Newton polygons for a variant of the Kloosterman family

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Abstract. We study the p-adic valuations of roots of L-functions associated with certain families of exponential sums of Laurent polynomials $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The families we consider are reflection and Kloosterman variants of diagonal polynomials. Using decomposition theorems of Wan, we determine the Newton and Hodge polygons of a non-degenerate Laurent polynomial in one of these families.

1. Introduction

Let q be a power of a prime p and $\mathbb{F}_q$ be the finite field of q elements. Let $\zeta_p \in \mathbb{C}$ be a fixed primitive pth root of unity. For $k \in \mathbb{N}$, consider the trace homomorphism $\mathrm{Tr}_k : \mathbb{F}_{q^k} \to \mathbb{F}_p$. Given a Laurent polynomial $f(x_1, \ldots, x_n) \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, its k-th exponential sum is

$$S_k^*(f) = \sum_{x_i \in \mathbb{F}_{q^k}^*} \zeta_p^{\mathrm{Tr}_k f(x_1, \ldots, x_n)} \in \mathbb{Q}(\zeta_p).$$

The L-function of the exponential sum of f is defined as

$$L^*(f, T) = \exp\left(\sum_{k=1}^{\infty} S_k^*(f) \frac{T^k}{k}\right).$$

A theorem of Dwork-Bombieri-Grothendieck states that

$$L^*(f, T) = \frac{\prod_{i=1}^{d_1}(1 - \alpha_i T)}{\prod_{j=1}^{d_2}(1 - \beta_j T)},$$

where α_i, β_j are non-zero algebraic integers for $1 \leq i \leq d_1$ and $1 \leq j \leq d_2$. Thus

$$S_k^*(f) = \beta_1^k + \cdots + \beta_{d_2}^k - \alpha_1^k - \cdots - \alpha_{d_1}^k.$$

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The values d_1 and d_2 depend on geometric and cohomological properties of the motive defined by f. A theorem of Deligne [**5**] implies that the complex absolute values satisfy $|\alpha_i| = q^{u_i/2}$ and $|\beta_j| = q^{v_j/2}$ for some weights $u_i, v_j \in \mathbb{Z} \cap [0, 2n]$. Also, for each prime $\ell \neq p$, the values α_i, β_j are ℓ-adic units.

There are many open questions about the p-adic valuation of the roots and poles of $L^*(f, T)$. Write $|\alpha_i|_p = q^{-r_i}, |\beta_j|_p = q^{-s_j}$, where the p-adic valuation is normalized such that $|q|_p = 1/q$. Deligne's integrality theorem implies that $r_i, s_j \in \mathbb{Q} \cap [0, n]$. If f is diagonal, then α_i, β_j are roots of products of Gauss sums and the slopes r_i, s_j can be determined using Stickelberger's theorem. In this paper, we use Wan's decomposition theory [**12**] to study two families of Laurent polynomials that are not diagonal. We briefly explain the results, referring to Section 2 for definitions and background material.

Given a Laurent polynomial f, one can define its Newton polytope Δ which is an n-dimensional integral convex polyhedron in $\mathbb{R}^n$ determined by the dominant terms of f. Using Δ, one can define a non-degeneracy condition on f. Also, one can assign a weight function to lattice points of $\mathbb{R}^n$. One can associate to Δ its Hodge numbers and Hodge polygon $\mathrm{HP}(\Delta)$, a lower convex polygon in $\mathbb{R}^2$ starting at the origin, by counting the number of lattice points of a given weight.

If f is non-degenerate and Δ is general enough, then $L^*(f, T)^{(-1)^{n-1}}$ is a polynomial of degree $n!V(\Delta)$ by results of Adolphson and Sperber [**1**]. In this case, information about the p-adic valuations of the roots of $L^*(f, T)^{(-1)^{n-1}}$ is encapsulated in the Newton polygon $\mathrm{NP}(f)$, another lower convex polygon in $\mathbb{R}^2$ starting at the origin. Grothendieck's specialization theorem implies that there exists a generic Newton polygon $\mathrm{GNP}(\Delta, \overline{\mathbb{F}}_p) := \inf_f \mathrm{NP}(f)$ where f ranges over all non-degenerate Laurent polynomials over $\overline{\mathbb{F}}_p$ with Newton polytope Δ. If f is nondegenerate and $\dim(\Delta) = n$, then by [**1**], the endpoints of the three polygons meet and

$$\mathrm{NP}(f) \geq \mathrm{GNP}(\Delta, \overline{\mathbb{F}}_p) \geq \mathrm{HP}(\Delta).$$

There are important theorems and open questions about when $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$ or $\mathrm{GNP}(\Delta, \overline{\mathbb{F}}_p) = \mathrm{HP}(\Delta)$, e.g., [**1**], [**10**]. In this paper, we consider two families of Laurent polynomials f that are deformations of diagonal polynomials. In Section 3, we apply Wan's decomposition theory [**12**] to determine congruence conditions on p for which $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$. In Section 4, we compute the Hodge numbers of $\mathrm{HP}(f)$ under certain numeric restrictions.

Here are the two families we consider. Fix $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and let $f_{n,\vec{m}} = x_1^{m_1} + \cdots + x_n^{m_n}$. For $1 \leq j \leq n$, define

$$G^j_{n,\vec{m}} = f_{n,\vec{m}} + x_1^{-m_1} + \cdots + x_j^{-m_j},$$

and

$$K^j_{n,\vec{m}} = f_{n,\vec{m}} + (x_1 \cdots x_j)^{-1}.$$

An effective lower bound for the Newton polygon for $\mathrm{NP}(G^j_{n,m})$ is given by Hodge-Stickelberger polygon as described in [**4**, Theorem 6.4], see also further results in [**3**]. We say that $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a *reflection variant* of $f_{n,\vec{m}}$ if $\Delta(f) = \Delta(G^j_{n,\vec{m}})$ for some $1 \leq j \leq n$. We say that $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a *Kloosterman variant* of $f_{n,\vec{m}}$ if $\Delta(f) = \Delta(K^j_{n,\vec{m}})$ for some $1 \leq j \leq n$.

Our motivation to study this problem came from the information that it yields about Newton polygons of varieties defined over $\mathbb{F}_q$. Consider the affine toric Artin-Schreier variety V_f in $\mathbb{A}^{n+1}$ defined by the affine equation $y^p - y = f(x_1, \ldots x_n)$

where $f(x_1,\ldots,x_n)\in\mathbb{F}_q[x_1^{\pm1},\ldots,x_n^{\pm1}]$ as above. The p-adic Newton polygons of $L(f/\mathbb{F}_q,T)$ and $L(V_f/\mathbb{F}_q,T)$ are the same after scaling by a factor of $p-1$, denoted by $\mathrm{NP}(V_f)=(p-1)\mathrm{NP}(f)$.

Further decomposition methods for Newton polygons are developed in [**9**]. Other related work can be found in [**6**], [**7**].

2. Background material

Consider a Laurent polynomial $f\in\mathbb{F}_q[x_1^{\pm1},\ldots,x_n^{\pm1}]$. Then f is of the form $f=\sum_{j=1}^{J}a_j\vec{x}^{V_j}$ where $a_j\neq0$, $V_j=(v_{1,j},\ldots,v_{n,j})\in\mathbb{Z}^n$, and $\vec{x}^{V_j}:=x_1^{v_{1,j}}\ldots x_n^{v_{n,j}}$ for $1\leq j\leq J$.

2.1. The Hodge polygon. The *Newton polytope* $\Delta(f)$ of f is the convex polygon generated by the origin $\vec{0}$ and the lattice points V_j. Note that Δ is an integral polytope, namely its vertices have integral coordinates. Without loss of generality, we assume that $\dim(\Delta)=n$. Let $V(\Delta)$ denote the volume of Δ. If δ is a subset of $\Delta(f)$, let $f^\delta=\sum\limits_{V_j\in\delta}a_jx^{V_j}$.

Definition 2.1. *A Laurent polynomial f is non-degenerate with respect to Δ and p if for each closed face δ of $\Delta(f)$ not containing $\vec{0}$, the partial derivatives $\{\frac{\partial f^\delta}{\partial x_1},\cdots,\frac{\partial f^\delta}{\partial x_n}\}$ have no common zeros with $x_1\cdots x_n\neq0$ over $\overline{\mathbb{F}}_q$.*

Let $\mathbb{A}(\Delta)$ denote the space of all Laurent polynomials with Newton polytope Δ, parametrized by their (non-vertex) coefficients (a_j). It is a smooth irreducible affine variety defined over $\mathbb{F}_p$. The subspace $\mathcal{M}_p(\Delta)\subset\mathbb{A}(\Delta)$ of all nondegenerate Laurent polynomials is the complement of a discriminant locus in $\mathbb{A}(\Delta)$. It is known that $\mathcal{M}_p(\Delta)$ is Zariski dense and open in $\mathbb{A}(\Delta)$ for each prime p; in other words, a generic Laurent polynomial with Newton polytope Δ is non-degenerate. We assume throughout that $f\in\mathcal{M}_p(\Delta)$.

Definition 2.2.
(1) *The cone $C(\Delta)=\sum_{v\in\Delta}v\mathbb{R}^{\geq0}$ of Δ is the monoid generated by vectors in Δ.*
(2) *If δ is a codimension one face of Δ, with equation $\sum_{i=1}^n c_ix_i=1$ for $c_i\in\mathbb{Q}$, the denominator $D(\delta)$ is $\min\{d\mid dc_i\in\mathbb{Z},\ 1\leq i\leq n\}$.*
(3) *The denominator $D(\Delta)$ is the least common multiple of $D(\delta)$ for all codimension one faces δ of Δ not containing $\vec{0}$.*
(4) *If $u=(u_1,\ldots,u_n)\in\mathbb{Q}^n$, the weight $w(u)$ is the smallest $c\in\mathbb{Q}^{\geq0}$ such that $u\in c\Delta:=\{c\vec{x}\mid\vec{x}\in\Delta\}$. (If there is no such rational number c, then $w(u)=\infty$).*

The weight $w(u)$ is finite if and only if $u\in C(\Delta)$. Here is an equivalent way to define the weight. If $u\in C(\Delta)$, then the ray $u\mathbb{R}^{\geq0}$ intersects a codimension one face of Δ not containing $\vec{0}$. If $\sum_{i=1}^n c_ix_i=1$ is the equation of δ, then $w(u)=\sum_{i=1}^n c_iu_i$. Thus $w(u)\in\frac{1}{D(\delta)}\mathbb{Z}^{\geq0}$.

We now define the Hodge numbers by counting the number of lattice points of a given weight k/D.

Definition 2.3. *If $k\in\mathbb{Z}^{\geq0}$,*
(1) *let $W_\Delta(k)=\#\{u\in\mathbb{Z}^n\mid w(u)=\frac{k}{D(\Delta)}\}$ be the number of lattice points in $\mathbb{Z}^n$ with weight $k/D(\Delta)$.*

(2) *let* $H_\Delta(k) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}W_\Delta(k - iD(\Delta))$ *(the Hodge number).*

For example, when $n = 2$, $H_\Delta(k) = W_\Delta(k) - 2W_\Delta(k-m) + W_\Delta(k-2m)$. The Hodge number $H_\Delta(k)$ is the number of lattice points of weight $k/D(\Delta)$ in a fundamental domain of Δ which corresponds to a basis of the p-adic cohomology used to compute the L-polynomial. Therefore, $H_\Delta(k) \geq 0$ if $k \geq 0$ and $H_\Delta(k) = 0$ if $k > nD(\Delta)$. Furthermore,

$$\sum_{k=0}^{nD(\Delta)} H_\Delta(k) = n!V(\Delta).$$

Definition 2.4. *The Hodge polygon* $\mathrm{HP}(\Delta)$ *is the lower convex polygon in* $\mathbb{R}^2$ *that starts at* $\vec{0}$ *and has a side of slope* k/D *with horizontal length* $H_\Delta(k)$ *for* $0 \leq k \leq nD$*. In other words, it is the polygon with vertices at the origin and, for* $0 \leq j \leq nD$*, at the point*

$$\left(\sum_{k=0}^{j} H_\Delta(k), \frac{1}{D(\Delta)}\sum_{k=0}^{j} kH_\Delta(k)\right).$$

2.2. Newton polygon. When f is nondegenerate with respect to $\Delta(f)$, then $L^*(f,T)^{(-1)^{n-1}}$ is a polynomial of degree $N = n!V(\Delta)$ [**1**, Corollary 3.14]. Write $L^*(f,T)^{(-1)^{n-1}} = 1 + C_1T + \cdots + C_NT^N$ with $C_i \in \mathbb{Z}[\zeta_p]$. For $C \in \mathbb{Z}[\zeta_p]$, write $\mathrm{ord}_q(C) = \mathrm{ord}_p(C)/\log_p(q)$ where $|C|_p = p^{-\mathrm{ord}_p(C)}$. The p-adic *Newton polygon* $\mathrm{NP}(f)$ of f is the lower convex hull in $\mathbb{R}^2$ of the points $(i, \mathrm{ord}_q(C_i))$ for $0 \leq i \leq N$. The Newton polygon $\mathrm{NP}(f)$ has a segment with slope α and horizontal length ℓ_α if and only if $L^*(f,T)^{(-1)^{n-1}}$ has a root of p-adic valuation $r_i = \alpha$ with multiplicity ℓ_α. Results about the slopes of the Newton polygon of f yield results about the p-adic Riemann hypothesis on the distribution of the roots of $L^*(f,T)^{(-1)^{n-1}}$ in $\overline{\mathbb{Q}}_p$.

By Grothendieck's specialization theorem, for each prime p, there exists a generic Newton polygon $\mathrm{GNP}(\Delta, \overline{\mathbb{F}}_p) := \inf_f \mathrm{NP}(f)$ where f ranges over all $f \in \mathcal{M}_p(\Delta)$ defined over $\overline{\mathbb{F}}_p$.

Theorem 2.5. [**1**, Corollary 3.11] *If* p *is prime and if* $f \in \mathcal{M}_p(\Delta)$*, then the endpoints of the three polygons meet and*

$$\mathrm{NP}(f) \geq \mathrm{GNP}(\Delta; \overline{\mathbb{F}}_p) \geq \mathrm{HP}(\Delta).$$

It is natural to ask what the slopes of $\mathrm{GNP}(\Delta, p)$ are and how they vary with p. In particular, it is natural to ask for which Δ and p the generic Newton polygon equals the Hodge polygon. Also, one would like to understand when the Newton polygon of f equals the Hodge polygon. In this context, Wan proved:

Theorem 2.6. [**10**, Theorem 3] *There is a computable integer* $D^*(\Delta) \equiv 0 \bmod D(\Delta)$ *such that if* $p \equiv 1 \bmod D^*(\Delta)$ *then* $\mathrm{GNP}(\Delta, \overline{\mathbb{F}}_p) = \mathrm{HP}(\Delta)$.

A non-degenerate Laurent polynomial f is *ordinary* if $\mathrm{NP}(f) = \mathrm{HP}(\Delta(f))$. In [**12**, Theorem 1.8], Wan gives conditions under which $\mathrm{NP}(f) = \mathrm{HP}(f)$ for all $f \in \mathcal{M}_p(\Delta)$, in other words, for which all non-degenerate f with $\Delta(f) = \Delta$ are ordinary.

The proofs of these results are quite deep. Wan constructs an overconvergent σ-module $\mathcal{E}(\Delta)$ of rank $n!\mathbf{V}(\Delta)$ on $\mathcal{M}_p(\Delta)$ such that the L-function of any non-degenerate f with Newton polytope Δ can be computed on the fiber $\mathcal{E}(\Delta)_f$ of $\mathcal{E}(\Delta)$ at the corresponding point of $\mathcal{M}_p(\Delta)$, i.e.,

$$L^*(f,T)^{(-1)^{n-1}} = \det(I - T\mathrm{Frob}_f|\mathcal{E}(\Delta)_f).$$

The Newton polygon of $L^*(f,T)^{(-1)^{n-1}}$ can be computed from the "linear algebra data" $\mathcal{E}(\Delta)_f$. A general theorem shows that for a family of F-crystals [**8**] or σ-modules [**11**], the Newton polygon goes up under specialization. This implies that there is a Zariski dense and open subspace $U \subset \mathcal{M}_p(\Delta)$ such that for every $f \in U$, the Newton polygon of $L^*(f,T)^{(-1)^{n-1}}$ equals $\mathrm{GNP}(\Delta,p)$.

3. Newton polygons of non-diagonal Laurent polynomials

In this section, we apply Wan's decomposition theory to study two families of non-diagonal Laurent polynomials. A Laurent polynomial f is *diagonal* if it is the sum of n monomials and $n = \dim(\Delta(f))$. We first survey some results about the diagonal case from [**12**, Section 2]. Suppose $f = \sum_{j=1}^{n} a_j \vec{x}^{V_j}$ where $a_j \neq 0$, $V_j = (v_{1,j},\dots,v_{n,j}) \in \mathbb{Z}^n$, and $\vec{x}^{V_j} := x_1^{v_{1,j}}\dots x_n^{v_{n,j}}$ for $1 \leq j \leq n$. Let $\Delta = \Delta(f)$ and suppose $\dim(\Delta) = n$. We will need the following definition.

Definition 3.1. *The polytope Δ is indecomposable if the $(n-1)$-dimensional face generated by $V_1,\dots,V_n$ contains no lattice points other than its vertices.*

Linear algebra techniques are useful for studying the Hodge polygon in the diagonal case. Let M be the non-singular $n \times n$ matrix $M = (V_1,\dots,V_n)$. The Laurent polynomial f is non-degenerate with respect to Δ and p if and only if $p \nmid \det(M)$. Integral lattice points $\vec{u}$ of the fundamental domain

$$\Gamma = \mathbb{R}V_1 + \cdots + \mathbb{R}V_n \bmod \mathbb{Z}V_1 + \cdots + \mathbb{Z}V_n$$

are in bijection with the set $S(\Delta)$ of solutions $\vec{r} = (r_1,\dots,r_n)$ of $M\vec{r}^T \equiv 0 \bmod 1$ with $r_j \in \mathbb{Q} \cap [0,1)$. This bijection preserves size in that the weight $w(\vec{u})$ equals the norm $|\vec{r}| = \sum_{j=1}^{n} r_i$. Now $S(\Delta)$ is a finite abelian group under addition modulo 1. Let D^* be its largest invariant factor. Consider the multiplication-by-p automorphism $[p]$ on $S(\Delta)$, denoted $\vec{r} \to \{p\vec{r}\}$. The automorphism $[p]$ is weight-preserving if $p \equiv 1 \bmod D^*$.

Using Gauss sums and the Stickelberger theorem, one proves that the p-adic valuation of a root α of $L^*(f/\mathbb{F}_q,T)^{(-1)^{n-1}}$ can be expressed in terms of the average norm of an element $\vec{r} \in S(\Delta)$ under $[p]$ [**12**, Corollary 2.3]. Specifically, the horizontal length of the slope s portion of the Newton polygon equals the number of elements $r \in S(\Delta)$ whose average norm is s [**12**, Corollary 2.4]. This yields the following.

Theorem 3.2. [**12**, Section 2.3] *Let Δ be a simplex containing $\vec{0}$ with* $\dim(\Delta) = n$. *Then*

(1) $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$ *for all $f \in \mathcal{M}_p(\Delta)$ supported only on the interior and vertices of Δ if $p \equiv 1 \bmod D^*$.*
(2) $\mathrm{GNP}(\Delta,\overline{\mathbb{F}}_p) = \mathrm{HP}(\Delta)$ *if $p \equiv 1 \bmod D^*$.*

For the main result, we need to strengthen Theorem 3.2 in a certain case. Suppose $\vec{m} = (m_1,\dots,m_n) \in \mathbb{N}^n$ and $f_{n,\vec{m}} = x_1^{m_1} + \cdots + x_n^{m_n}$. Suppose f is a

Laurent polynomial such that $\Delta(f) = \Delta(f_{n,\vec{m}})$. Notice that f is non-degenerate with respect to Δ and p if and only if $p \nmid D^* = \mathrm{LCM}(m_1, \ldots m_n)$.

Lemma 3.3. *Let* $\Delta = \Delta(f_{n,\vec{m}})$ *with* $f_{n,\vec{m}} = x_1^{m_1} + \cdots + x_n^{m_n}$.

(1) *Suppose* $f \in \mathcal{M}_p(\Delta)$ *is supported only on the interior and vertices of* Δ. *Then* $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$ *if and only if* $p \equiv 1 \bmod D^*$.
(2) *If* $m_1, \ldots, m_n$ *are pairwise relatively prime, then* $\mathrm{GNP}(\Delta, \overline{\mathbb{F}}_p) = \mathrm{HP}(\Delta)$ *if and only if* $p \equiv 1 \bmod D^*$.

Proof. (1) The sufficiency statement follows from Theorem 3.2. For the other direction, if f is ordinary then each boundary restriction $x_i^{m_i}$ is ordinary by Wan's boundary decomposition theorem [**10**, Section 5]. Hence $p \equiv 1 \bmod m_i$ for $1 \le i \le n$ which implies $p \equiv 1 \bmod D^*$.
(2) The polytope Δ is indecomposable if and only if $m_1, \ldots, m_n$ are pairwise relatively prime. Then the statement follows from part (1) and Theorem 3.2.

□

The facial decomposition theory of Wan allows one to study the Newton polygon of a non-diagonal Laurent polynomial by dividing Δ into smaller diagonal polytopes.

Theorem 3.4. [**10**, Theorem 8] *Suppose* f *is non-degenerate and* $\dim(\Delta(f)) = n$. *Let* $\delta_1, \ldots, \delta_h$ *be the codimension* 1 *faces of* $\Delta(f)$ *which do not contain* $\vec{0}$. *Then* f *is ordinary if and only if* f^{δ_i} *is ordinary for each* i.

As illustrations of Wan's facial decomposition theory, we study two deformation families of basic diagonal polynomials.

Definition 3.5. *Fix* $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ *and let* $f_{n,\vec{m}} = x_1^{m_1} + \cdots + x_n^{m_n}$. *A Laurent polynomial* $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ *is:*

(1) *a reflection variant of* $f_{n,\vec{m}}$ *if* $\Delta(f) = \Delta(G^j_{n,\vec{m}})$ *for some* $1 \le j \le n$ *where*

$$G^j_{n,\vec{m}} = f_{n,\vec{m}} + x_1^{-m_1} + \cdots + x_j^{-m_j}.$$

(2) *a Kloosterman variant of* $f_{n,\vec{m}}$ *if* $\Delta(f) = \Delta(K^j_{n,\vec{m}})$ *for some* $1 \le j \le n$ *where*

$$K^j_{n,\vec{m}} = f_{n,\vec{m}} + (x_1 \cdots x_j)^{-1}.$$

If $n = 2$ and $m_1 = m_2 = 1$, then $K^2_{2,(1,1)}$ is the classical Kloosterman polynomial, and it is well-known in this case that the Newton polygon has slopes 0 and 1 each with multiplicity one. Pictures and basic facts about the polytopes for $G^j_{n,\vec{m}}$ and $K^j_{n,\vec{m}}$ can be found in Section 4. Here is our main result.

Corollary 3.6. *Suppose* f *is a reflection variant or a Kloosterman variant of* $f_{n,\vec{m}}$ *for some* $1 \le j \le n$. *Write* $\Delta = \Delta(G^j_{n,\vec{m}})$ *or* $\Delta = \Delta(K^j_{n,\vec{m}})$ *as appropriate.*

(1) *Then* f *is non-degenerate if and only if* $p \nmid D^* = \mathrm{LCM}(m_1, \ldots, m_n)$.
(2) $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$ *for all* $f \in \mathcal{M}_p(\Delta)$ *supported only on the interior and vertices of* Δ *if and only if* $p \equiv 1 \bmod D^*$.
(3) *If* $m_1, \ldots, m_n$ *are pairwise relatively prime, then* $\mathrm{GNP}(\Delta, \overline{\mathbb{F}}_p) = \mathrm{HP}(\Delta)$ *if and only if* $p \equiv 1 \bmod D^*$.

PROOF. This proof follows essentially from Lemma 3.3. The proof of each part relies on the decomposition of Δ into different faces. By [**10**], one can measure whether f is non-degenerate, whether the generic Newton polygon and the Hodge polygon coincide, and whether the Newton polygon and the Hodge polygon coincide by seeing whether these properties are true for the restriction f^δ of f to each face δ of Δ.

For the reflection case, after a change of variables of the form $x_i \mapsto x_i^{\pm 1}$, one can restrict to the face of $f_{n,\vec{m}} = G^0_{n,\vec{m}}$ not containing $\vec{0}$. The result then follows from Lemma 3.3.

For the Kloosterman case, there is a unique face not containing $-\vec{1}_j = -\sum_{i=1}^j e_i = (-1,\dots,-1,0,\dots,0)$. It is the same face as in Lemma 3.3; in particular, $D^* = \mathrm{LCM}(m_1,\dots,m_n)$ for this face and its vertices are the only lattice points with integral coordinates on this face if and only if $m_1,\dots,m_n$ are pairwise relatively prime.

There are j other faces of Δ not containing $\vec{0}$. We consider the face δ through $-\vec{1}_j$ and $v_i = m_i e_i$ for $2 \le i \le n$. The argument for the other faces is similar. By Lemma 4.8, δ is contained in the hyperplane

$$\frac{1}{m_2}x_2 + \cdots \frac{1}{m_n}x_n - \frac{m+n-1}{m}x_1 = 1.$$

The integral lattice points $\vec{u}$ of the fundamental domain

$$\Gamma = \mathbb{R}(-\vec{1}_j) + \mathbb{R}v_2 + \cdots + \mathbb{R}v_n \bmod \mathbb{Z}(-\vec{1}_j) + \mathbb{Z}v_2 + \cdots + \mathbb{Z}v_n$$

are the set

$$\{(0,u_2,\dots,u_n) \in \mathbb{Z}^n \mid 0 \le u_i < m_i\}.$$

Thus $\Gamma \simeq \times_{i=2}^n \mathbb{Z}/m_i$ and $D_1^* = \mathrm{LCM}(m_2,\dots,m_n)$ is the largest invariant factor of Γ. The multiplication-by-p map on Γ is thus weight-preserving if $p \equiv 1 \bmod D_1^*$. Since D_1^* divides D^*, the face δ places no new constraints on the condition $\mathrm{GNP}(\Delta, \overline{\mathbb{F}}_p) = \mathrm{HP}(\Delta)$. Furthermore, if δ does not contain $\vec{0}$, then there are no lattice points on δ other than the vertices. Thus the face δ places no new constraints on the condition $\mathrm{NP}(f) = \mathrm{HP}(\Delta)$ for all $f \in \mathcal{M}_p(\Delta)$.

Conversely, if f is ordinary then its restriction to each face f^δ is ordinary. Then $p \equiv 1 \bmod D^*$ by Lemma 3.3. □

REMARK 3.7. *By* [**1**, Corollary 3.14]*, if f is non-degenerate, then $L^*(f,T)^{(-1)^{n-1}}$ is a polynomial of degree $n!V(\Delta)$. In the reflection case,*

$$V(\Delta(G^j_{n,\vec{m}})) = 2^j V(G^0_{n,\vec{m}}) = 2^j \prod_{j=1}^n m_j/n!.$$

For the Kloosterman case, write s_k for the kth symmetric product in $m_1,\dots,m_j$. For example, $s_j = \prod_{i=1}^j m_i$. Then, see Lemma 4.8,

$$V(\Delta(K^j_{n,\vec{m}})) = \left(s_j + \sum_{i=1}^{j-1}(-1)^i i s_{j-1-i}\right) \prod_{i=j+1}^n m_i/n!.$$

4. Computation of Hodge polygons

In this section, we describe the Hodge polygons for two types of Laurent polynomials: the reflection variants $G^j_{n,\vec{m}}$ in Section 4.2; and the Kloosterman variants $K^j_{n,\vec{m}}$ in Section 4.3. Each of these is a generalization of the diagonal case which we review in Section 4.1. We give explicit formulae for the Hodge numbers under certain numeric restrictions on $\vec{m}$.

Fix $n \in \mathbb{N}$ and $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$. Let $v_i = m_i \vec{e}_i$ where $\vec{e}_i$ is the standard basis vector of $\mathbb{R}^n$; in other words, $v_1 = (m_1, 0 \ldots, 0)$, $v_2 = (0, m_2, 0, \ldots, 0)$, etc. Write $\vec{x} = (x_1, \ldots, x_n)$.

4.1. Diagonal Case. Recall that a Laurent polynomial $f \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is diagonal if it is the sum of n monomials and $\dim(\Delta(f)) = n$. If f is diagonal, each reciprocal zero of its L-function can be computed using Gauss sums, yielding a theoretical understanding of the Newton Polygon of the diagonal case. The diagonal case is still interesting, however, since nontrivial combinatorial and arithmetic problems arise in computing the Newton Polygon.

Let $f = \sum_{j=1}^{n} a_j x^{V_j}$, with $a_j \in \mathbb{F}_q$, be a diagonal, non-degenerate Laurent polynomial. Let's recall the definition of Gauss sums.

Definition 4.1. *Let χ be the Teichmuller character of $\mathbb{F}_q^*$. For $0 \le k \le q-2$, the Gauss sum $G_k(q)$ over $\mathbb{F}_q$ is defined as:*

$$G_k(q) = -\sum_{a \in \mathbb{F}_q^*} \chi(a)^{-k} \zeta_p^{\mathrm{Tr}(a)}.$$

Gauss sums satisfy certain interpolation relations which yield formulas for the exponential sums $S_k^*(f)$ [**12**, 16]. For example,

$$S_1^*(f) = \sum_{x_j \in \mathbb{F}_q^*} \zeta_p^{\mathrm{Tr}(f(x))} = (-1)^n \sum_{k_1 V_1 + \ldots + k_n V_n \equiv 0 \bmod q-1} \prod_{i=1}^{n} \chi(a_i)^k G_{k_i}(q).$$

Combining this with the Hasse-Davenport relation, Wan obtains an explicit formula for $L^*(f,T)^{(-1)^{n-1}}$ in [**12**, Theorem 2.1]. By applying Stickelberger's Theorem, it is possible to determine the p-adic absolute values of the reciprocal zeros of $L^*(f,T)^{(-1)^{n-1}}$. In particular, the Newton Polygon is independent of the coefficients a_j and one can suppose $f = \sum_{j=1}^{n} x^{V_j}$ without loss of generality.

We now restrict to the special case of Laurent polynomials of the form $f_{n,\vec{m}} = \sum_{i=1}^{n} x_i^{m_i}$. The vertices of the polytope $\Delta := \Delta(f_{n,\vec{m}})$ are $\{v_1, \ldots, v_n, \vec{0}\}$ and the volume is $V(\Delta) = \prod_{j=1}^{n} m_j / n!$. The denominator is $D(\Delta) = \mathrm{LCM}(m_1, \ldots, m_n)$. The numeric restriction in Section 4.1.1 is that $m_i = m_j$ for all $1 \le i, j \le n$ and in Section 4.1.2 is that $n = 2$ and $\gcd(m_1, m_2) = 1$.

4.1.1. *General dimension, equilateral.* For later use, we review some results about the Hodge numbers of the diagonal polynomials

$$G^0_{n,m} = x_1^m + \cdots + x_n^m.$$

Lemma 4.2. *The weight numbers for $G^0_{n,m}$ are:*

$$W(k) = \binom{n-1+k}{n-1}.$$

The Hodge numbers for $G^0_{n,m}$ are:

$$H(k) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n-1+k-im}{n-1}.$$

PROOF. The face of Δ not containing $\vec{0}$ is the hyperplane

$$\frac{1}{m}x_1 + \cdots + \frac{1}{m}x_n = 1.$$

Thus $D(\Delta) = m$. The cone $c(\Delta)$ is $\{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_i \geq 0\}$. The weight of a vector is given by the formula: $w(\vec{x}) = \frac{1}{m}x_1 + \cdots + \frac{1}{m}x_n$. The number $W(k)$ of points in $c(\Delta)$ with weight k/m is the number of solutions to

$$x_1 + x_2 + \cdots + x_n = k,$$

which yields the formula for $W(k)$. The formula for $H(k)$ follows from Definition 2.3. □

REMARK 4.3. *The vertices of* $\mathrm{HP}(\Delta(G^0_{n,m}))$ *are at* $(0,0)$ *and* (x_j, y_j) *where*

$$x_j = \sum_{i=0}^{\lfloor j/m \rfloor} (-1)^i \binom{n}{i} \binom{n+j-im}{n},$$

and

$$y_j = \frac{1}{m} \sum_{i=0}^{\lfloor j/m \rfloor} (-1)^i \binom{n}{i} \left(n \cdot \binom{n+j-im}{n+1} + im \cdot \binom{n+j-im}{n} \right).$$

4.1.2. *Dimension two, non-equilateral.* Suppose $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ with $m_1, \ldots m_n \in \mathbb{N}$ pairwise relatively prime. Let $W^0_{n,\vec{m}}(k) := W_{\Delta(f_{n,\vec{m}})}(k)$. Let $M_j := \prod_{i=1, i\neq j}^{n} m_i$. Then

$$W^0_{n,\vec{m}}(k) = \#\{(x_1, \ldots, x_n) \in \mathbb{N}^n \mid \sum_{i=1}^{n} M_i x_i = k\}.$$

These restricted partition functions can be computed using Dedekind sums [**2**].

Restricting to the case $n = 2$, then

$$W^0_{2,\vec{m}}(k) = \#\{(x_1, x_2) \in \mathbb{N}^2 \mid m_2 x_1 + m_1 x_2 = k\}.$$

Consider the generating function:

$$\frac{1}{1-z^{m_1}} \frac{1}{1-z^{m_2}} = \sum_{x_2=0}^{\infty} z^{m_1 x_2} \sum_{x_1=0}^{\infty} z^{m_2 x_1} = \sum_{k \geq 0} W^0_{2,\vec{m}}(k) z^k.$$

In this case, Popoviciu used partial fractions to give the following formula for $W^0_{2,\vec{m}}(k)$. For $x \in \mathbb{Q}$, let $\{x\} = \lfloor x \rfloor - x$ denote the fractional part of x.

THEOREM 4.4. [**2**, Section 1.4] *Given* $m_1, m_2 \in \mathbb{N}$ *with* $\gcd(m_1, m_2) = 1$*, let* $m_1^{-1}, m_2^{-1} \in \mathbb{N}$ *be such that:*

(1) $1 \leq m_1^{-1} < m_2$ *and* $m_1 m_1^{-1} \equiv 1 \bmod m_2$ *and*

(2) $1 \leq m_2^{-1} < m_1$ *and* $m_2 m_2^{-1} \equiv 1 \bmod m_1$.

Then

$$W^0_{2,(m_1,m_2)}(k) = \frac{k}{m_1m_2} - \left\{\frac{m_2^{-1}k}{m_1}\right\} - \left\{\frac{m_1^{-1}k}{m_2}\right\} + 1.$$

Using Theorem 4.4, one can explicitly compute all Hodge numbers $H^0(k)$ for $W^0_{2,(m_1,m_2)} = x_1^{m_1} + x_2^{m_2}$ when $\gcd(m_1, m_2) = 1$. Note that the sum of the Hodge numbers is

$$\sum_{k=0}^{2m_1m_2} H^0(k) = m_1m_1 = 2V(\Delta(f_{2,(m_1,m_2)})).$$

k	$0, 1, \ldots, m_1m_2 - 1$	m_1m_2	$m_1m_2 + t; 0 < t < m_1m_2$	$2m_1m_2$
$H^0(k)$	$W^0_{2,(m_1,m_2)}(k)$	0	$1 - W^0_{2,(m_1,m_2)}(t)$	0

TABLE 1. Hodge Numbers for $x_1^{m_1} + x_2^{m_2}$ if $\gcd(m_1, m_2) = 1$

REMARK 4.5. *The method for $n = 2$ can be generalized to higher dimensions; complicated formulas for $W^0_{n,\vec{m}}(k)$ can be found in terms of Dedekind sums* [**2**, Theorem 1.7]. *For instance, when $n = 3$ and m_1, m_2, m_3 are pairwise relatively prime then*

$$\begin{aligned} W^0_{3,(m_1,m_2,m_3)}(k) &= \frac{k^2}{m_1m_2m_3} + \frac{k}{2}\left(\frac{1}{m_1m_2} + \frac{1}{m_1m_3} + \frac{1}{m_2m_3}\right) \\ &+ \frac{1}{12}\left(\frac{3}{m_1} + \frac{3}{m_2} + \frac{3}{m_3} + \frac{m_1}{m_2m_3} + \frac{m_2}{m_1m_3} + \frac{m_3}{m_1m_1}\right) \\ &+ \varphi_{m_1}(m_2, m_3)(k) + \varphi_{m_2}(m_1, m_3)(k) + \varphi_{m_3}(m_1, m_2)(k), \end{aligned}$$

where $\varphi_a(b, c)(k) := \frac{1}{c}\sum_{i=1}^{c-1}[(1 - \zeta_c^{ib})(1 - \zeta_c^{ia})\zeta_c^i k]^{-1}$.

4.2. Reflection variant Laurent polynomials. Suppose $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and let

$$G^0_{n,\vec{m}} = x_1^{m_1} + \ldots + x_n^{m_n}.$$

The polytope $\Delta^0_{n,\vec{m}}$ for $G^0_{n,\vec{m}}$ has vertices $\vec{0}$ and v_i for $1 \le i \le n$.

We consider reflections of $\Delta^0_{n,\vec{m}}$ across coordinate hyperplanes. After a permutation of the variables, it is no loss of generality to reflect across the hyperplanes $x_i = 0$ for $1 \le i \le j$. Let

$$G^j_{n,\vec{m}} = x_1^{m_1} + \ldots + x_n^{m_n} + x_1^{-m_1} + \ldots + x_j^{-m_j}.$$

Let $\Delta^j_{n,\vec{m}}$ be the polytope of $G^j_{n,\vec{m}}$. For example, $G^1_{n,\vec{m}} = x_1^{m_1} + \ldots + x_n^{m_n} + x_1^{-m_1}$ and $\Delta^1_{n,\vec{m}}$ is the polygon in $\mathbb{R}^n$ with vertices v_i for $1 \le i \le n$ and $-v_1$. Then $\Delta^j_{n,\vec{m}}$ has $n + j$ vertices other than $\vec{0}$ and

$$\mathrm{Vol}(\Delta^j_{n,\vec{m}}) = 2^j \cdot \mathrm{Vol}(\Delta^0_{n,\vec{m}}) = 2^j \prod_{i=1}^{n} m_i/n!.$$

Using the inclusion-exclusion principle, there is a recursive formula for the weight numbers of $\Delta^j_{n,\vec{m}}$:

$$W_{\Delta^j_{n,\vec{m}}}(k) = 2W_{\Delta^{j-1}_{n,\vec{m}}}(k) - W_{\Delta^{j-1}_{n-1,(m_1,\ldots,\hat{m}_j,\ldots,m_n)}}(k), \tag{1}$$

where the notation $\hat{m}_j$ means that the jth variable is omitted. Using this recursive formula, it is possible to obtain the weights for a general reflection case in terms of the weights for the base case $j = 0$.

4.2.1. *General dimension, equilateral.* Suppose $\vec{m} = (m, \ldots, m)$ and write

$$G^j_{n,m} = x_1^m + \ldots + x_n^m + x_1^{-m} + \ldots + x_j^{-m}.$$

The polytope $\Delta^j_{n,m} = \Delta(G^j_{n,m})$ is obtained by reflecting $\Delta^0_{n,m}$ across the hyperplanes $x_i = 0$ for $1 \le i \le j$, see Figure 1.

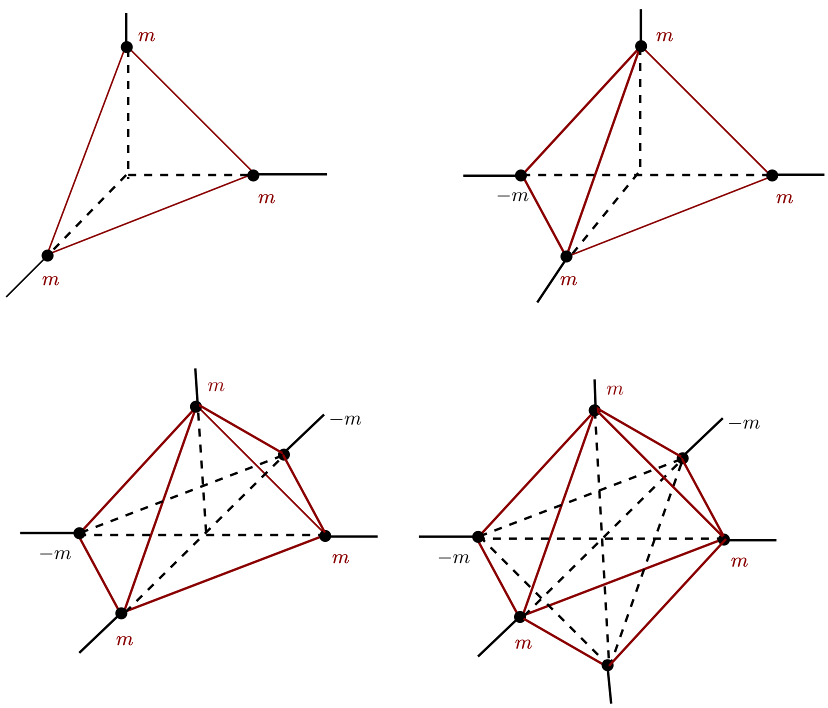

FIGURE 1. $\Delta^j_{3,m}$ for $0 \le j \le 3$

In this case, $\mathrm{Vol}(\Delta^j_{n,m}) = 2^j m^n/n!$ and Equation (1) yields the recursive formula

$$W_{\Delta^j_{n,m}}(k) = 2W_{\Delta^{j-1}_{n,m}}(k) - W_{\Delta^{j-1}_{n-1,m}}(k). \tag{2}$$

We obtain the following closed form for the weight numbers:

PROPOSITION 4.6. *The weight numbers for $G^j_{n,m}$ are given by:*

$$W_{\Delta^j_{n,m}}(k) = \sum_{i=0}^{j} 2^{j-i}(-1)^i \binom{j}{i} W_{\Delta^0_{n-i,m}}(k).$$

PROOF. First, the formula holds when $j = 0$.

To show the formula satisfies the recursion in (2), we compute

$$2W_{\Delta^{j-1}_{n,m}}(k) = \sum_{i=0}^{j-1} 2^{j-i}(-1)^i \binom{j-1}{i} W_{\Delta^0_{n-i,m}}(k),$$

and

$$
\begin{aligned}
-W_{\Delta^{j-1}_{n-1,m}}(k) &= \sum_{i=0}^{j-1} 2^{j-1-i}(-1)^{i+1}\binom{j-1}{i} W_{\Delta^0_{n-1-i,m}}(k) \\
&= \sum_{i=0}^{j-1} 2^{j-(i+1)}(-1)^{i+1}\binom{j-1}{i} W_{\Delta^0_{n-(i+1),m}}(k) \\
&= \sum_{i=1}^{j} 2^{j-i}(-1)^{i}\binom{j-1}{i-1} W_{\Delta^0_{n-i,m}}(k).
\end{aligned}
$$

Then $2W_{\Delta^{j-1}_{n,m}}(k) - W_{\Delta^{j-1}_{n-1,m}}(k)$ equals

$$
\begin{aligned}
&= \sum_{i=0}^{j-1} 2^{j-i}(-1)^{i}\binom{j-1}{i} W_{\Delta^0_{n-i,m}}(k) + \sum_{i=1}^{j} 2^{j-i}(-1)^{i}\binom{j-1}{i-1} W_{\Delta^0_{n-i,m}}(k) \\
&= 2^j W_{\Delta^0_{n,m}}(k) + \sum_{i=1}^{j-1} 2^{j-i}(-1)^{i}\left(\binom{j-1}{i} + \binom{j-1}{i-1}\right) W_{\Delta^0_{n-i,m}}(k) + (-1)^j W_{\Delta^0_{n-j,m}}(k) \\
&= \sum_{i=0}^{j} 2^{j-i}(-1)^{i}\binom{j}{i} W_{\Delta^0_{n-i,m}}(k) = W_{\Delta^j_{n,m}}(k).
\end{aligned}
$$

□

EXAMPLE 4.7. *Weight and Hodge numbers for $G^j_{2,m}$ with $0 \le j \le 2$.*

k	0	1	...	$m-1$	m	$m+1$	...	$2m-1$	$2m$
$W(k)$	1	2	...	m	$m+1$	$m+2$	...	$2m$	$2m+1$
$H(k)$	1	2	...	m	$m-1$	$m-2$	...	0	0

TABLE 2. Hodge Numbers for $G^0_{2,m} = x_1^m + x_2^m$

k	0	1	...	$m-1$	m	$m+1$	...	$2m-1$	$2m$
$W(k)$	1	3	...	$2m-1$	$2m+1$	$2m+3$	...	$4m-1$	$4m+1$
$H(k)$	1	3	...	$2m-1$	$2m-1$	$2m-3$	...	1	0

TABLE 3. Hodge Numbers for $G^1_{2,m} = x_1^m + x_2^m + x_1^{-m}$

k	0	1	...	$m-1$	m	$m+1$	...	$2m-1$	$2m$
$W(k)$	1	4	...	$4(m-1)$	$4(m+1)$	$4(m+3)$	...	$4(2m-1)$	$8m$
$H(k)$	1	4	...	$4(m-1)$	$4m-2$	$4(m-1)$	...	4	1

TABLE 4. Hodge Numbers for $G^2_{2,m} = x_1^m + x_2^m + x_1^{-m} + x_2^{-m}$.

4.2.2. *Dimension two, non-equilateral.* Suppose $n = 2$ and $\gcd(m_1, m_2) = 1$. Let $W^j(k) := W_{\Delta^j_{2,(m_1,m_2)}}(k)$ for $0 \le j \le 2$. Then $W^0(k)$ can be computed using Theorem 4.4. Equation (1) gives recursive formulae $W^1(k) = 2W^0(k) - 1$ and $W^2(k) = 2W^1(k) - 2 = 4W^0(k) - 4$.

The Hodge numbers are computed in Tables 5 and 6. Note that the sum of the Hodge numbers in Table 5 is

$$\sum_{k=0}^{2m_1m_2} H^1(k) = 2m_1m_2 = 2V(\Delta^1_{2,(m_1,m_2)}),$$

and in Table 6 is

$$\sum_{k=0}^{2m_1m_2} H^2(k) = 4m_1m_2 = 2V(\Delta^2_{2,(m_1,m_2)}).$$

k	$0,1,\ldots,m_1m_2-1$	m_1m_2	$m_1m_2+t; 0<t<m_1m_2$	$2m_1m_2$
$H^1(k)$	$2W^0(k)-1$	1	$3-2W^0(t)$	0

TABLE 5. Hodge Numbers for $G^1_{2,(m_1,m_2)} = x_1^{m_1} + x_2^{m_2} + x_1^{-m_1}$

k	$0,1,\ldots,m_1m_2-1$	m_1m_2	$m_1m_2+t; 0<t<m_1m_2$	$2m_1m_2$
$H^2(k)$	$4W^0(k)-4$	4	$8-4W^0(t)$	0

TABLE 6. Hodge Numbers for $G^2_{2,(m_1,m_2)} = \sum_{i=1}^2 (x_i^{m_i} + x_i^{-m_i})$

4.3. Kloosterman variant Laurent polynomials. Fix $n \in \mathbb{N}$, $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $1 \le j \le n$. In this section, let Δ denote the polytope of the Laurent polynomial

$$K^j_{n,\vec{m}} = x_1^{m_1} + \cdots + x_n^{m_n} + (x_1 \cdots x_j)^{-1}.$$

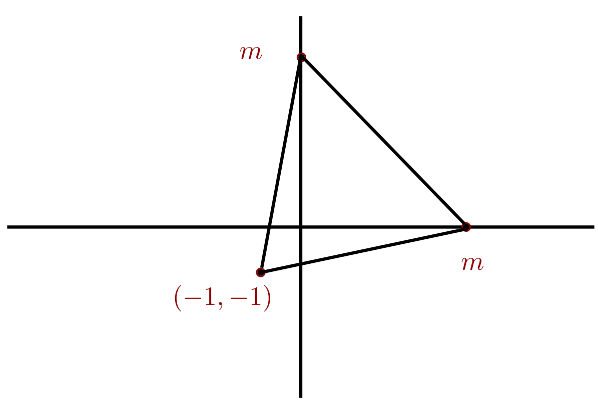

FIGURE 2. The polytope $\Delta(K^2_{2,(m,m)})$

The vertices of Δ are $-\vec{1}_j := -\sum_{i=1}^{j} e_i = (-1, \dots, -1, 0, \dots, 0)$ and $v_1, \dots, v_n$. The cone is $c(\Delta) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ j+1 \leq i \leq n\}$.

The vectors with initial point $-\vec{1}_j$ along the edges of Δ are, for $1 \leq \ell \leq n$,

$$w_\ell = \sum_{i=1}^{j} e_j + m_\ell e_\ell.$$

The volume of the polytope $\Delta = \Delta(K^j_{n,\vec{m}})$ is $V(\Delta) = \frac{1}{n!}\det(w_1, \dots, w_n)$. Write s_k for the kth symmetric product in $m_1, \dots, m_j$. Then

$$V(\Delta(K^j_{n,\vec{m}})) = [s_j + \sum_{i=1}^{j-1}(-1)^i i s_{j-1-i}] \prod_{i=j+1}^{n} m_i/n!.$$

The denominator of Δ is $D = \mathrm{LCM}(m_1, \dots, m_n)$.

LEMMA 4.8. (1) *Suppose $1 \leq \ell \leq j$. Let δ_ℓ be the face of Δ containing the vertices $-\vec{1}_j$ and v_i for $1 \leq i \leq n$ and $i \neq \ell$. Then δ_ℓ is contained in the hyperplane:*

$$\sum_{i \neq \ell} \frac{1}{m_i} x_i - (1 + \sum_{i \neq \ell} \frac{1}{m_i}) x_\ell = 1.$$

(2) *The other faces of Δ are contained in the hyperplanes $\sum_{i=1}^{n} \frac{1}{m_i} x_i = 1$ and $x_i = 0$ for $j+1 \leq i \leq n$.*

4.3.1. *General dimension, equilateral case.* Suppose $\vec{m} = (m_1, \dots, m_n)$ and write $K^j_{n,m} := K^j_{n,\vec{m}}$.

PROPOSITION 4.9. *For $0 \leq k \leq nm$, the weight numbers for $K^j_{n,m}$ are:*

$$W(k) = \binom{n-1+k}{n-1} + \sum_{s=1}^{j} \beta(j,s) \sum_{\ell=1}^{n} \binom{k - \ell m + (n-j-1)}{n-s-1} + \alpha(j,k),$$

where $\beta(j,s) = \binom{j}{s}$ unless $j = s = n$ in which case $\beta(n,n) = 0$ and $\alpha(j,k) = 0$ unless $j = n$ and $0 < k \equiv 0 \bmod m$ in which case $\alpha(j,k) = 1$.

PROOF. The lattice points $\{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_i \geq 0\}$ have the same weight as in the diagonal case. This contributes $\binom{n-1+k}{n-1}$ to $W(k)$.

Thus it suffices to consider the weight of $\vec{x} = (x_1, \dots, x_n)$ when at least one coordinate is negative. By symmetry, it suffices to first focus on the points $\vec{x}$ closest to the face δ of Δ containing the vertices $-\vec{1}_j$ and v_i for $2 \leq i \leq n$. This face is contained in the hyperplane

$$\frac{1}{m} \sum_{i=2}^{n} x_i - \frac{m + (j-1)}{m} x_1 = 1.$$

These points satisfy the conditions: $x_1 < 0$ and $x_i \geq x_1$ for $2 \leq i \leq j$, and $x_i \geq 0$ for $j+1 \leq i \leq n$.

The condition $k \leq nm$ implies that $x_1 \in \{-1, \dots, -n\}$. Fix $-\ell \in \{-1, \dots, -n\}$ and let $x_1 = -\ell$. First suppose $x_i > x_1$ for all $2 \leq i \leq j$. The smallest weight k possible for this set of points is

$$-(j-1)(\ell-1) + (m+j-1)(\ell) + (n-j)(0) = m\ell + j - 1,$$

occurring when $x_i = -(\ell-1)$ for $2 \le i \le j$ and $x_i = 0$ for $j+1 \le i \le n$. To increase this value to k, one needs to add a combined total of $k-(m\ell+j-1)$ to $\{x_i \mid i \ge 2\}$. For $1 \le \ell \le n$, there are

$$\binom{k-(m\ell+j-1)+(n-2)}{n-2} = \binom{k-m\ell+n-j-1}{n-2}$$

ways to do this, which is the number of points of weight k with $x_1 = -\ell$, $x_i > x_1$ for $2 \le i \le j$ and $x_i \ge 0$ for $j+1 \le i \le n$.

Next, let $2 \le s \le j$ and suppose $\#\{i \le j \mid x_i = -\ell\} = s$. Recall that $-\ell \in \{-1,\dots,-n\}$. (This is the case where $\vec{x}$ is equidistant to more than one face of Δ containing $-\vec{1}_j$.) For ease of notation, suppose $x_i = -\ell$ for $1 \le i \le s$. Recall that $-\ell \in \{-1,\dots,-n\}$, and $x_i \ge -(\ell-1)$ for $s+1 \le i \le j$ and $x_i > 0$ for $j+1 \le i \le n$. The smallest weight k possible for this set of points is

$$-(j-s)(\ell-1)-(s-1)(\ell)+(m+j-1)(\ell)+0(n-j) = m\ell+j-s,$$

occurring when $x_i = -(\ell-1)$ for $s+1 \le i \le j$ and $x_i = 0$ for $j+1 \le i \le n$. To increase this value to k, one needs to add a combined total of $k-(m\ell+j-s)$ to $\{x_i \mid i \ge s+1\}$. Thus, for $1 \le \ell \le n$, outside the case $s=j=n$, there are

$$\binom{k-(m\ell+j-s)+(n-s-1)}{n-s-1} = \binom{k-m\ell+n-j-1}{n-1-s}$$

ways to do this, which is the number of points of weight k with $x_i = -\ell$ for $1 \le i \le s$, and $x_i > -\ell$ for $s+1 \le i \le j$, and $x_i \ge 0$ for $j+1 \le i \le n$. Let $C_s(k)$ denote the set of lattice points $\vec{x}$ of weight k such that $\#\{i \mid x_i = \min(x_1,\dots,x_n)\} = s$. The conclusion is that, outside the case $s=j=n$,

$$\#C_s(k) = \binom{j}{s}\sum_{\ell=1}^{n}\binom{k-m\ell+n-j-1}{n-s-1}.$$

If $s=j=n$, none of the sets C_s include the points $\vec{x}$ which are a multiple of $-\vec{1}_j$. There is one such point of weight $m\ell$ for each $1 \le \ell \le n$. This contributes one point of weight k only when $0 < k \equiv 0 \bmod m$. This is accounted for by the definitions of $\beta(j,s)$ and $\alpha(j,k)$. □

EXAMPLE 4.10. *Let $n=2$. The difference between the number of lattice points of weight k/m for $K^2_{2,m}$ and $G^0_{2,m}$ is zero if $0 \le k < m$, is one if $k=m$, is two if $m < k < 2m$, and is three if $k=2m$.*

k	0	1	...	$m-1$	m	$m+1$	$m+2$	...	$2m-1$	$2m$
$W(k)$	1	2	...	m	$m+2$	$m+4$	$m+5$	...	$2m+2$	$2m+4$
$H(k)$	1	2	...	m	m	m	$m-1$	...	2	1

TABLE 7. Hodge Numbers for $K^2_{2,m} = x_1^m + x_2^m + (x_1x_2)^{-1}$

EXAMPLE 4.11. *Let $n=3$. Table 8 shows the difference $\tau(k,m)$ between the number of lattice points of weight k/m for $K^3_{3,m}$ and $G^0_{3,m}$.*

k	$0 \le k < m$	$k = m$	$k = m + \epsilon$ $0 < \epsilon < m$	$k = 2m$	$k = 2m + \epsilon$ $0 < \epsilon < m$	$k = 3m$
$\tau(k, m)$	0	1	3ϵ	$3m + 1$	$3m + 6\epsilon$	$9m + 1$

TABLE 8. The difference between $W_{K^3_{3,m}}(k)$ and $W_{G^0_{3,m}}(k)$

4.3.2. *Dimension two, non-equilateral case.* Let $n = 2$ and $\gcd(m_1, m_2) = 1$. Recall that one can explicitly compute the weight number for the diagonal polytope $x_1^{m_1} + x_2^{m_2}$ from Theorem 4.4 and Table 1. In this section, we denote that weight number by $d(k)$.

Let Δ^2 denote the polytope of $K^2_{2,(m_1,m_2)} = x_1^{m_1} + x_2^{m_2} + (x_1x_2)^{-1}$. It has vertices at $(m_1, 0), (0, m_2)$, and $(-1, -1)$ and denominator $D = m_1m_2$. Similarly, let Δ^1 denote the polytope of $K^1_{2,(m_1,m_2)} = x_1^{m_1} + x_2^{m_2} + (x_1)^{-1}$ which has vertices at $(m_1, 0), (0, m_2)$ and $(-1, 0)$ and denominator $D = m_1m_2$.

We compute the weight numbers $W(k)$ for $K^j_{2,(m_1,m_2)}$ for $j = 1, 2$.

PROPOSITION 4.12. *Let $d(k)$ denote the weight number for the diagonal polytope $x_1^{m_1} + x_2^{m_2}$. For $0 \le k \le 2m_1m_2$ and $1 \le j \le 2$, the weight numbers for $K^j_{2,(m_1,m_2)}$ are*

$$W_j(k) = d(k) + \begin{cases} 1 + j & \text{if } k = 2m_1m_2, \\ 1 & \text{if } m_1m_2 \le k < 2m_1m_2 \text{ and } \gcd(k, m_1m_2) > 1, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The contribution to $W_j(k) - d(k)$ comes from points (x_1, x_2) with at least one negative coordinate. Then $x_1, x_2 \ge -2$ since $k \le 2m_1m_2$.

When $j = 2$, there are $2 + m_1 + m_2$ new lattice points having at least one negative coordinate: $(-1, -1)$ with weight 1, $(-2, -2)$ with weight 2, $(-1, \ell)$ for $0 \le \ell < m_2$ and $(\ell, -1)$ for $0 \le \ell < m_1$. The equation of the face δ through the vertices $(-1, -1)$ and $(m_1, 0)$ is $\frac{1}{m_1}x_1 - \frac{m_1+1}{m_1}x_2 = 1$. Using this, the weight of $(\ell, -1)$ is k/m_1m_2 with $k = m_1m_2 + (\ell + 1)m_2$. Similarly, the weight of $(-1, \ell)$ is k/m_1m_2 with $k = m_1m_2 + (\ell + 1)m_1$.

When $j = 1$, there are exactly $m_2 + 2$ new lattice points having at least one negative coordinate: $(-1, 0)$ with weight 1, $(-2, 0)$ with weight 2, and the points $(-1, \ell)$ for $1 \le \ell \le m_2$. Using the equation $-x_1 + \frac{1}{m_2}x_2 = 1$ of the corresponding face δ, the weight of $(-1, \ell)$ is k/m_1m_2 with $k = m_1m_2 + m_1\ell$. □

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