_\lambda, c_\lambda}$ over $\mathbb{F}_p$, with dimension a_λ, codimension c_λ and, thus, height b_λ [18]. If G is any p-divisible group over an algebraically closed field k, then there is an isogeny $G \to \oplus_\lambda H_\lambda^{\oplus m_G(\lambda)}$. The isogeny itself is not canonical, but the collection of all nonnegative integers $\{m_G(\lambda)\}$ is independent of all choices. Let $e_G(\lambda) = b_\lambda m_G(\lambda)$.

Now let X/K be an abelian variety of dimension g; the Newton polygon of X is that of its p-divisible group $X[p^\infty]_{\overline{K}}$. It is not hard to verify that $e_{X[p^\infty]}$ is an admissible symmetric Newton polygon of height $2g$. In the special case where $C/\mathbb{F}_q$ is a smooth projective curve over a finite field, the Newton polygon of C (as defined in Section 2.2.1) coincides with that of its Jacobian. In general, we define the Newton polygon of C/K as that of its Jacobian.

The Newton polygon of a p-divisible group, and thus of an abelian variety, is invariant under isogeny. (Note, however, that even if $X[p^\infty]$ and $Y[p^\infty]$ are isogenous, it does *not* follow that X and Y are isogenous.) Moreover, if G and H are p-divisible groups, it follows immediately from the definition that

$$m_{G \oplus H}(\lambda) = m_G(\lambda) + m_H(\lambda).$$

The symmetry condition $m(\lambda) = m(1 - \lambda)$ forces the inequality $0 \leq f_X \leq \dim X$ noted in Section 2.1.

An abelian variety X/K is ordinary if and only if all its slopes are 0 and 1. The p-rank of X is equal to the multiplicity $m(0)$ of the slope 0 in the Newton polygon. An abelian variety is *supersingular* if all the slopes of its Newton polygon equal 1/2. Thus, if an abelian variety is supersingular, then it has p-rank zero. However, in dimension at least three, the converse is false. For example, there are abelian varieties with p-rank 0 whose Newton polygons have slopes $\frac{1}{g}$ and $\frac{g-1}{g}$ and are thus not supersingular for $g \geq 3$.

2.3. **Semicontinuity and purity.** We now consider a family of p-divisible groups, such as that coming from a family of abelian varieties in characteristic p. It is not too hard to show that the p-rank is a lower semicontinuous function, i.e., that it can only decrease under specialization. In fact, if the p-rank does change, it does so in codimension one:

Lemma 2.2. [22, Lemma 1.6] *Let $X \to S$ be an abelian variety over an integral scheme in positive characteristic, and suppose that X has generic p-rank f. Let $S^{<f} \subset S$ denote the locus parametrizing those s such that X_s has p-rank strictly less than f. Then either $S^{<f}$ is empty or $S^{<f}$ is pure of codimension one.*

There are analogous semicontinuity and purity results for Newton polygons, although the former requires more notation to state, and the proof of the latter is much deeper than that of Lemma 2.2.

The partial ordering on Newton polygons is defined as follows. Let

$$\nu = \{\lambda_1^{\oplus m_{\lambda_1}}, \ldots, \lambda_r^{\oplus m_{\lambda_r}}\},$$

where $\lambda_i < \lambda_{i+1}$. Let $\Gamma(\nu) \subset \mathbb{R}^2$ be the convex hull of $(0,0)$ and

$$\{(\sum_{i=1}^{j} m_{\lambda_i} c_{\lambda_i}, \sum_{i=1}^{j} m_{\lambda_i} a_{\lambda_i}) : 1 \leq j \leq r\}.$$

If μ and ν are two Newton polygons, we will write $\mu \preceq \nu$ if $\Gamma(\mu)$ and $\Gamma(\nu)$ have the same endpoints and if all points of $\Gamma(\mu)$ lie on or above those of $\Gamma(\nu)$. (This convention may seem a little surprising, but it has the pleasant consequence that "smaller" Newton strata are in the closures of "larger" ones.) See Figure 2.2 for the $g = 4$ case of the poset of admissible symmetric Newton polygons.

Newton polygons have the following semicontinuity property [16, Theorem 2.3.1]: Let $S = \mathrm{Spec}(R)$ be the spectrum of a local ring, with geometric generic point $\bar{\eta}$ and geometric closed point $\bar{s}$. If G is a p-divisible group over S, then $\nu(G_{\bar{s}}) \preceq \nu(G_{\bar{\eta}})$. Like the p-rank, the Newton polygon is a discrete invariant which changes (if at all) in codimension one.

Proposition 2.3. [10] *Let S be an integral, excellent scheme. Let $G \to S$ be a p-divisible group over S, and let $U \subset S$ be the largest dense set on which the Newton polygon is constant. Then either $U = S$ or* $\mathrm{codim}((S \setminus U), S) = 1$.

2.4. **Notation on stratifications and Newton polygons.** Let $X \to S$ be any family of abelian varieties of relative dimension g in positive characteristic. For any Newton polygon ν, let S^ν be the reduced subspace such that $s \in S^\nu$ if and only if the Newton polygon of X_s is ν. In general, the Newton stratification refines the p-rank stratification $S = \sqcup S^f$, where $s \in S^f$ if and only if the p-rank of X_s is f.

If $C \to S$ is a family of smooth, projective curves, then the Newton polygon and p-rank strata of S are those corresponding to the relative Jacobian $\mathrm{Jac}(C) \to S$.

Suppose that S is irreducible and $X \to S$ is a family of abelian varieties. Since there are only finitely many symmetric admissible Newton polygons of any particular height, by semicontinuity there is a nonempty open subset U over which the Newton polygon of X is constant. We call the Newton polygon of (any geometric fiber of) $X_U \to U$ the generic Newton polygon of $X \to S$, or simply of S if the family of abelian varieties is clear from context.

Definition 2.4. For each $0 \leq f \leq g$, define a Newton polygon ν_g^f as follows:

$$\nu_g^g = \{0^{\oplus g}, 1^{\oplus g}\}$$

$$\nu_g^0 = \begin{cases} \{\frac{1}{2}^{\oplus g}\} & g \leq 2 \\ \{\frac{1}{g}, \frac{g-1}{g}\} & g \geq 3 \end{cases}$$

and if $0 < f < g$, set

$$\nu_g^f = \nu_f^f \oplus \nu_{g-f}^0.$$

This is the largest ("most generic") admissible symmetric Newton polygon of height $2g$ and p-rank f. Similarly, let

$$\sigma_g = \{\frac{1}{2}^{\oplus g}\}$$

be the supersingular Newton polygon of height $2g$.

$$\nu_3^0 \oplus \sigma_1 \xleftarrow{\quad} \sigma_4$$

Figure layout:

$$
\begin{array}{ccc}
 & \nu_1^1 \oplus \sigma_3 & \\
 \nearrow & & \searrow \\
\sigma_4 \longrightarrow \nu_3^0 \oplus \sigma_1 & & \nu_1^1 \oplus \nu_3^0 \longrightarrow \nu_4^2 \longrightarrow \nu_4^3 \longrightarrow \nu_4^4 \\
 \searrow & & \nearrow \\
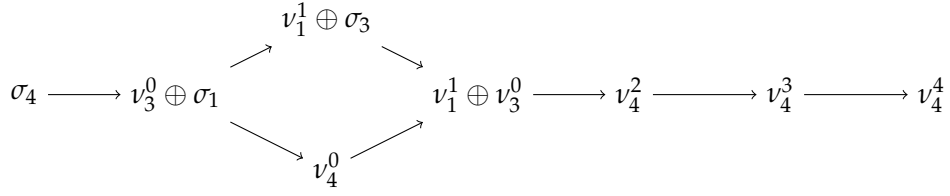 & \nu_4^0 &
\end{array}
$$

FIGURE 2.2. Symmetric admissible Newton polygons of height 8. There is a path from ν to ν' if and only if $\nu \prec \nu'$.

With this notation, we depict the poset of Newton polygons for $g = 4$ in Figure 2.2.

For a symmetric admissible Newton polygon ν of height $2g$, let codim(ν) be the length of any path from ν_g^g to ν in the poset of all such Newton polygons.

3. STRATIFICATIONS ON THE MODULI SPACE OF ABELIAN VARIETIES

The p-rank and Newton stratifications of $\mathcal{A}_g$ are well understood. We review some of their features here, for two different reasons.

First, for $1 \leq g \leq 3$, $\mathcal{M}_g \hookrightarrow \mathcal{A}_g$ is an open immersion, and $\mathcal{M}_g^* \to \mathcal{A}_g$ is an isomorphism. Consequently, in small genus, the stratifications on $\mathcal{A}_g$ are perfectly mirrored in $\mathcal{M}_g$. This information is used directly below.

Second, as we shall see, $\mathcal{A}_g$ is in some sense more highly structured than $\mathcal{M}_g$. Thus, the results described here for $\mathcal{A}_g$ might elicit some of the most optimistic conjectures one might make about such stratifications on $\mathcal{M}_g$. (Indeed, some "obvious" conjectural statements are too strong to be true; see Section 7.1.)

3.1. **The p-ranks of abelian varieties.**

Theorem 3.1. [21] *Let $g \geq 1$ and $0 \leq f \leq g$. Then $\mathcal{A}_g^f$ is nonempty and pure of codimension $g - f$ in $\mathcal{A}_g$.*

The nonemptiness is easy to see; it suffices to take a product of f ordinary elliptic curves and $g - f$ supersingular elliptic curves, equipped with the product principal polarization.

Corollary 3.2. *Let $g \geq 1$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{A}_g^f$.*

(a) *If $f < g$, then S is in the closure of $\mathcal{A}_g^{f+1}$.*

(b) *If $f > 0$, then the closure of S contains an irreducible component of $\mathcal{A}_g^{f-1}$.*

Proof. Part (a) follows from the dimension count of Theorem 3.1 and purity for p-ranks (Lemma 2.2).

We do not know an elementary proof of part (b), although it is an immediate consequence of Theorem 3.3. $\qquad\square$

Note that, unless $(g, f) = (1, 0)$ or $(2, 0)$, Theorem 3.3 implies that $\mathcal{A}_g^f$ is actually irreducible.

3.2. **Newton polygons of abelian varieties.** Thanks especially to work of Oort and of Chai and Oort, the Newton stratification on $\mathcal{A}_g$ is well-understood.

Theorem 3.3. [7] *Let ν be a symmetric admissible Newton polygon of height $2g$.*

(a) *The stratum $\mathcal{A}_g^\nu$ is nonempty and has codimension $\mathrm{codim}(\nu)$ in $\mathcal{A}_g$.*

(b) *If $\nu \neq \sigma_g$, then $\mathcal{A}_g^\nu$ is irreducible.*

(c) *The supersingular locus $\mathcal{A}_g^{\sigma_g}$ is connected but reducible.*

In analyzing Newton strata $\mathcal{A}_g^\nu$, two key tools are the Serre-Tate theorem and the action of Hecke operators. In concert with Dieudonné theory, the first tool allows one to use (semi-)linear algebra to study the deformation space of an abelian variety. The second tool, and in particular the fact that Newton strata are stable under this large group of partial symmetries of $\mathcal{A}_g$, allows one to deduce global information about Newton strata.

In general, both of these structures are missing from $\mathcal{M}_g$, and the state of our knowledge is, correspondingly, much cruder. For a given symmetric admissible Newton polygon ν of height $2g$, it is typically not even known if $\mathcal{M}_g^\nu$ is nonempty, let alone pure or even irreducible. *Degeneration* (as in Theorem 4.2) and *low genus phenomena* (e.g., the fact that every principally polarized abelian threefold is the Jacobian of a stable curve) are among the few tools we have at our disposal. In the second half of this paper, we show how these techniques can be combined to yield information about stratifications on $\mathcal{M}_g$.

4. THE p-RANK STRATIFICATION OF THE MODULI SPACE OF STABLE CURVES

Let $\mathcal{M}_g/\mathbb{F}_p$ be the moduli stack of smooth proper curves of genus g. Even if one is intrinsically only interested in smooth curves, one is quickly led, following Deligne and Mumford, to study $\overline{\mathcal{M}}_g$, the moduli stack of stable proper curves of genus g [11].

4.1. **The moduli space of stable curves.** It turns out that $\mathcal{M}_g$ is open in $\overline{\mathcal{M}}_g$, which is proper. The boundary $\partial\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_g \smallsetminus \mathcal{M}_g$ is a union $\partial\overline{\mathcal{M}}_g = \cup_{0 \le i \le \lfloor g/2 \rfloor}\Delta_i[\overline{\mathcal{M}}_g]$, whose construction we briefly recall.

For a natural number r, let $\overline{\mathcal{M}}_{g;r}$ be the moduli stack of r-labeled stable curves of genus g. There are finite clutching morphisms

$$\overline{\mathcal{M}}_{g_1;1} \times \overline{\mathcal{M}}_{g_2;1} \xrightarrow{\ \kappa_{g_1,g_2}\ } \overline{\mathcal{M}}_{g_1+g_2}$$

$$\overline{\mathcal{M}}_{g;2} \xrightarrow{\ \kappa_g\ } \overline{\mathcal{M}}_{g+1}$$

in which the labeled points are identified; see [17, Section 3] for more details. Here, we simply record the following facts.

Suppose that, for $i \in \{1, 2\}$, (C, P_i) is a smooth, pointed curve with moduli point $s_i \in \mathcal{M}_{g_i;1}(k)$. Then $\kappa_{g_1,g_2}(s_1, s_2)$ is the moduli point of the curve C of genus $g_1 + g_2$ obtained by identifying P_1 and P_2. One has

$$(4.1) \qquad\qquad\qquad \mathrm{Pic}^0(C) \cong \mathrm{Pic}^0(C_1) \oplus \mathrm{Pic}^0(C_2).$$

Now suppose that $(C, P, Q) \in \mathcal{M}_{g;2}(k)$ is a smooth, 2-pointed curve. Then $\kappa_g(s)$ is the moduli point of the curve $\widetilde{C}$ obtained from C by identfying P and Q, and there is an exact sequence

$$(4.2) \qquad\qquad 1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Pic}^0(\widetilde{C}) \longrightarrow \mathrm{Pic}^0(C) \longrightarrow 1.$$

Let $\Delta_0[\overline{\mathcal{M}}_g] = \kappa_{g-1}(\overline{\mathcal{M}}_{g-1;2})$, and for $1 \le i \le g-1$ let $\Delta_i[\overline{\mathcal{M}}_g] = \kappa_{i,g-i}(\overline{\mathcal{M}}_{i;1} \times \overline{\mathcal{M}}_{g-i;1})$. If S is a stack equipped with a morphism $S \to \overline{\mathcal{M}}_g$, we let $\Delta_i[S] = S \times_{\overline{\mathcal{M}}_g} \Delta_i[\overline{\mathcal{M}}_g]$. Let $\mathcal{M}_g^* = \overline{\mathcal{M}}_g \smallsetminus \Delta_0$; this is precisely the locus of curves whose Picard varieties are actually abelian varieties, and not merely semiabelian varieties. The Torelli map $\tau : \mathcal{M}_g^* \to \mathcal{A}_g$ is a birational morphism onto its image, and contracts fibers on the boundary. More precisely, (the Torelli theorem states that) τ is injective on $\mathcal{M}_g$. On the boundary, τ forgets the identification point; if P_1 and Q_1 are points on a curve C_1, then $\tau((C_1, P_1), (C_2, P_2)) = \tau((C_1, Q_1), (C_2, P_2))$.

4.2. **The p-rank stratification of $\overline{\mathcal{M}}_g$.** The notion of p-rank makes sense for stable, and not just smooth, curves.

On Δ_0 we find, in the notation of (4.2), that

$$(4.3) \qquad\qquad f_{\tilde{C}} = f_C + 1;$$

while on Δ_i with $i > 0$ we find, in the notation of (4.1), that

$$(4.4) \qquad\qquad f_C = f_{C_1} + f_{C_2}.$$

Thus, the p-rank stratification extends to $\overline{\mathcal{M}}_g$. We emphasize that $\overline{\mathcal{M}}_g^f$ parametrizes those stable curves whose p-rank is f, while $\overline{\mathcal{M}}_g^f$ is the closure of the set of smooth curves of genus g and p-rank f. In particular, if $f > 0$, one has $\overline{\mathcal{M}}_g^f \subsetneq \overline{\mathcal{M}}_g^f$.

From (4.3) and (4.4) it immediately follows that

$$\kappa_{g_1,g_2}(\overline{\mathcal{M}}_{g_1}^{f_1} \times \overline{\mathcal{M}}_{g_2}^{f_2}) \subseteq \overline{\mathcal{M}}_{g_1+g_2}^{f_1+f_2}$$
$$\kappa_{g-1}(\overline{\mathcal{M}}_{g-1}^f) \subseteq \overline{\mathcal{M}}_g^{f+1}.$$

Faber and van der Geer exploit this structure to show:

Theorem 4.1. [12] *Suppose $g \geq 1$ and $0 \leq f \leq g$. Then $\mathcal{M}_g^f$ is nonempty and pure of codimension $g - f$ in $\mathcal{M}_g$.*

In fact, much more is true. First, using relations (4.3)-(4.4) and the fundamental dimension count supplied by Theorem 4.1, it is not hard to see that $\mathcal{M}_g^f$ is dense in $\overline{\mathcal{M}}_g^f$; every stable curve of genus g and p-rank f is a limit of smooth curves with the same discrete parameters. Moreover, the recursive structure of the boundary is compatible with the p-rank stratification:

Theorem 4.2. [2, Lemma 3.2 and Prop. 3.4] *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{M}_g^f$.*

(a) *If $f > 0$, then $\overline{S}$ contains the image of an irreducible component of $\overline{\mathcal{M}}_{g-1;2}^{f-1}$ under κ_{g-1}.*

(b) *Suppose $1 \leq i \leq g - 1$. Let f_1 and f_2 be nonnegative integers such that $0 \leq f_1 \leq i$; $0 \leq f_2 \leq g - i$; and $f_1 + f_2 = f$. Then $\overline{S}$ contains the image of an irreducible component of $\overline{\mathcal{M}}_i^{f_1} \times \overline{\mathcal{M}}_{g-i}^{f_2}$ under $\kappa_{i,g-i}$.*

Consequently, closures of components of p-rank strata contain chains of elliptic curves:

Corollary 4.3. [2, Corollary 3.6] *Suppose $g \geq 2$, $0 \leq f \leq g$, and $A \subset \{1, \cdots, g\}$ has cardinality f. Let S be an irreducible component of $\mathcal{M}_g^f$. Then $\overline{S}$ contains the moduli point of a chain of elliptic curves $E_1, \cdots, E_g$, where E_j is ordinary if and only if $j \in A$.*

4.3. **Connectedness of p-rank strata.** We combine Theorem 4.1 with degeneration techniques to prove:

Corollary 4.4. *Suppose $g \geq 1$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{M}_g^f$.*

(a) *If $f < g$, then S is in the closure of $\mathcal{M}_g^{f+1}$.*

(b) *If $f > 0$, then $\overline{S}$ contains an irreducible component of $\mathcal{M}_g^{f-1}$.*

Proof. Part (a) is a direct consequence of purity for p-rank (Lemma 2.2) and the dimension count Theorem 4.1; the proof proceeds by induction on $g - f$.

For part (b), the statement is clearly true for $g = 1$, or more generally when $f = g$. Now suppose $g \geq 2$. By Theorem 4.2(b), $\overline{S}$ contains an irreducible component of $\overline{\mathcal{M}}_1^1 \times \overline{\mathcal{M}}_{g-1}^{f-1}$ under the

image of $\kappa_{1,g-1}$. Since $\mathcal{M}_1^1$ is irreducible, its closure contains points of p-rank zero, and $\Delta_1[\overline{S}]^{f-1}$ is nonempty. Therefore, $\overline{S}^{<f}$ is nonempty and, by Lemma 2.2, has codimension one in $\overline{S}$. For each i between 0 and g, $\Delta_i[\overline{S}]^{<f}$ has codimension two in $\overline{S}$. Therefore, $\overline{S}^{<f}$ is the closure of $S^{<f}$, and the latter has codimension one in S. The basic dimension count (Theorem 4.1) now shows that $\overline{S}$ contains an irreducible component of $\mathcal{M}_g^{f-1}$. $\square$

In contrast with the theory of p-rank strata of $\mathcal{A}_g$, at present we have essentially no nontrivial information about irreducibility of various $\mathcal{M}_g^f$. Still, the method of degeneration detects a small amount of the topology of these strata. Recall that $\overline{\mathcal{M}}_g^f$ denotes the p-rank f stratum of the moduli space of stable curves of genus g.

Corollary 4.5. *If $g \geq 2$ and $0 \leq f \leq g$, then $\overline{\mathcal{M}}_g^f$ is connected.*

Proof. Since p-rank strata in $\mathcal{M}_3^*$ coincide with those of $\mathcal{A}_3$, and since p-rank strata in $\mathcal{A}_g$ are irreducible (Theorem 3.3), each $\mathcal{M}_3^f$ is irreducible. Similarly, $\mathcal{M}_2^1$ and $\mathcal{M}_2^2$ are irreducible, while $\overline{\mathcal{M}}_2^0$ is connected.

We next prove by induction on g that, for each $g \geq 4$ and each $0 \leq f \leq g$, $\overline{\mathcal{M}}_g^f$ is connected. Fix integers f_1 and f_2 with $0 \leq f_1 \leq 2$, $0 \leq f_2 \leq g-2$, and $f_1 + f_2 = f$. Let S_1 and S_2 be irreducible components of $\mathcal{M}_g^f$. By Theorem 4.2(b), each closure $\overline{S}_i$ contains an irreducible component of $\kappa_{2,g-2}(\overline{\mathcal{M}}_2^{f_1} \times \overline{\mathcal{M}}_{g-2}^{f-2})$. By the inductive hypothesis, $\overline{\mathcal{M}}_2^{f_1} \times \overline{\mathcal{M}}_{g-2}^{f-2}$ is connected. Consequently, $\overline{S_1 \cup S_2}$ is connected. The theorem now follows by considering each of the (finitely many) irreducible components of $\mathcal{M}_g^f$. $\square$

4.4. **Open questions about the p-rank stratification.** The results above indicate that the p-rank stratification is in some ways "transverse" to other interesting loci in $\mathcal{M}_g$. For example, as long as $(g, f) \neq (1, 0)$ or $(2, 0)$, then there exists a smooth projective curve over $\overline{\mathbb{F}}_p$ of genus g and p-rank f whose automorphism group is trivial [1]. More generally, there is a precise sense in which a randomly chosen Jacobian of genus g and p-rank f behaves like a randomly selected principally polarized abelian variety of dimension g [2].

On the other hand, there is a non-trivial interplay between the p-rank and the automorphism group of a curve. For example, there are constraints on the p-rank of a cyclic tame cover of $\mathbb{P}^1$ [5]. There are even stronger constraints in the case of a wildly ramified Galois action. The Deuring-Shafarevich formula places severe limitations on the p-rank for a p-group cover of curves [9, 27]. The p-rank stratification of the moduli space of Artin-Schreier curves is discovered in [25]. See also [13, 14].

Here are several other open questions about the p-ranks of Jacobians of curves.

Question 4.6. *Suppose $g \geq 4$ and $0 \leq f \leq g-1$. Is $\mathcal{M}_g^f$ irreducible?*

Question 4.7. *Suppose $g \geq 3$ and $0 \leq f \leq g-1$. Does there exist a curve of genus g and p-rank f defined over $\mathbb{F}_p$?*

5. STRATIFICATION BY NEWTON POLYGON

In this section, we explain how Theorem 4.2 yields information about generic Newton polygons, beginning with the cases $g \leq 4$, and extending to arbitrary g. Recall (Definition 2.4) the Newton polygons ν_g^f.

5.1. **Newton polygons of curves of small genus.**

Lemma 5.1. *The p-rank zero locus $\mathcal{M}_3^0$ is irreducible, with generic Newton polygon $\nu_3^0 = \{1/3, 2/3\}$.*

Proof. The only symmetric Newton polygons of height 6 and p-rank zero are ν_3^0 and $\sigma_3 = \{\frac{1}{2}^{\oplus 3}\}$; but the supersingular locus in $\mathcal{A}_3^0$ is pure of dimension 2 (3.3), while $\mathcal{M}_3^0$ is pure of dimension 3 (4.1). The result now follows from the Torelli theorem and Theorem 3.3. $\square$

Lemma 5.2. *Let S be an irreducible component of $\mathcal{M}_4^0$. Then the generic Newton polygon of S lies on or below $\sigma_1 \oplus \nu_3^0 = \{1/3, 1/2, 2/3\}$.*

Proof. By Theorem 4.2, $\overline{S}$ contains a component of $\kappa_{1,3}(\overline{\mathcal{M}}_{1;1}^0 \times \overline{\mathcal{M}}_{3;1}^0)$, which has generic Newton polygon $\{1/2\} \oplus \{1/3, 2/3\}$ (Lemma 5.1). The result for S follows from the semicontinuity of Newton polygons. $\square$

Lemma 5.3. *In $\mathcal{M}_4^0$,*

    (a) *there is at least one irreducible component with generic Newton polygon ν_4^0; and*
    (b) *there is at most one irreducible component with generic Newton polygon $\nu_3^0 \oplus \sigma_1$.*

Proof. Let S be an irreducible component of $\mathcal{M}_4^0$; then S has dimension 5 (Theorem 4.1). Suppose the generic Newton polygon of S is $\nu_3^0 \oplus \sigma_1$. The locus in $\mathcal{A}_4$ of abelian fourfolds with Newton polygon $\nu_3^0 \oplus \sigma_1$ is irreducible of dimension 5 (Theorem 3.3). Thus S, or rather its image $\tau(S)$ under the Torelli map, must coincide with this locus. In particular, such an S, if it exists, is unique; this proves (b).

For part (a), it suffices to show that there exists some curve whose Jacobian has Newton polygon ν_4^0. Consider the moduli space $\mathcal{Z}$ of principally polarized abelian fourfolds equipped with an action by $\mathbb{Z}[\zeta_3]$ of signature $(3, 1)$. Then $\mathcal{Z}$ is contained in the (compactified) Torelli locus.

This has been known for some time, but we provide a sketch here. For a squarefree polynomial $f(x)$ of degree 6, let C_f denote the curve with affine equation $y^3 = f(x)$. Then $\mathrm{Jac}(C_f)$ has an action by $\mathbb{Z}[\zeta_3]$ of signature $(3, 1)$. On one hand, the parameter space for such curves has dimension $6 - \dim \mathrm{PGL}_2 = 3$. On the other hand, $\mathcal{Z}$ itself is irreducible (since $\mathbb{Q}(\zeta_3)$ has class number one) of dimension $3 \cdot 1 = 3$. Consequently, $\tau(\mathcal{M}_4)$ contains an open, dense subspace of U, and $\tau(\mathcal{M}_4^*)$, which is equal to the closure of $\mathcal{M}_4$ in $\mathcal{A}_4$, contains all of $\mathcal{Z}$.

It now suffices to show that there exists a point on $\mathcal{Z}$ parametrizing an abelian variety with Newton polygon ν_4^0. If p splits in $\mathbb{Q}(\zeta_3)$, this follows from [19, Section 2.2]. (In the notation of [19], $(q, h - q) = (1, 3)$, and α is the Newton polygon of slope $1/4$.) If p is inert in $\mathbb{Q}(\zeta_3)$, this follows from [6, Section 5.4]. (In the notation of [6], $(n - s, s) = (3, 1)$, $\rho = 4$.) For $p = 3$, this follows from Example 2.1. $\square$

5.2. **Generic Newton polygons.**

Proposition 5.4. *Let $g_0, f \in \mathbb{N}$ and let $g = g_0 + f$.*

    (a) *If the generic Newton polygon of every component of $\mathcal{M}_{g_0}^0$ is $\nu_{g_0}^0$, then the generic Newton polygon of every component of $\mathcal{M}_g^f$ is ν_g^f.*
    (b) *If the generic Newton polygon of at least one component of $\mathcal{M}_{g_0}^0$ is $\nu_{g_0}^0$ then the generic Newton polygon of at least one component of $\mathcal{M}_g^f$ is ν_g^f.*

Proof. For (a), let S be a component of $\mathcal{M}_g^f$ and consider its closure in $\overline{\mathcal{M}}_g^f$. By Theorem 4.2(b), $\overline{S}$ contains the image of an irreducible component of $\mathcal{M}_{f;1}^f \times \mathcal{M}_{g_0;1}^0$. Now $\mathcal{M}_{f;1}^f$ is irreducible with generic Newton polygon ν_g^g. By hypothesis, every component of $\mathcal{M}_{g_0;1}^0$ has generic Newton polygon $\nu_{g_0}^0$. By semicontinuity, the generic Newton polygon ν of S lies on or below ν_g^f. On the

other hand, the p-rank f condition implies that ν has f slopes of 0 and 1. This is only possible if $\nu = \nu_g^f$.

The proof of (b) is by induction on f. The base case is the hypothesis that there exists a component S_{g_0} of $\mathcal{M}_{g_0}^0$ with generic Newton polygon ν_{g_0}. Now suppose, as inductive hypothesis, that S_{g-1} is a component of $\mathcal{M}_{g-1}^{f-1}$ which has generic Newton polygon ν_{g-1}^{f-1}. We add a labeling of two points to each curve represented by a point of S_{g-1} by letting $T_{g-1} = S_{g-1} \times_{\mathcal{M}_{g-1}} \mathcal{M}_{g-1,2}^{f-1}$. Then consider the image under the clutching morphism $Z_g = \kappa_{g-1}(T_{g-1})$ which is contained in $\Delta_0[\overline{\mathcal{M}}_g^f]$. By [2, Lemma 3.2], there exists an irreducible component S_g of $\mathcal{M}_g^f$ such that $\overline{S}_g$ contains Z_g. By semicontinuity, the generic Newton polygon ν of S_g lies on or below ν_g^f. Then $\nu = \nu_g^f$ by the p-rank f condition. $\square$

If $g \leq 2$, then the p-rank of an abelian variety determines its Newton polygon. More generally, if $f \in \{g-2, g-1, g\}$, then there is a unique symmetric admissible Newton polygon of height $2g$ and p-rank f. In contrast, if $f < g-2$ then the p-rank constrains, but does not determine, the Newton polygon.

Corollary 5.5. *Let $g \geq 3$. If S is a component of $\mathcal{M}_g^{g-3}$, then S has generic Newton polygon $\nu_g^{g-3} = \{0^{\oplus g-3}, 1/3, 2/3, 1^{\oplus g-3}\}$.*

Proof. By Lemma 5.1, the generic Newton polygon of every component of $\mathcal{M}_3^0$ is ν_3^0. The result follows from Proposition 5.4(a). $\square$

We obtain partial information in the next case when $f = g - 4$.

Corollary 5.6. *Let $g \geq 4$. There exists a component of $\mathcal{M}_g^{g-4}$ with generic Newton polygon ν_g^{g-4}. In particular, there is a smooth projective curve whose Jacobian has Newton polygon ν_g^{g-4}.*

Proof. By Lemma 5.3(a), there exists a component S_4 of $\mathcal{M}_4^0$ with generic Newton polygon ν_4^0. The result follows from Proposition 5.4(b). $\square$

A better understanding of $\mathcal{M}_5^0$ would allow one to prove results for arbitrary g when $f = g - 5$.

6. HYPERELLIPTIC CURVES

Recall that a hyperelliptic curve is a smooth projective curve C which can be realized as a double cover $C \to \mathbb{P}^1$ of the projective line. Among all curves, hyperelliptic curves have enjoyed special attention. On the practical side, algorithmic methods for handling hyperelliptic curves over finite fields are much more highly developed than they are for arbitrary curves. On the theoretical side, hyperelliptic curves over finite fields are a natural function-field analogue of quadratic number fields, and thus an attractive site for investigation of conjectures. In this section, we briefly sketch the extent to which the ideas and results surveyed here for $\mathcal{M}_g$ extend to the moduli space of hyperelliptic curves. Throughout, we assume that the characteristic p is *odd*.

The boundary of $\mathcal{H}_g$. Let $\mathcal{H}_g$ be the moduli space of hyperelliptic curves. Any given curve admits at most one hyperelliptic involution ι, and thus there is an inclusion $\mathcal{H}_g \hookrightarrow \mathcal{M}_g$. Let $\overline{\mathcal{H}}_g$ be the closure of $\mathcal{H}_g$ in $\mathcal{M}_g$. The boundary ∂H_g necessarily admits a decomposition $\partial H_g = \cup \Delta_i[\overline{\mathcal{H}}_g]$, but constructing the full boundary is somewhat delicate. Briefly, if two hyperelliptic curves (C_1, P_1) and (C_2, P_2), with hyperelliptic involutions ι_1 and ι_2, are clutched, then the resulting curve is hyperelliptic if and only if each P_i is fixed by ι_i. Consequently, for $0 < i < g$, $\Delta_i[\overline{\mathcal{H}}_g]$ is described as the image of

$$\widetilde{\mathcal{H}}_i \times \widetilde{\mathcal{H}}_{g-i} \longrightarrow \widetilde{\mathcal{H}}_g \longrightarrow \overline{\mathcal{H}}_g$$

where $\widetilde{\mathcal{H}}_g$ is the moduli space of hyperelliptic curves equipped with a labeling of the $2g+2$ points of the smooth ramification locus, and $\widetilde{\mathcal{H}}_g \to \overline{\mathcal{H}}_g$ is the (finite-to-one) forgetful map.

The description of irreducible components of $\Delta_0[\overline{\mathcal{H}}_g]$ is somewhat more intricate, since the curve obtained by identifying points P and Q of the hyperelliptic curve C is hyperelliptic if and only if $\iota(P) = Q$. In fact, there is a decomposition of $\Delta_0[\overline{\mathcal{H}}_g]$ as a union

$$\Delta_0[\overline{\mathcal{H}}_g] = \kappa_{g-1}(\overline{\mathcal{H}}_{g-1};1) \bigcup_{1 \leq i \leq \lfloor g \rfloor} \lambda_{i,g-1-i}(\overline{\mathcal{H}}_{i;1} \times \overline{\mathcal{H}}_{g-1-i;1})$$

of irreducible divisors. We refer to [3, 30] for more details.

The fact that, for $i > 0$, $\Delta_i[\overline{\mathcal{H}}_g]$ is parametrized by hyperelliptic curves with a *discrete* choice of point makes it much more difficult to unravel the p-rank stratification $\partial\mathcal{H}_g$.

As in Theorem 4.1, it turns out that if $0 \leq f \leq g$, then $\mathcal{H}_g^f$ is nonempty and pure of codimension $g - f$ [15].

This is used in proving the currently optimal hyperelliptic analogue to Theorem 4.2:

Theorem 6.1. [3, Lemma 3.4] *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$.*

(a) *If $f > 0$, then each irreducible component of $\Delta_0[\overline{S}]$ contains either an irreducible component of $\kappa_{g-1}(\overline{\mathcal{H}}_{g-1;1}^{f-1})$ or of some $\lambda_{i,g-1-i}(\overline{\mathcal{H}}_{i;1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i;1}^{f_2})$ with $0 \leq f_1 \leq i$, $0 \leq f_2 \leq g-1-i$, $f_1 + f_2 = f$.*

(b) *If $f = 0$, then $\overline{S}$ contains the image of an irreducible component of $\widetilde{\mathcal{H}}_i^0 \times \widetilde{\mathcal{H}}_{g-i}^0$ for some $1 \leq i \leq g - 1$.*

In contrast to the case of general curves (Corollary 4.3), at present one only knows that hyperelliptic curves degenerate to *trees* of elliptic curves:

Corollary 6.2. [3, Theorem 3.11(c)] *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$. Then $\overline{S}$ contains the moduli point of some tree of elliptic curves, of which f are ordinary and $g - f$ are supersingular.*

Stratification by p-rank. To some extent, the p-rank stratification of $\mathcal{H}_g$ is known to be much like that of $\mathcal{M}_g$.

Corollary 4.4 holds, *mutatis mutandis*, for $\mathcal{H}_g$. (Part (a) relies only on dimension counting; part (b), whose proof relies on degeneration, already appears as [3, Corollary 3.15].)

However, even for $g = 3$, we start losing information. It is not known if $\mathcal{H}_3^0$ is irreducible. While $\mathcal{H}_2^0$ is connected, it is not clear if $\widetilde{\mathcal{H}}_2^0$ is connected, making it more difficult to prove a result analogous to Corollary 4.5 for hyperelliptic curves.

Stratification by Newton polygon. Less is known about the Newton stratification of $\mathcal{H}_g$ than that of $\mathcal{M}_g$. However, it *is* known that, for each irreducible component of $\mathcal{H}_3^0$, the generic Newton polygon is ν_3^0 [23].

The analogue of Proposition 5.4 is valid for $\mathcal{H}_g$, too. In part (a), even though one has less control over degenerations, one still knows that, for an irreducible component S of $\mathcal{H}_g^f$, $\overline{S}$ contains an irreducible component of $\kappa_{1,g-1}(\widetilde{\mathcal{H}}_1 \times \widetilde{\mathcal{H}}_{g-1}^{f-1})$. Part (b) is valid for $\mathcal{H}_g$, as well. The key observation is that, if S is an irreducible component of $\mathcal{H}_{g-1}^{f-1}$, then $\kappa_{g-1}(S \times_{\mathcal{H}_{g-1}} \mathcal{H}_{g-1;1})$ is in the boundary of *some* irreducible component of $\mathcal{H}_g^f$.

We conclude that the generic Newton polygon of each component of $\mathcal{H}_g^{g-3}$ is ν_g^{g-3}.

7. SOME CONJECTURES ABOUT NEWTON POLYGONS OF CURVES

In this section, we discuss variations of the following question.

Question 7.1. *Given g and p, does every symmetric admissible Newton polygon of height $2g$ occur for the Jacobian of a smooth projective curve of genus g?*

The first open case of this question is when $g = 4$, for the Newton polygons σ_4, $\nu_3^0 \oplus \sigma_1$, and $\nu_1^1 \oplus \sigma_3$. Using Theorem 4.2, it is not hard to see that each of these Newton polygons is realized by a singular curve of compact type. It is unlikely that appeal to Shimura varieties, as in the proof of Lemma 5.3, will resolve this question for all p. For example, if p is inert in $\mathbb{Q}(\zeta_3)$, then no abelian variety with p-rank one has an action by $\mathbb{Z}[\zeta_3]$. In particular, $\nu_1^1 \oplus \sigma_3$ is not the Newton polygon of any abelian variety with moduli point in $\mathcal{Z}$.

7.1. **Non-existence philosophy.** In [24], Oort explains why the answer to Question 7.1 could be *no*. Consider the partial ordering of Newton polygons. Suppose that:

: (i) $\operatorname{codim}(\xi) \geq 3g - 3$, i.e., the length of the longest chain of Newton polygons connecting ξ and ν_g^g is larger than $3g - 3$; and
: (ii) the denominators of ξ are "large".

Then [24, Expectation 8.5.4] states that one expects that there is no curve of compact type whose Jacobian has Newton polygon ξ. By this reasoning, one expects that there is no curve of genus 11 whose Jacobian has Newton polygon $\xi = G_{5,6} \oplus G_{6,5}$, with slopes $5/11$ and $6/11$.

The dimension of $\mathcal{A}_g$ is $g(g+1)/2$ and the dimension of its supersingular locus is $\lfloor g^2/4 \rfloor$. The length of the longest chain of Newton polygons connecting the supersingular Newton polygon σ_g and the ordinary Newton polygon ν_g^g is the difference between these, which is greater than $3g - 3$ when $g \geq 9$. It is still possible that every Newton polygon in the chain occurs for a Jacobian, but if so, then there are two Newton polygons ξ_1 and ξ_2, such that $\mathcal{A}_g^{\xi_1}$ is in the closure of $\mathcal{A}_g^{\xi_2}$ but $\mathcal{M}_g^{\xi_1}$ is not in the closure of $\mathcal{M}_g^{\xi_2}$. In other words, under Condition (i), $\mathcal{M}_g$ does not admit a perfect stratification by Newton polygon.

The Newton polygon of a curve of compact type is the join of the Newton polygons of its components. If the denominators of ξ are all less than g, then one can try to construct a singular curve with Newton polygon ξ from curves of smaller genus. For example, it is easy to see that σ_g is the Newton polygon of a tree of supersingular elliptic curves. If ξ is indecomposable as a symmetric Newton polygon, then it cannot occur for the Jacobian of a singular curve of compact type. As a means of making Condition (ii) more precise, one could restrict to the case that ξ is indecomposable.

Another variation is to restrict to the Jacobians of smooth curves. In fact, Oort conjectures that if ξ_i is the Newton polygon of a point of $\mathcal{M}_{g_i}$ for $i = 1, 2$, then the join of ξ_1 and ξ_2 occurs as the Newton polygon of a point of $\mathcal{M}_{g_1 + g_2}$, in other words, as the Newton polygon of the Jacobian of a *smooth* curve [24, Conjecture 8.5.7].

Remark 7.2. The original motivation for this non-existence expectation was the following. Let $Y^{(cu)}(K, g)$ denote the statement: There exists an abelian variety A of dimension g defined over K which is not isogenous to the Jacobian of any curve of compact type; (here, this is an isogeny of abelian varieties without polarization). There is an expectation[1] that $Y^{(cu)}(\overline{\mathbb{Q}}, g)$ is true for every $g \geq 4$. To prove $Y^{(cu)}(\overline{\mathbb{Q}}, g)$, it suffices to find a prime p and a Newton polygon ξ of height $2g$ such that ξ does not occur as the Newton polygon of a Jacobian of a curve [24, (8.5.1)]. It turns out that $Y^{(cu)}(\overline{\mathbb{Q}}, g)$ was proven by other methods [8, 28].

7.2. **Supersingular curves.** Recall that $\sigma_g = \{\frac{1}{2}^{\oplus g}\}$. The supersingular locus $\mathcal{M}_g^{\sigma_g}$ has been studied extensively. When $p = 2$, Van der Geer and Van der Vlugt proved that there is a smooth curve of every genus which is supersingular. More generally, the same methods show there exist supersingular curves of arbitrarily large genus defined over $\overline{\mathbb{F}}_p$ for all primes p.

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[1]attributed to Katz and to Oort by, respectively, Oort and Katz

Example 7.3. Let $R(x) \in \overline{\mathbb{F}}_p[x]$ be an additive polynomial, i.e., a polynomial of the form $R[x] = a_0 + a_1 x^p + \cdots + a_d x^{p^d}$. Consider the Artin-Schreier curve Y with affine equation $y^p - y = xR(x)$; it has genus $(p-1)(p^d+1)/2$. Then Y is supersingular by [29, Theorem 13.7].

7.3. **Other non-existence results.**

To this date, the only non-existence results about Newton polygons are for Jacobians of curves with automorphisms. Newton polygons of degree p covers of $\mathbb{P}^1$ have been studied using techniques for exponential sums and Dwork cohomology. For example [26], when $p = 2$, there are no hyperelliptic curves of genus $2^n - 1$ which are supersingular. More generally, there are conditions on the first slope of the Newton polygon of a degree p cover of the projective line [4].

REFERENCES

[1] Jeffrey D. Achter, Darren Glass, and Rachel Pries, *Curves of given p-rank with trivial automorphism group*, Michigan Math. J. **56** (2008), no. 3, 583–592.

[2] Jeffrey D. Achter and Rachel Pries, *Monodromy of the p-rank strata of the moduli space of curves*, Int. Math. Res. Not. IMRN (2008), no. 15, Art. ID rnn053, 25.

[3] _____, *The p-rank strata of the moduli space of hyperelliptic curves*, Adv. Math. **227** (2011), no. 5, 1846–1872, 10.1016/j.aim.2011.04.004.

[4] Régis Blache, *Valuation of exponential sums and the generic first slope for Artin-Schreier curves*, J. Number Theory **132** (2012), no. 10, 2336–2352, 10.1016/j.jnt.2012.04.017.

[5] Irene I. Bouw, *The p-rank of ramified covers of curves*, Compositio Math. **126** (2001), no. 3, 295–322, 10.1023/A:1017513122376.

[6] Oliver Bültel and Torsten Wedhorn, *Congruence relations for Shimura varieties associated to some unitary groups*, J. Inst. Math. Jussieu **5** (2006), no. 2, 229–261.

[7] Ching-Li Chai and Frans Oort, *Monodromy and irreducibility of leaves*, Ann. of Math. (2) **173** (2011), no. 3, 1359–1396, 10.4007/annals.2011.173.3.3.

[8] _____, *Abelian varieties isogenous to a Jacobian*, Ann. of Math. (2) **176** (2012), no. 1, 589–635, 10.4007/annals.2012.176.1.11.

[9] Richard M. Crew, *Etale p-covers in characteristic p*, Compositio Math. **52** (1984), no. 1, 31–45.

[10] A. J. de Jong and F. Oort, *Purity of the stratification by Newton polygons*, J. Amer. Math. Soc. **13** (2000), no. 1, 209–241.

[11] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 75–109.

[12] Carel Faber and Gerard van der Geer, *Complete subvarieties of moduli spaces and the Prym map*, J. Reine Angew. Math. **573** (2004), 117–137.

[13] Massimo Giulietti and Gábor Korchmáros, *Automorphism groups of algebraic curves with p-rank zero*, J. Lond. Math. Soc. (2) **81** (2010), no. 2, 277–296, 10.1112/jlms/jdp066.

[14] Darren Glass, *The 2-ranks of hyperelliptic curves with extra automorphisms*, Int. J. Number Theory **5** (2009), no. 5, 897–910, 10.1142/S1793042109002468.

[15] Darren Glass and Rachel Pries, *Hyperelliptic curves with prescribed p-torsion*, Manuscripta Math. **117** (2005), no. 3, 299–317.

[16] Nicholas M. Katz, *Slope filtration of F-crystals*, Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I, Soc. Math. France, Paris, 1979, pp. 113–163.

[17] Finn F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$*, Math. Scand. **52** (1983), no. 2, 161–199.

[18] Yuri I. Manin, *Theory of commutative formal groups over fields of finite characteristic*, Uspehi Mat. Nauk **18** (1963), no. 6 (114), 3–90.

[19] Elena Mantovan, *On certain unitary group Shimura varieties*, Astérisque (2004), no. 291, 201–331, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales.

[20] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.

[21] Peter Norman and Frans Oort, *Moduli of abelian varieties*, Ann. of Math. (2) **112** (1980), no. 3, 413–439.

[22] Frans Oort, *Subvarieties of moduli spaces*, Invent. Math. **24** (1974), 95–119.

[23] _____, *Hyperelliptic supersingular curves*, Arithmetic algebraic geometry (Texel, 1989), Progr. Math., vol. 89, Birkhäuser Boston, Boston, MA, 1991, pp. 247–284.

[24] _____, *in problems from the Workshop on Automorphisms of Curves*, Rend. Sem. Mat. Univ. Padova **113** (2005), 129–177.

[25] Rachel Pries and Hui June Zhu, *The p-rank stratification of Artin-Schreier curves*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 2, 707–726, 10.5802/aif.2692.

[26] Jasper Scholten and Hui June Zhu, *Hyperelliptic curves in characteristic 2*, Int. Math. Res. Not. (2002), no. 17, 905–917, 10.1155/S1073792802111160.

[27] Doré Subrao, *The p-rank of Artin-Schreier curves*, Manuscripta Math. **16** (1975), no. 2, 169–193.

[28] Jacob Tsimerman, *The existence of an abelian variety over $\overline{\mathbb{Q}}$ isogenous to no Jacobian*, Ann. of Math. (2) **176** (2012), no. 1, 637–650, 10.4007/annals.2012.176.1.12.

[29] Gerard van der Geer and Marcel van der Vlugt, *Reed-Muller codes and supersingular curves. I*, Compositio Math. **84** (1992), no. 3, 333–367.

[30] Kazuhiko Yamaki, *Cornalba-Harris equality for semistable hyperelliptic curves in positive characteristic*, Asian J. Math. **8** (2004), no. 3, 409–426.

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