EKEDAHL-OORT STRATA OF HYPERELLIPTIC CURVES IN CHARACTERISTIC TWO

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ABSTRACT. Suppose X is a hyperelliptic curve of genus g defined over an algebraically closed field k of characteristic $p = 2$. We prove that the de Rham cohomology of X decomposes into pieces indexed by the branch points of the hyperelliptic cover. This allows us to compute the isomorphism class of the 2-torsion group scheme $J_X[2]$ of the Jacobian of X in terms of the Ekedahl-Oort type. The interesting feature is that $J_X[2]$ depends only on some discrete invariants of X, namely, on the ramification invariants associated with the branch points. We give a complete classification of the group schemes which occur as the 2-torsion group schemes of Jacobians of hyperelliptic k-curves of arbitrary genus, showing that only relatively few of the possible group schemes actually do occur.

1. INTRODUCTION

Suppose k is an algebraically closed field of characteristic $p > 0$. There are several important stratifications of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g defined over k, including the Ekedahl-Oort stratification. The Ekedahl-Oort type characterizes the p-torsion group scheme of the corresponding abelian varieties, and, in particular, determines invariants of the group scheme such as the p-rank and a-number. It is defined by the interaction between the Frobenius F and Verschiebung V operators on the p-torsion group scheme. Very little is known about how the Ekedahl-Oort strata intersect the Torelli locus of Jacobians of curves. In particular, one would like to know which group schemes occur as the p-torsion $J_X[p]$ of the Jacobian J_X of a curve X of genus g.

In this paper, we completely answer this question for hyperelliptic k-curves X of arbitrary genus when k has characteristic $p = 2$, a case which is amenable to calculation because of the confluence of hyperelliptic and Artin-Schreier properties. We first prove a decomposition result about the structure of $\mathrm{H}^1_{\mathrm{dR}}(X)$ as a module under the actions of F and V, where the pieces of the decomposition are indexed by the branch points of the hyperelliptic cover. This is the only decomposition result about the de Rham cohomology of Artin-Schreier curves that we know of, although the action of V on $\mathrm{H}^0(X, \Omega^1)$ and the action of F on $\mathrm{H}^1(X, \mathcal{O})$ have been studied for Artin-Schreier curves under less restrictive hypotheses (e.g., [15, 25]).

The second result of this paper is a complete classification of the isomorphism classes of group schemes which occur as the 2-torsion group scheme $J_X[2]$ for a hyperelliptic k-curve X of arbitrary genus when $\mathrm{char}(k) = 2$. The group schemes which occur decompose into pieces indexed by the branch points of the hyperelliptic cover, and we determine the Ekedahl-Oort types of these pieces. In particular, we determine which a-numbers occur for the 2-torsion group schemes of hyperelliptic k-curves of arbitrary genus when $\mathrm{char}(k) = 2$. Before describing the result precisely, we note that it shows that the group scheme $J_X[2]$ depends only on some discrete invariants of X and not on the location of the branch points or the equation of the hyperelliptic cover. This is in sharp contrast

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with the case of hyperelliptic curves in odd characteristic p, where even the p-rank depends on the location of the branch points, [27].

Here is some notation needed to describe the results precisely.

Notation 1.1. Suppose k is an algebraically closed field of characteristic $p = 2$. Let X be a k-curve of genus g which is hyperelliptic, in other words, for which there exists a degree two cover $\pi : X \to \mathbb{P}^1$. Let $B \subset \mathbb{P}^1(k)$ denote the set of branch points of π and let $r := \#B - 1$. After a fractional linear transformation, one may suppose that $0 \in B$ and $\infty \notin B$.

For $\alpha \in B$, the *ramification invariant* d_α is the largest integer for which the higher ramification group of π above α is non-trivial. By [23, Prop. III.7.8], d_α is odd. Let $c_\alpha := (d_\alpha - 1)/2$ and let $x_\alpha := (x - \alpha)^{-1}$.

The cover π is given by an affine equation of the form $y^2 - y = f(x)$ for some non-constant rational function $f(x) \in k(x)$. After a change of variables of the form $y \mapsto y + \epsilon$, one may suppose that the partial fraction decomposition of $f(x)$ has the form:

$$f(x) = \sum_{\alpha \in B} f_\alpha(x_\alpha), \tag{1.1}$$

where $f_\alpha(x) \in xk[x^2]$ is a polynomial of degree d_α containing no monomials of even exponent. In particular, the divisor of poles of $f(x)$ on $\mathbb{P}^1$ has the form

$$\mathrm{div}_\infty(f(x)) = \sum_{\alpha \in B} d_\alpha \alpha.$$

By the Riemann-Hurwitz formula [22, IV, Prop. 4], the genus g of X satisfies

$$2g + 2 = \sum_{\alpha \in B} (d_\alpha + 1).$$

Recall that the 2-*rank* of (the Jacobian of) the k-curve X is $\dim_{\mathbb{F}_2} \mathrm{Hom}(\mu_2, J_X[2])$ where μ_2 is the kernel of Frobenius on $\mathbb{G}_m$. By the Deuring-Shafarevich formula [24, Theorem 4.2] or [5, Cor. 1.8], the 2-rank of X is r. Note that $g = r + \sum_{\alpha \in B} c_\alpha$. The implication of these formulae is that, for a given genus g (and 2-rank r), there is an additional discrete invariant of the hyperelliptic k-curve X, namely a partition of $2g + 2$ into $(r + 1)$ positive even integers $d_\alpha + 1$. In Section 5.1, we show that the Ekedahl-Oort type of X depends only on this discrete invariant.

Theorem 1.2. *Suppose X is a hyperelliptic curve, defined over an algebraically closed field k of characteristic 2, with affine equation $y^2 - y = f(x)$, branch locus B, and polynomials f_α for $\alpha \in B$ as described in Notation 1.1. For $\alpha \in B$, consider the Artin-Schreier k-curve Y_α with affine equation $y^2 - y = f_\alpha(x)$. Let E be an ordinary elliptic k-curve. As a module under the actions of Frobenius F and Verschiebung V, the de Rham cohomology of X decomposes as:*

$$\mathrm{H}^1_{\mathrm{dR}}(X) \cong \mathrm{H}^1_{\mathrm{dR}}(E)^{\#B-1} \oplus \bigoplus_{\alpha \in B} \mathrm{H}^1_{\mathrm{dR}}(Y_\alpha).$$

As an application of Theorem 1.2, we give a complete classification of the Ekedahl-Oort types which occur for hyperelliptic k-curves. Recall that the 2-torsion group scheme $J_X[2]$ of the Jacobian of a k-curve is a polarized BT_1 group scheme over k (short for polarized Barsotti-Tate truncated level 1 group scheme), and that the isomorphism class of a BT_1 group scheme determines and is determined by its Ekedahl-Oort type; see Section 2 for more details. For $p = 2$ and a natural number c, let G_c be the polarized BT_1 group scheme of rank p^{2c} with Ekedahl-Oort type $[0, 1, 1, 2, 2, \ldots, \lfloor c/2 \rfloor]$. For example, G_1 is the 2-torsion group scheme of a supersingular elliptic k-curve. The group scheme G_2 occurs as the 2-torsion of a supersingular non-superspecial abelian surface over k. The group scheme G_c is not necessarily indecomposable. More explanation about G_c is given in Sections 2.3 and 5.2.

Before stating the classification result, we note that it also includes a complete description of which a-numbers occur for the Jacobians of hyperelliptic k-curves. Recall that the a-number of X is defined as $a_X := \dim_k \mathrm{Hom}(\alpha_2, J_X[2])$, where α_2 is the kernel of Frobenius on $\mathbb{G}_a$.

Theorem 1.3. *Suppose X is a hyperelliptic k-curve with affine equation $y^2 - y = f(x)$ defined over an algebraically closed field of characteristic 2 as described in Notation 1.1. Then the 2-torsion group scheme of the Jacobian variety of X is*

$$J_X[2] \simeq (\mathbb{Z}/2 \oplus \mu_2)^r \oplus \bigoplus_{\alpha \in B} G_{c_\alpha},$$

and the a-number of X is $a_X = (g + 1 - \#\{\alpha \in B \mid d_\alpha \equiv 1 \bmod 4\})/2$.

Theorem 1.3 is stated without proof in [26, 3.2] for the special case when $f(x) \in k[x]$, i.e., $r = 0$.

There are two interesting things about Theorem 1.3. First, it shows that the Ekedahl-Oort type of $X : y^2 - y = f(x)$ depends only on the orders of the poles of $f(x)$. This is in sharp contrast with the case of hyperelliptic curves in odd characteristic p, where even the p-rank depends on $f(x)$ and the location of the branch points, [27]. Similarly, it differs from the results of [3], [8], [12], all of which give bounds for the p-rank and a-number of various kinds of curves that depend strongly on the coefficients of their equations. Likewise, preliminary calculations indicate that it is in contrast with the situation for Artin-Schreier curves in odd characteristic.

Secondly, Theorem 1.3 is interesting because it shows that most of the possibilities for the 2-torsion group scheme of an abelian variety over k do not occur for Jacobians of hyperelliptic k-curves when $\mathrm{char}(k) = 2$. Specifically, there are 2^g possibilities for the 2-torsion group scheme of a g-dimensional abelian variety over k. We determine a subset of these of cardinality equal to the number $P(g+1)$ of partitions of $g+1$, and prove that the group schemes in this subset are exactly those which occur as the 2-torsion $J_X[2]$ for a hyperelliptic k-curve X of genus g. Recall [11] that $P(g+1)$ grows asymptotically like $e^{\pi\sqrt{2(g+1)/3}}/4\sqrt{3}(g+1)$ as g goes to infinity. Also, Theorem 1.3 gives the non-trivial bounds $(g - r)/2 \leq a_X \leq (g + 1)/2$ for the a-number.

An earlier non-existence result of this type can be found in [7], where the author proved that a curve X of genus $g > p(p - 1)/2$ in characteristic $p > 0$ cannot be superspecial, and thus $a_X < g$. There are also other recent results about Newton polygons of hyperelliptic (i.e., Artin-Schreier) curves in characteristic 2, including several non-existence results, [1], [2], [21]. In addition, there are closed formulae for the number of hyperelliptic curves of genus 3 with given 2-rank over each finite field of characteristic 2 [17].

Here is an outline of this paper: Section 2 contains notation and background. Results on $\mathrm{H}^0(X, \Omega^1)$ and the a-number are in Section 3. Theorem 1.2 is with the material on the de Rham cohomology in Section 4. Section 5 contains the results about the Ekedahl-Oort type, including Theorem 1.3.

2. Background

In this paper, all objects are defined over an algebraically closed field k of characteristic $p > 0$ and all curves are smooth, projective, and connected. This section includes background on p-torsion group schemes, Ekedahl-Oort types, the de Rham cohomology, and Frobenius and Verschiebung.

2.1. **The p-torsion group scheme.** Suppose A is a principally polarized abelian variety of dimension g defined over k. For example, A could be the Jacobian of a k-curve of genus g. Consider the multiplication-by-p morphism $[p] : A \to A$ which is a finite flat morphism of degree p^{2g}. It factors as $[p] = V \circ F$. Here $F : A \to A^{(p)}$ is the relative Frobenius morphism coming from the p-power map on the structure sheaf; it is purely inseparable of degree p^g. Furthermore, $V : A^{(p)} \to A$ is the Verschiebung morphism.

The *p-torsion group scheme* of A, denoted $A[p]$, is the kernel of $[p]$. It is a finite commutative group scheme annihilated by p, again having morphisms F and V. By [19, 9.5], the p-torsion group scheme $A[p]$ is a polarized BT_1 group scheme over k (short for polarized Barsotti-Tate truncated level 1 group scheme), as defined in [19, 2.1, 9.2]. The rank of $A[p]$ is p^{2g}.

We now give a brief summary of the classification [19, Theorem 9.4 &12.3] of polarized BT_1 group schemes over k in terms of Dieudonné modules and Ekedahl-Oort type; other useful references are [13] (unpublished - without polarization) and [16] (for $p \geq 3$).

2.2. **The Dieudonné module and polarizations.** It is useful to describe the group scheme $A[p]$ using (the modulo p reduction of) the *covariant Dieudonné module*, see e.g., [19, 15.3]. This is the dual of the contravariant theory found in [6]. Briefly, consider the non-commutative ring $\mathbb{E} = k[F, V]$ generated by semi-linear operators F and V with the relations $FV = VF = 0$ and $F\lambda = \lambda^p F$ and $\lambda V = V\lambda^p$ for all $\lambda \in k$. Let $\mathbb{E}(A, B)$ denote the left ideal $\mathbb{E}A + \mathbb{E}B$ of $\mathbb{E}$ generated by A and B. A deep result is that the Dieudonné functor D gives an equivalence of categories between BT_1 group schemes over k (with rank p^{2g}) and finite left $\mathbb{E}$-modules (having dimension $2g$ as a k-vector space). We use the notation $D(\mathbb{G})$ to denote the Dieudonné module of $\mathbb{G}$. For example, the Dieudonné module of the p-torsion group scheme of an ordinary elliptic curve is $D(\mathbb{Z}/p \oplus \mu_p) \simeq \mathbb{E}/\mathbb{E}(F, 1-V) \oplus \mathbb{E}/\mathbb{E}(V, 1-F)$, [9, Ex. A.5.1 & 5.3].

The polarization of A induces a symmetry on $A[p]$ as defined in [19, 5.1], namely an anti-symmetric isomorphism from $A[p]$ to the Cartier dual group scheme $A[p]^{\text{dual}}$ of $A[p]$. Unfortunately, in characteristic 2, there may be anti-symmetric morphisms $A[p] \to A[p]^{\text{dual}}$ which do not come from a polarization. Luckily, this issue can be resolved by defining a polarization on $A[p]$ in terms of a non-degenerate alternating pairing on $D(A[p])$, [19, 9.2, 9.5, 12.2].

2.3. **The Ekedahl-Oort type.** As in [19, Sections 5 & 9], the isomorphism type of a BT_1 group scheme $\mathbb{G}$ over k can be encapsulated into combinatorial data. If $\mathbb{G}$ is symmetric with rank p^{2g}, then there is a *final filtration* $N_1 \subset N_2 \subset \cdots \subset N_{2g}$ of $\mathbb{G}$ as a k-vector space which is stable under the action of V and F^{-1} such that $i = \dim(N_i)$, [19, 5.4]. If w is a word in V and F^{-1}, then $wD(\mathbb{G})$ is an object in the filtration; in particular, $N_g = VD(\mathbb{G}) = F^{-1}(0)$.

The *Ekedahl-Oort type* of $\mathbb{G}$, also called the *final type*, is $\nu = [\nu_1, \ldots, \nu_g]$ where $\nu_i = \dim(V(N_i))$. The Ekedahl-Oort type of $\mathbb{G}$ does not depend on the choice of a final filtration. There is a restriction $\nu_i \leq \nu_{i+1} \leq \nu_i + 1$ on the final type. There are 2^g Ekedahl-Oort types of length g since all sequences satisfying this restriction occur. By [19, 9.4, 12.3], there are bijections between (i) Ekedahl-Oort types of length g; (ii) polarized BT_1 group schemes over k of rank p^{2g}; and (iii) principal quasi-polarized Dieudonné modules of dimension $2g$ over k.

2.4. **The p-rank and a-number.** Two invariants of (the p-torsion of) an abelian variety are the p-rank and a-number. The *p-rank* of A is $r = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, A[p])$ where μ_p is the kernel of Frobenius on $\mathbb{G}_m$. Then p^r is the cardinality of $A[p](k)$. The *a-number* of A is $a = \dim_k \mathrm{Hom}(\alpha_p, A[p])$ where α_p is the kernel of Frobenius on $\mathbb{G}_a$. It is well-known that $0 \leq f \leq g$ and $1 \leq a + f \leq g$. The p-rank of $A[p]$ equals the dimension of $V^g D(\mathbb{G})$. The a-number of $A[p]$ equals $g - \dim(V^2 D(\mathbb{G}))$ [14, 5.2.8]. The p-rank equals $\max\{i \mid \nu_i = i\}$ and the a-number equals $g - \nu_g$.

2.5. **The de Rham cohomology.** Suppose X is a k-curve of genus g and recall the definition of the non-commutative ring $\mathbb{E} = k[F, V]$ from Section 2.2. By [18, Section 5], there is an isomorphism of $\mathbb{E}$-modules between the Dieudonné module of the p-torsion group scheme $J_X[p]$ and the de Rham cohomology group $\mathrm{H}^1_{\mathrm{dR}}(X)$. In particular, $\ker(F) = \mathrm{H}^0(X, \Omega^1) = \mathrm{im}(V)$. Recall that $\dim_k \mathrm{H}^1_{\mathrm{dR}}(X) = 2g$.

In [18, Section 5], there is the following description of $\mathrm{H}^1_{\mathrm{dR}}(X)$. Let $\mathcal{U} = \{U_i\}$ be a covering of X by affine open subvarieties and let $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$. For a sheaf $\mathcal{F}$ on X, let

$$\begin{aligned} \mathrm{C}^0(\mathcal{U},\mathcal{F}) &:= \{\kappa = (\kappa_i)_i \mid \kappa_i \in \Gamma(U_i,\mathcal{F})\}, \\ \mathrm{C}^1(\mathcal{U},\mathcal{F}) &:= \{\phi = (\phi_{ij})_{i<j} \mid \phi_{ij} \in \Gamma(U_{ij},\mathcal{F})\}, \\ \mathrm{C}^2(\mathcal{U},\mathcal{F}) &:= \{\psi = (\psi_{ijk})_{i<j<k} \mid \psi_{ijk} \in \Gamma(U_{ijk},\mathcal{F})\}. \end{aligned}$$

For convenience, let $\phi_{ii} := 0$ for any $\phi \in \mathrm{C}^1(\mathcal{U},\mathcal{F})$. There are coboundary operators $\delta : \mathrm{C}^0(\mathcal{U},\mathcal{F}) \to \mathrm{C}^1(\mathcal{U},\mathcal{F})$ defined by $(\delta\kappa)_{i<j} = \kappa_i - \kappa_j$; and $\delta : \mathrm{C}^1(\mathcal{U},\mathcal{F}) \to \mathrm{C}^2(\mathcal{U},\mathcal{F})$ by $(\delta\phi)_{i<j<k} = \phi_{ij} - \phi_{ik} + \phi_{jk}$. All other maps are applied to $\mathrm{C}^m(\mathcal{U},\mathcal{F})$ elementwise, e.g., $(F\phi)_i := F\phi_i$. As expected, $\delta^2 = 0$.

The de Rham cocycles are defined by

$$\mathrm{Z}^1_{\mathrm{dR}}(\mathcal{U}) := \left\{(\phi,\omega) \in \mathrm{C}^1(\mathcal{U},\mathcal{O}) \times \mathrm{C}^0(\mathcal{U},\Omega^1) \mid \delta\phi = 0,\ d\phi = \delta\omega\right\},$$

that is, $\phi_{ij} - \phi_{ik} + \phi_{jk} = 0$ and $d\phi_{ij} = \omega_i - \omega_j$ for all indices $i < j < k$. The de Rham coboundaries are defined by

$$\mathrm{B}^1_{\mathrm{dR}}(\mathcal{U}) := \left\{(\delta\kappa, d\kappa) \in \mathrm{Z}^1_{\mathrm{dR}}(\mathcal{U}) \mid \kappa \in C^0(\mathcal{U},\mathcal{O})\right\}.$$

Finally,

$$\mathrm{H}^1_{\mathrm{dR}}(X) \cong \mathrm{H}^1_{\mathrm{dR}}(\mathcal{U}) := \mathrm{Z}^1_{\mathrm{dR}}(\mathcal{U})/\mathrm{B}^1_{\mathrm{dR}}(\mathcal{U}).$$

There is an injective homomorphism $\lambda : \mathrm{H}^0(X,\Omega^1) \to \mathrm{H}^1_{\mathrm{dR}}(X)$ denoted informally by $\omega \mapsto (0,\omega)$ where the second coordinate is defined by $\omega_i = \omega|_{U_i}$. This map is well-defined since $d(0) = \omega|_{U_i} - \omega|_{U_j} = (\delta\omega)_{i<j}$. It is injective because, if $(0,\omega_1) \equiv (0,\omega_2) \bmod \mathrm{B}^1_{\mathrm{dR}}(\mathcal{U})$, then $\omega_1 - \omega_2 = d\kappa$ where $\kappa \in C^0(\mathcal{U},\mathcal{O})$ is such that $\delta\kappa = 0$; thus $\kappa \in \mathrm{H}^0(\mathcal{U},\mathcal{O}) \simeq k$ is a constant function on X and so $\omega_1 - \omega_2 = 0$.

There is another homomorphism $\gamma : \mathrm{H}^1_{\mathrm{dR}}(X) \to \mathrm{H}^1(X,\mathcal{O})$ sending the cohomology class of (ϕ,ω) to the cohomology class of ϕ. The choice of cocycle (ϕ,ω) does not matter, since the coboundary conditions on $\mathrm{H}^1_{\mathrm{dR}}(X)$ and $\mathrm{H}^1(X,\mathcal{O})$ are compatible. The homomorphisms λ and γ fit into a short exact sequence

$$0 \to \mathrm{H}^0(X,\Omega^1) \xrightarrow{\lambda} \mathrm{H}^1_{\mathrm{dR}}(X) \xrightarrow{\gamma} \mathrm{H}^1(X,\mathcal{O}) \to 0 \tag{2.1}$$

of k-vector spaces. In Subsections 4.4 and 4.6, we construct a suitable section $\sigma : \mathrm{H}^1(X,\mathcal{O}) \to \mathrm{H}^1_{\mathrm{dR}}(X)$ of γ when X is a hyperelliptic k-curve with $\mathrm{char}(k) = 2$.

2.6. **Frobenius and Verschiebung.** The Cartier operator $\mathscr{C}$ on the sheaf Ω^1 is defined in [4]. Its three principal properties are that it annihilates exact differentials, preserves logarithmic ones, and induces a p^{-1}-linear map on $\mathrm{H}^0(X,\Omega^1)$. The Cartier operator can be computed as follows. Let $x \in k(X)$ be an element which forms a p-basis of $k(X)$ over $k(X)^p$, i.e., an element such that every $z \in k(X)$ can be written as

$$z := z_0^p + z_1^p x + \cdots + z_{p-1}^p x^{p-1}$$

for uniquely determined $z_0, \ldots, z_{p-1} \in k(X)$. Then

$$\mathscr{C}(z\,dx/x) := z_0\,dx/x.$$

The Frobenius operator F on the structure sheaf $\mathcal{O}$ of X induces a p-linear map F on $\mathrm{H}^1(X,\mathcal{O})$. By Serre duality, the $k[F]$-module $\mathrm{H}^1(X,\mathcal{O})$ is dual to the $k[\mathscr{C}]$-module $\mathrm{H}^0(X,\Omega^1)$.

The p-linear operator F and the p^{-1}-linear operator V are defined on $\mathrm{H}^1_{\mathrm{dR}}(X)$ as follows. Let $V(\omega) := \mathscr{C}(\omega)$ and $F(\omega) := 0$ for $\omega \in \mathrm{H}^0(X,\Omega^1)$ and $V(f) := 0$ for $f \in \mathrm{H}^1(X,\mathcal{O})$. Then

$$F(f,\omega) := (F(f),F(\omega)) = (f^p,0) \quad \text{and} \quad V(f,\omega) := (V(f),V(\omega)) = (0,\mathscr{C}(\omega)).$$

The short exact sequence (2.1) is an exact sequence of $\mathbb{E}$-modules, where $\mathbb{E} = k[F,V]$ is defined in Section 2.3. However, the section σ of (2.1) constructed in Section 4.4 is not a splitting of $\mathbb{E}$-modules.

3. Results about regular 1-forms and the a-number

We specialize to the case when the algebraically closed field k has characteristic $p = 2$. Consider a hyperelliptic k-curve X with affine equation $y^2 - y = f(x)$ as described in Section 1. For each branch point $\alpha \in B$, recall the definitions of the ramification invariant $d_\alpha = 2c_\alpha + 1$, the function $x_\alpha = (x-\alpha)^{-1}$, and the polynomial $f_\alpha(x_\alpha)$ appearing in the partial fraction decomposition of $f(x)$. Important facts mentioned in Section 1 are that the genus is determined from the ramification invariants by the formula $2g + 2 = \sum_{\alpha \in B}(d_\alpha + 1)$ and that the 2-rank of J_X equals $r = \#B - 1$.

For $\alpha \in B$, let $P_\alpha := \pi^{-1}(\alpha) \in X(k)$ be the ramification point above α, and define the divisor $D_\infty := \pi^{-1}(\infty)$ on X. Recall that $0 \in B$ and $\infty \notin B$, and let $B_\infty := B \cup \{\infty\}$ and $B' := B - \{0\}$.

3.1. The space $\mathrm{H}^0(X, \Omega^1)$.

For an integer j and for $\alpha \in B$, consider the following 1-forms on X:

$$\omega_{\alpha,j} := x_\alpha^{j-1}\, dx_\alpha.$$

Note that $\omega_{\alpha,j} = -(x-\alpha)^{-j-1}dx$ and, if $\alpha \in B'$, then $\omega_{\alpha,0} - \omega_{0,0} = -\alpha dx/x(x-\alpha)$.

For completeness, we prove the next lemma, a variation of a special case of [25, Lemma 1(c)].

Lemma 3.1. *A basis for $\mathrm{H}^0(X, \Omega^1)$ is given by the 1-forms $\omega_{\alpha,j}$ for $\alpha \in B$ and $1 \le j \le c_\alpha$ and $\omega_{\alpha,0} - \omega_{0,0}$ for $\alpha \in B'$.*

Proof. For $\alpha \in B$, one can calculate the following divisors on X: $\mathrm{div}(x_\alpha) = D_\infty - 2P_\alpha$ and

$$\mathrm{div}(dx_\alpha) = (d_\alpha - 3)P_\alpha + \sum_{\beta \in B - \{\alpha\}} (d_\beta + 1)P_\beta \tag{3.1}$$

and

$$\mathrm{div}(\omega_{\alpha,j}) = 2(c_\alpha - j)P_\alpha + (j-1)D_\infty + \sum_{\beta \in B - \{\alpha\}} (d_\beta + 1)P_\beta. \tag{3.2}$$

Thus $\omega_{\alpha,j}$ is regular for $1 \le j \le c_\alpha$. Also, $\omega_{\alpha,0} - \omega_{0,0}$ is regular for $\alpha \in B'$ because

$$\mathrm{div}(\omega_{\alpha,0} - \omega_{0,0}) = 2c_\alpha P_\alpha + 2c_0 P_0 + \sum_{\beta \in B - \{0,\alpha\}} (d_\beta + 1)P_\beta.$$

This set of regular differentials of X is linearly independent because the corresponding set of divisors is linearly independent over $\mathbb{Z}$. It forms a basis since the set has cardinality $r + \sum_{\alpha \in B} c_\alpha = g$. □

Lemma 3.2. *If $\alpha \in B$, then*

$$\mathscr{C}(\omega_{\alpha,j}) = \begin{cases} \omega_{\alpha,j/2} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$$

In particular, $\mathscr{C}(\omega_{\alpha,0} - \omega_{0,0}) = \omega_{\alpha,0} - \omega_{0,0}$ for all $\alpha \in B'$.

Proof. Using the properties of the Cartier operator found in Section 2.6, one computes when j is even that

$$\mathscr{C}\left(x_\alpha^{j-1}\, dx_\alpha\right) = x_\alpha^{j/2}\mathscr{C}\left(dx_\alpha/x_\alpha\right) = x_\alpha^{j/2-1}\, dx_\alpha,$$

and when j is odd that

$$\mathscr{C}\left(x_\alpha^{j-1}\, dx_\alpha\right) = x_\alpha^{(j-1)/2}\mathscr{C}(dx_\alpha) = 0.$$

□

For $\alpha \in B'$, let $W'_{\alpha,\mathrm{ss}} := \langle \omega_{\alpha,0} - \omega_{0,0} \rangle$, and for $\alpha \in B$, let $W'_{\alpha,\mathrm{nil}} := \langle \omega_{\alpha,j} \mid 1 \le j \le c_\alpha \rangle$, where $\langle \cdot \rangle$ denotes the k-span. These subspaces are invariant under the Cartier operator by Lemma 3.2.

Lemma 3.3. *The subspaces $W'_{\alpha,\mathrm{ss}}$ and $W'_{\alpha,\mathrm{nil}}$ of $\mathrm{H}^0(X,\Omega^1)$ are stable under the action of Verschiebung for each $\alpha \in B$. There is an isomorphism of V-modules:*

$$\mathrm{H}^0(X,\Omega^1) \simeq \bigoplus_{\alpha\in B'} W'_{\alpha,\mathrm{ss}} \oplus \bigoplus_{\alpha \in B} W'_{\alpha,\mathrm{nil}}.$$

Proof. This follows immediately from Lemmas 3.1 and 3.2. □

3.2. **Application: the a-number.**

Proposition 3.4. *Suppose X is a hyperelliptic k-curve with affine equation $y^2 - y = f(x)$ as described in Notation 1.1. If $\mathrm{div}_\infty(f(x)) = \sum_{\alpha\in B} d_\alpha \alpha$ is the divisor of poles of $f(x)$ on $\mathbb{P}^1$, then the a-number of X is*

$$a_X = \frac{g+1-\#\{\alpha \in B \mid d_\alpha \equiv 1 \bmod 4\}}{2}.$$

Proof. The a-number of $\mathbb{G} = J_X[2]$ is $a_X = g - \dim(V^2 D(\mathbb{G}))$, see [14, 5.2.8]. The action of V on $VD(\mathbb{G})$ is the same as the action of the Cartier operator $\mathscr{C}$ on $\mathrm{H}^0(X,\Omega^1)$. So a_X equals the dimension of the kernel of $\mathscr{C}$ on $\mathrm{H}^0(X,\Omega^1)$. By Lemma 3.2, the kernel of $\mathscr{C}$ on $\mathrm{H}^0(X,\Omega^1)$ is spanned by $\omega_{\alpha,j}$ for $\alpha \in B$ and j odd with $1 \le j \le c_\alpha = (d_\alpha - 1)/2$. Thus the contribution to the a-number from each $\alpha \in B$ is $\lfloor (d_\alpha+1)/4 \rfloor$. In other words, if $d_\alpha \equiv 1 \bmod 4$, the contribution is $(d_\alpha - 1)/4$ and if $d_\alpha \equiv 3 \bmod 4$, the contribution is $(d_\alpha+1)/4$. Since $g+1 = \sum_{\alpha\in B}(d_\alpha+1)/2$, this yields

$$2a_X = (g+1) - \#\{\alpha \in B \mid d_\alpha \equiv 1 \bmod 4\}.$$

□

3.3. **Examples with large p-rank.** Let A be a principally polarized abelian variety over k with dimension g and p-rank r. If $r = g$, then $A[p] \simeq (\mathbb{Z}/p \oplus \mu_p)^g$ and the a-number is $a = 0$. If $r = g-1$ then $A[p] \simeq (\mathbb{Z}/p \oplus \mu_p)^{g-1} \oplus E[p]$ where E is a supersingular elliptic curve and the a-number is $a = 1$. So the first case where $A[p]$ and a are not determined by the p-rank is when $r = g-2$.

Example 3.5. Let $g \ge 2$. There are two possibilities for the p-torsion group scheme of a principally polarized abelian variety over k with dimension g and p-rank $g-2$. When $p = 2$, both of these occur as the 2-torsion group scheme $J_X[2]$ of the Jacobian of a hyperelliptic k-curve X of genus g.

Proof. If A is a principally polarized abelian variety over k with dimension g and p-rank $g-2$, then $A[p] \simeq (\mu_p \oplus \mathbb{Z}/p)^{g-2} \oplus \mathbb{G}$ where $\mathbb{G}$ is isomorphic to the p-torsion group scheme of an abelian surface Z with p-rank 0. The abelian surface can be superspecial or merely supersingular. In the superspecial case, $\mathbb{G} = (G_1)^2$, where G_1 denotes the p-torsion group scheme of a supersingular elliptic k-curve; in the merely supersingular case, we denote the group scheme by G_2, see [9, Ex. A.3.15] or [20, Ex. 2.3] for a complete description of G_2.

To prove the second claim, consider the two possibilities for a partition of $2g+2$ into $r+1 = g-1$ even integers, namely (A) $\{2,2,\ldots,2,4,4\}$ or (B) $\{2,2,\ldots,2,2,6\}$. In case (A), consider $f(x) \in k(x)$ with $g-1$ poles, such that 0 and 1 are poles of order 3 and the other poles are simple. In case (B), consider $f(x) \in k(x)$ with $g-1$ poles, such that 0 is a pole of order 5 and the other poles are simple. The kernel of the Cartier operator on $\mathrm{H}^0(X,\Omega^1)$ is spanned by dx/x^2 and $dx/(x-1)^2$ in case (A) and by dx/x^2 in case (B). Thus the a-number equals 2 in case (A) and equals 1 in case (B). In both cases, this completely determines the group scheme. Namely, the group scheme $J_X[2]$ is isomorphic to $(\mathbb{Z}/2 \oplus \mu_2)^{g-2} \oplus (G_1)^2$ in case (A) and to $(\mathbb{Z}/2 \oplus \mu_2)^{g-2} \oplus G_2$ in case (B). □

For $g \ge 3$ and $r \le g-3$, the action of V on $\mathrm{H}^0(X,\Omega^1)$ (and, in particular, the value of the a-number) is not sufficient to determine the isomorphism class of the group scheme $J_X[2]$. To determine this group scheme, in the next section we study the $\mathbb{E}$-module structure of $\mathrm{H}^1_{\mathrm{dR}}(X)$.

4. Results on the de Rham cohomology

4.1. **An open covering.** Let $V' = \mathbb{P}^1 - B_\infty$ and $U' = \pi^{-1}(V') = X - \pi^{-1}(B_\infty)$. For $\alpha \in B_\infty$, let $V_\alpha = V' \cup \{\alpha\}$ and $U_\alpha = U' \cup \{\pi^{-1}(\alpha)\}$. The collection $\mathcal{U} := \{U_\alpha \mid \alpha \in B_\infty\}$ is a cover of X by open affine subvarieties. By construction, if $\alpha, \beta \in B_\infty$ are distinct, then $V_{\alpha\beta} := V_\alpha \cap V_\beta = V'$ and $U_{\alpha\beta} := U_\alpha \cap U_\beta = U'$. In particular, the subvarieties $U_{\alpha\beta}$ do not depend on the choice of α and β.

For a sheaf $\mathcal{F}$, let $\mathrm{Z}^1(\mathcal{U}, \mathcal{F})$ and $\mathrm{B}^1(\mathcal{U}, \mathcal{F})$ denote the closed cocycles and coboundaries of $\mathcal{F}$ with respect to $\mathcal{U}$. Recall the definition of the non-commutative ring $\mathbb{E} = k[F, V]$ and the notation about $\mathrm{H}^1_{\mathrm{dR}}(X)$ from Section 2.5. In this section, we compute $\mathrm{H}^1(X, \mathcal{O}) \simeq \mathrm{H}^1(\mathcal{U}, \mathcal{O})$ and $\mathrm{H}^1_{\mathrm{dR}}(X) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathcal{U})$ with respect to the open covering $\mathcal{U}$ of X.

4.2. **Defining components.** Given a sheaf $\mathcal{F}$ and a cocycle $\phi \in \mathrm{Z}^1(\mathcal{U}, \mathcal{F})$, consider its components $\phi_{\alpha\infty} \in \Gamma(U', \mathcal{F})$ for $\alpha \in B$. We call $\{\phi_{\alpha\infty} \mid \alpha \in B\}$ the set of *defining components* of ϕ. The reason is that the remaining components of ϕ are determined by the coboundary condition $\phi_{\alpha\beta} = \phi_{\alpha\infty} - \phi_{\beta\infty}$. A collection of sections $\{\phi_{\alpha\infty} \in \Gamma(U', \mathcal{F}) \mid \alpha \in B\}$ determines a unique closed cocycle $\phi \in \mathrm{Z}^1(\mathcal{U}, \mathcal{F})$. Thus,

$$\mathrm{Z}^1(\mathcal{U}, \mathcal{F}) \cong \bigoplus_{\alpha \in B} \Gamma(U', \mathcal{F}). \tag{4.1}$$

For $\beta \in B$, consider the natural k-linear map

$$\varphi_\beta : \Gamma(U', \mathcal{O}) \to \mathrm{Z}^1(\mathcal{U}, \mathcal{O}),$$

whose defining components for $\alpha \in B$ are

$$(\varphi_\beta(h))_{\alpha\infty} := \begin{cases} h & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Also, consider the k-linear map $\varphi_\infty : \Gamma(U', \mathcal{O}) \to \mathrm{Z}^1(\mathcal{U}, \mathcal{O})$ defined by:

$$(\varphi_\infty(h))_{\alpha\infty} := -h \quad \text{for all } \alpha \in B.$$

Observe that if $h \in \Gamma(U', \mathcal{O})$, then

$$\sum_{\beta \in B_\infty} \varphi_\beta(h) = 0. \tag{4.2}$$

For $\beta \in B_\infty$, consider the natural k-linear map

$$\psi_\beta : \Gamma(U_\beta, \mathcal{O}) \to \mathrm{C}^0(\mathcal{U}, \mathcal{O})$$

given for $\alpha \in B_\infty$ by

$$(\psi_\beta(h))_\alpha := \begin{cases} h & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

It is straightforward to verify the next lemma.

Lemma 4.1. *Suppose $\beta \in B_\infty$ and $h \in \Gamma(U_\beta, \mathcal{O})$ (i.e., h is regular at P_β if $\beta \neq \infty$ and h is regular at the two points in the support of D_∞ if $\beta = \infty$). Then $\varphi_\beta(h|_{U'}) = \delta\psi_\beta(h)$ is a coboundary.*

4.3. **The space** $\mathrm{H}^1(X, \mathcal{O})$**.** In this section, we find an F-module decomposition of $\mathrm{H}^1(X, \mathcal{O}) \simeq \mathrm{H}^1(\mathcal{U}, \mathcal{O})$. The results could be deduced from Section 3.1 using the duality between $\mathrm{H}^1(X, \mathcal{O})$ and $\mathrm{H}^0(X, \Omega^1)$. Instead, we take a direct approach, because an explicit description of $\mathrm{H}^1(X, \mathcal{O})$ is helpful for studying $\mathrm{H}^1_{\mathrm{dR}}(X)$ in Section 4.6. The following lemmas will be useful.

Lemma 4.2. *Write $D_\infty = P_{\infty,1} + P_{\infty,2}$. Then $\mathrm{ord}_{P_{\infty,1}}(y) = 0$ and $\mathrm{ord}_{P_{\infty,2}}(y) = s$ for some $s \geq 0$ (possibly after reordering). For $\alpha \in B$ and $j \in \mathbb{Z}$, the divisor of poles on X of the function $yx_\alpha^{-j} = y(x-\alpha)^j$ satisfies*

$$\mathrm{div}_\infty(y(x-\alpha)^j) = \max(d_\alpha - 2j, 0)P_\alpha + \max(j,0)P_{\infty,1} + \max(j-s,0)P_{\infty,2} + \sum_{\beta \in B - \{\alpha\}} d_\beta P_\beta.$$

Proof. Recall that $\mathrm{div}_\infty(y) = \sum_{\beta \in B} d_\beta P_\beta$. Note that $\mathrm{ord}_{P_{\infty,i}}(y) \geq 0$ for $i = 1, 2$ since $\infty \notin B$. If $\mathrm{ord}_{P_{\infty,2}}(y) > 0$, i.e., if y has a zero at $P_{\infty,2}$, then the value of y is one at the Galois conjugate $P_{\infty,1}$ of $P_{\infty,2}$. Thus y cannot have a zero at both points in the support of D_∞. The second claim follows from the additional fact that $\mathrm{div}(x-\alpha) = 2P_\alpha - D_\infty$ for $\alpha \in B$. □

Lemma 4.2 implies that $y(x-\alpha)^j \in \Gamma(U', \mathcal{O})$ for all $\alpha \in B$ and $j \in \mathbb{Z}$.

Lemma 4.3. *With notation as above:*

(i) $\mathrm{Z}^1(\mathcal{U}, \mathcal{O}) = \langle \varphi_\beta((x-\alpha)^j), \varphi_\beta(y(x-\alpha)^j) \mid \alpha, \beta \in B,\ j \in \mathbb{Z} \rangle$.

(ii) *If $\alpha \in B$, then $\langle \varphi_\alpha(y(x-\beta)^j) \mid j \geq 0 \rangle = \langle \varphi_\alpha(y(x-\alpha)^j) \mid j \geq 0 \rangle$ as subspaces of $\mathrm{Z}^1(\mathcal{U}, \mathcal{O})$ for each $\beta \in B$.*

Proof. (i) This is immediate from Equation (4.1) because

$$\mathrm{Z}^1(\mathcal{U}, \mathcal{O}) = \bigoplus_{\beta \in B} \langle \varphi_\beta(h) \mid h \in \Gamma(U', \mathcal{O}) \rangle.$$

(ii) Both are equal to the subspace $\{\varphi_\alpha(yh(x)) \mid h(x) \in k[x]\}$.

□

Lemma 4.4. *Let $\alpha \in B \subset k$ and $j \in \mathbb{Z}$. Then:*

(i) *$\varphi_\beta((x-\alpha)^j) \in \mathrm{B}^1(\mathcal{U}, \mathcal{O})$ for all $\beta \in B_\infty$.*

(ii) *$\varphi_\alpha(y(x-\alpha)^j) \in \mathrm{B}^1(\mathcal{U}, \mathcal{O})$ if $j > c_\alpha$.*

(iii) *$\varphi_\infty(y(x-\alpha)^j) \in \mathrm{B}^1(\mathcal{U}, \mathcal{O})$ if $j \leq 0$.*

Proof. (i) Suppose that $\beta \in B$. If $\beta \neq \alpha$ or if $j \geq 0$, then $(x-\alpha)^j$ is regular at P_β and so $\varphi_\beta((x-\alpha)^j) \in \mathrm{B}^1(\mathcal{U}, \mathcal{O})$ by Lemma 4.1. For $j \geq 0$, it follows from this and Equation (4.2) that the cocycle $\varphi_\infty((x-\alpha)^j) = -\sum_{\beta \in B} \varphi_\beta((x-\alpha)^j)$ is a coboundary. If $j < 0$, then $\varphi_\infty((x-\alpha)^j) \in \mathrm{B}^1(\mathcal{U}, \mathcal{O})$ by Lemma 4.1.

Finally, if $\beta = \alpha \neq \infty$ and $j < 0$, then $(x-\alpha)^j \in \Gamma(U_\gamma, \mathcal{O})$ for all $\gamma \in B_\infty - \{\alpha\}$. By Equation (4.2),

$$\varphi_\alpha((x-\alpha)^j) = -\sum_{\gamma \in B_\infty - \{\alpha\}} \varphi_\gamma\big((x-\alpha)^j\big) = -\sum_{\gamma \in B_\infty - \{\alpha\}} \delta\psi_\gamma((x-\alpha)^j), \tag{4.4}$$

which is a coboundary.

(ii) If $j > c_\alpha$, then $y(x-\alpha)^j \in \Gamma(U_\alpha, \mathcal{O})$ and $\varphi_\alpha(y(x-\alpha)^j) = \delta\psi_\alpha(y(x-\alpha)^j)$.

(iii) If $j \leq 0$, then $y(x-\alpha)^j \in \Gamma(U_\infty, \mathcal{O})$, and $\varphi_\infty(y(x-\alpha)^j) = \delta\psi_\infty(y(x-\alpha)^j)$.

□

Consider the cocycles $\phi_{\alpha,j} \in \mathrm{Z}^1(\mathcal{U}, \mathcal{O})$ for $\alpha \in B$ and $j \in \mathbb{Z}$ defined by

$$\phi_{\alpha,j} := \varphi_\alpha(y(x-\alpha)^j).$$

Given $\phi \in \mathrm{Z}^1(\mathcal{U}, \mathcal{O})$, denote by $\tilde{\phi}$ the cohomology class of ϕ in $\mathrm{H}^1(\mathcal{U}, \mathcal{O})$. For $\alpha \in B_\infty$, define the map

$$\tilde{\varphi}_\alpha : \Gamma(U', \mathcal{O}) \to \mathrm{H}^1(\mathcal{U}, \mathcal{O}), \quad f \mapsto \varphi_\alpha(f) \bmod \mathrm{B}^1(\mathcal{U}, \mathcal{O}).$$

We now study $\mathrm{H}^1(\mathcal{U}, \mathcal{O})$; the following lemma is a variant of a special case of [15, Lemma 6].

Lemma 4.5. *A basis for* $\mathrm{H}^1(\mathcal{U},\mathcal{O})$ *is given by the cohomology classes* $\tilde{\phi}_{\alpha,j}$ *for* $\alpha \in B$ *and* $1 \le j \le c_\alpha$, *and* $\tilde{\phi}_{\alpha,0}$ *for* $\alpha \in B'$.

Proof. The set of cohomology classes $S = \{\tilde{\phi}_{\alpha,j} \mid \alpha \in B,\ 1 \le j \le c_\alpha\} \cup \{\tilde{\phi}_{\alpha,0} \mid \alpha \in B'\}$ has cardinality $r + \sum_{\alpha \in B} c_\alpha = g$. By Lemmas 4.3(i) and 4.4(i), it suffices to show that $\varphi_\beta(y(x-\alpha)^j)$ is in the span of S for $\alpha, \beta \in B$ and $j \in \mathbb{Z}$. By Lemmas 4.3(ii) and 4.4(ii), it suffices to show that the span of S contains $\tilde{\phi}_{0,0}$ and $\tilde{\varphi}_\beta(y(x-\alpha)^{-j})$ for $\alpha, \beta \in B$ and $j > 0$.

The cocycle $\varphi_\infty(y)$ is a coboundary by Lemmas 4.1 and 4.2. Using this and Equation (4.2), one computes in $\mathrm{H}^1(\mathcal{U},\mathcal{O})$ that

$$\tilde{\phi}_{0,0} = \tilde{\varphi}_0(y) + \tilde{\varphi}_\infty(y) = -\sum_{\beta \in B'} \tilde{\varphi}_\beta(y) = -\sum_{\beta \in B'} \tilde{\phi}_{\beta,0},$$

which is in the span of S.

Now consider $\tilde{\varphi}_\beta(y(x-\alpha)^{-j})$ for $\alpha, \beta \in B$ and $j > 0$. If $0 = r := \#B - 1$, then this cocycle is a coboundary by Equation (4.2) and Lemma 4.4(iii).

Let $r > 0$ and first suppose that $\alpha \ne \beta$. Consider the rational function $h = (x-\alpha)^{-j}$ which has no pole at β. Write $h = T + E$ where T is the degree c_β Taylor polynomial of h at β. Then $\varphi_\beta(yh) = \varphi(yT) + \varphi(yE)$. Note that the function E on $\mathbb{P}^1$ has a zero at β of order at least $c_\beta + 1$. Recall that $\mathrm{ord}_{P_\beta}(x-\beta) = 2$ and observe that $\mathrm{ord}_{P_\beta}(E) \ge 2(c_\beta+1) = d_\beta + 1$ on X. Since $\mathrm{ord}_{P_\beta}(y) = -d_\beta$, it follows that $yE \in \Gamma(U_\beta, \mathcal{O})$ and thus $\varphi_\beta(yE) \in \mathrm{B}^1(\mathcal{U},\mathcal{O})$ by Lemma 4.1. The term $\varphi_\beta(yT)$ is, by construction, a linear combination of $\varphi_\beta(y(x-\beta)^j) = \phi_{\beta,j}$ for $0 \le j \le c_\beta$. Thus $\tilde{\varphi}_\beta(yh)$ is in the span of S, which completes the case when $\alpha \ne \beta$.

If $\alpha = \beta$ and $j > 0$, one can reduce to the previous case by adding the coboundary $\varphi_\infty(y(x-\alpha)^{-j})$ to $\varphi_\alpha(y(x-\alpha)^{-j})$ and using Equation (4.2) to see that

$$\tilde{\varphi}_\alpha(y(x-\alpha)^{-j}) = -\sum_{\gamma \in B - \{\alpha\}} \tilde{\varphi}_\gamma(y(x-\alpha)^{-j}).$$

□

The next lemma is important for describing the F-module structure of $\mathrm{H}^1(\mathcal{U},\mathcal{O})$.

Lemma 4.6. *If* $\alpha \in B$ *and* $j \ge 0$, *then*

$$F\tilde{\phi}_{\alpha,j} = \begin{cases} \tilde{\phi}_{\alpha,2j} & \text{if } 2j \le c_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $(F\phi_{\alpha,j})_{\beta\gamma} = (\phi_{\alpha,j})^2_{\beta\gamma}$, one computes that

$$\begin{aligned} (y(x-\alpha)^j)^2 &= (y + f(x))(x-\alpha)^{2j} \\ &= y(x-\alpha)^{2j} + f(x)(x-\alpha)^{2j}. \end{aligned}$$

The statement follows from the definition of $\tilde{\phi}_{\alpha,j}$ and Lemma 4.4(i). □

Now define

$$\begin{aligned} W''_{\alpha,\mathrm{ss}} &:= \langle \tilde{\phi}_{\alpha,0} \rangle \quad \text{for } \alpha \in B', \text{ and} \\ W''_{\alpha,\mathrm{nil}} &:= \langle \tilde{\phi}_{\alpha,j} \mid 1 \le j \le c_\alpha \rangle \quad \text{for } \alpha \in B. \end{aligned}$$

Lemma 4.7. *The subspaces* $W''_{\alpha,\mathrm{ss}}$ *and* $W''_{\alpha,\mathrm{nil}}$ *of* $\mathrm{H}^1(\mathcal{U},\mathcal{O})$ *are stable under the action of Frobenius for each* $\alpha \in B$. *There is an isomorphism of* F*-modules:*

$$\mathrm{H}^1(\mathcal{U},\mathcal{O}) \simeq \bigoplus_{\alpha \in B'} W''_{\alpha,\mathrm{ss}} \oplus \bigoplus_{\alpha \in B} W''_{\alpha,\mathrm{nil}}.$$

Proof. This follows immediately from Lemmas 4.5 and 4.6. □

4.4. **Auxiliary map.** The next goal is to define a section $\sigma : \mathrm{H}^1(X, \mathcal{O}) \to \mathrm{H}^1_{\mathrm{dR}}(X)$. To do this, the first step will be to define a homomorphism $\rho : \mathrm{Z}^1(\mathcal{U}, \mathcal{O}) \to \mathrm{C}^0(\mathcal{U}, \Omega^1)$ by defining its components $\rho_\alpha : \mathrm{Z}^1(\mathcal{U}, \mathcal{O}) \to \Gamma(U_\beta, \Omega^1)$ for $\alpha, \beta \in B$. Given $\phi \in \mathrm{Z}^1(\mathcal{U}, \mathcal{O})$ and $\alpha \in B$, the idea is to separate $d\phi$ into two parts: the first part will be regular at P_α and thus belong to $\Gamma(U_\alpha, \Omega^1)$; the second part will be regular away from P_α and hence belong to $\Gamma(U_\beta, \Omega^1)$ for every $\beta \neq \alpha$.

Notation 4.8. Define the *truncation* operator $\Theta_{\geq i} : k[x, x^{-1}] \to k[x, x^{-1}]$ by

$$\Theta_{\geq i}\left(\sum_j a_j x^j\right) := \sum_{j \geq i} a_j x^j.$$

Operators $\Theta_{>i}, \Theta_{\leq i}, \Theta_{<i} : k[x, x^{-1}] \to k[x, x^{-1}]$ can be defined analogously. These operators can also be defined on $k[x_\alpha, x_\alpha^{-1}]$. To clarify some ambiguity in notation, if $m(x_\alpha) \in k[x_\alpha, x_\alpha^{-1}]$, then let $\Theta_{\geq i}(m(x_\alpha))$ denote $\Theta_{\geq i}(m(x))|_{x = x_\alpha}$.

Recall that $x_\alpha := (x - \alpha)^{-1}$, and so $\phi_{\alpha,j} = \varphi_\alpha(y x_\alpha^{-j})$. Then

$$d(y x_\alpha^{-j}) = -j x_\alpha^{-j-1} y \, dx_\alpha + x_\alpha^{-j} dy. \tag{4.5}$$

Using partial fractions and the fact that $dy = -d(f(x))$, one sees that

$$dy = -\sum_{\beta \in B} f'_\beta(x_\beta) \, dx_\beta. \tag{4.6}$$

In light of these facts, consider the following definition.

Notation 4.9. For $\alpha \in B$ and $j \geq 0$, define

$$\begin{aligned} R_{\alpha,j} &:= \Theta_{\geq 0}\Big(x_\alpha^{-j} f'_\alpha(x_\alpha)\Big) \, dx_\alpha; \\ S_{\alpha,j} &:= d(y x_\alpha^{-j}) + R_{\alpha,j}. \end{aligned}$$

Remark 4.10. Let $a_{\alpha,i} \in k$ be the coefficients of the (odd-power) monomials of the polynomials $f_\alpha(x_\alpha)$ defined in the partial fraction decomposition (1.1):

$$f_\alpha(x_\alpha) = \sum_{i=0}^{c_\alpha} a_{\alpha,i} x_\alpha^{2i+1}.$$

Then

$$\begin{aligned} R_{\alpha,j} &= \sum_{j/2 \leq i \leq c_\alpha} a_{\alpha,i} x_\alpha^{2i-j} \, dx_\alpha \\ &= \sum_{j/2 \leq i \leq c_\alpha} a_{\alpha,i} \omega_{\alpha, 2i-j+1}. \end{aligned}$$

Lemma 4.11. *Let $\alpha \in B$ and $j \geq 0$.*

(1) *The differential form $R_{\alpha,j}$ is regular away from P_α, i.e., $R_{\alpha,j} \in \Gamma(U_\beta, \Omega^1)$ for all $\beta \in B_\infty - \{\alpha\}$*
(2) *The differential form $S_{\alpha,j}$ is regular at P_α for $0 \leq j \leq c_\alpha$, i.e., $S_{\alpha,j} \in \Gamma(U_\alpha, \Omega^1)$.*

Proof. Part (1) follows from Remark 4.10 and Equation (3.2).

For part (2), by Notation 4.8, Notation 4.9 and Equations (4.5) and (4.6), one sees that

$$\begin{aligned} S_{\alpha,j} &= d(y x_\alpha^{-j}) + \Theta_{\geq 0}(x_\alpha^{-j} f'_\alpha(x_\alpha)) \, dx_\alpha & (4.7) \\ &= -j x_\alpha^{-j-1} y \, dx_\alpha - \Theta_{<0}(x_\alpha^{-j} f'_\alpha(x_\alpha)) \, dx_\alpha - \sum_{\beta \in B - \{\alpha\}} x_\alpha^{-j} f'_\beta(x_\beta) \, dx_\beta. & (4.8) \end{aligned}$$

In the first part of Equation (4.8), note that the order of vanishing of $x_\alpha^{-j-1} y\, dx_\alpha$ at P_α is $2d_\alpha - 1 + 2j$ by Lemma 4.2 and Equation (3.1), and so this term is regular at P_α.

In the second part of Equation (4.8), note that $\Theta_{<0}(x_\alpha^{-j} f'_\alpha(x_\alpha))$ is contained in $x_\alpha^{-1} k[x_\alpha^{-1}]$. Thus $\Theta_{<0}(x_\alpha^{-j} f'_\alpha(x_\alpha))$ has a zero of order at least 2 at P_α. As seen in the proof of Lemma 3.1, dx_α has a zero of order $d_\alpha - 3$ at P_α. Thus $\Theta_{<0}(x_\alpha^{-j} f'_\alpha(x_\alpha))\, dx_\alpha$ is regular at P_α.

The last part of Equation (4.8) is regular at P_α since x_α^{-1} and $f'_\beta(x_\beta)\, dx_\beta$ are regular at P_α. □

4.5. **Definition of ρ.** We define a k-linear morphism

$$\rho : \mathrm{Z}^1(\mathcal{U}, \mathcal{O}) \to \mathrm{C}^0(\mathcal{U}, \Omega^1)$$

as follows.

4.5.1. *Definition of ρ on $\mathrm{B}^1(\mathcal{U}, \mathcal{O})$:* If $\phi \in \mathrm{B}^1(\mathcal{U}, \mathcal{O})$, then $\phi = \delta\kappa$ for some $\kappa \in \mathrm{C}^0(\mathcal{U}, \mathcal{O})$. Define

$$\rho(\phi) := d\kappa,$$

with differentiation performed component-wise. This map is well-defined, since if κ is regular at $P \in X(k)$, then so is $d\kappa$. Moreover, if κ' is another element such that $\phi = \delta\kappa'$, then $\delta(\kappa - \kappa') = 0$ and therefore $\kappa - \kappa' \in \mathrm{H}^0(\mathcal{U}, \mathcal{O})$ is constant and annihilated by d. Let $\rho_\beta(\phi)$ denote $(\rho(\phi))_\beta$.

It follows from the definition that $\mathscr{C}(\rho(\mathrm{B}^1(\mathcal{U}, \mathcal{O}))) = 0$, since the Cartier operator annihilates all exact differential forms. Explicitly, the map ρ is computed as follows.

Lemma 4.12. (i) *If $\alpha \in B_\infty$ and $h \in \Gamma(U_\alpha, \mathcal{O})$, then $\rho\varphi_\alpha(h|_{U'}) = d\psi_\alpha(h)$.*

(ii) *If $\alpha \in B$ and $j \leq 0$, then*

$$\rho\varphi_\alpha((x-\alpha)^j) = -\sum_{\gamma \in B_\infty - \{\alpha\}} d\psi_\gamma((x-\alpha)^j).$$

Proof. Part (i) is immediate from the definition of the map ρ and Lemma 4.1.

Part (ii) follows from part (i), Equation (4.4) and the definition of ρ. □

Example 4.13. The value of ρ on the 1-coboundary $\varphi_\alpha(f(x)x_\alpha^{-j})$ if $\alpha \in B$ and $j \geq 0$: Let

$$r_{\alpha,j} := \Theta_{>0}\big(x_\alpha^{-j} f_\alpha(x_\alpha)\big) \text{ and } s_{\alpha,j} := \Theta_{\leq 0}\big(x_\alpha^{-j} f_\alpha(x_\alpha)\big) + \sum_{\beta \neq \alpha} x_\alpha^{-j} f_\beta(x_\beta).$$

Then

$$f(x)x_\alpha^{-j} = r_{\alpha,j} + s_{\alpha,j},$$

and $r_{\alpha,j}$ has a pole at P_α, but is regular everywhere else, while $s_{\alpha,j}$ is regular at P_α. Thus,

$$\varphi_\alpha(f(x)x_\alpha^{-j}) = \delta\psi_\alpha(s_{\alpha,j}) - \sum_{\beta \in B_\infty - \{\alpha\}} \delta\psi_\beta(r_{\alpha,j}).$$

Therefore, for $\beta \neq \alpha$, by Lemma 4.12, $\rho_\beta\varphi_\alpha(f(x)x_\alpha^{-j}) = -d(r_{\alpha,j})$. Since $f_\alpha(x_\alpha) \in x_\alpha k[x_\alpha^2]$, this simplifies to

$$\rho_\beta\varphi_\alpha(f(x)x_\alpha^{-j}) = \begin{cases} -R_{\alpha,j} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases} \tag{4.9}$$

Similarly,

$$\rho_\alpha\varphi_\alpha(f(x)x_\alpha^{-j}) = \begin{cases} -S_{\alpha,j} & \text{if } j \text{ is even,} \\ d\big(f(x)x_\alpha^{-j}\big) & \text{if } j \text{ is odd.} \end{cases} \tag{4.10}$$

4.5.2. *Definition of ρ_β on $\mathrm{Z}^1(\mathcal{U},\mathcal{O})$:* By Lemma 4.5, $\mathrm{Z}^1(\mathcal{U},\mathcal{O})$ is generated by $\mathrm{B}^1(\mathcal{U},\mathcal{O})$ and $\phi_{\alpha,j}$ for $\alpha \in B$ and $0 \le j \le c_\alpha$. For $\alpha, \beta \in B$, define

$$\rho_\beta(\phi_{\alpha,j}) = \begin{cases} R_{\alpha,j} & \text{if } \beta \neq \alpha, \\ S_{\alpha,j} & \text{if } \beta = \alpha, \end{cases}$$

and extend ρ_β to $\mathrm{Z}^1(\mathcal{U},\mathcal{O})$ linearly. For all $\beta \in B - \{\alpha\}$, note that

$$\rho_\alpha(\phi_{\alpha,j}) = d(yx_\alpha^{-j}) + \rho_\beta(\phi_{\alpha,j}).$$

Lemma 4.14. *There is a well-defined map $\rho : \mathrm{Z}^1(\mathcal{U},\mathcal{O}) \to \mathrm{C}^0(\mathcal{U},\Omega^1)$ given by*

$$\rho := \bigoplus_{\beta \in B_\infty} \rho_\beta.$$

Proof. If $\beta \in B_\infty$, then $\rho_\beta(\mathrm{Z}^1(\mathcal{U},\mathcal{O})) \subset \Gamma(U_\beta, \Omega^1)$ by Section 4.5.1 and Lemma 4.11. □

Here is an example of a computation of the map ρ.

Lemma 4.15. *Let $\alpha \in B$ and $j \ge 0$. For each $\beta \in B$, in $\Gamma(U_\beta,\Omega^1)$,*

$$\rho_\beta\varphi_\alpha(y^2x_\alpha^{-2j}) = \begin{cases} 0 & \text{if } 0 \le 2j \le c_\alpha, \\ -R_{\alpha,2j} & \text{if } 2j > c_\alpha. \end{cases}$$

In particular, $\rho\varphi_\alpha(y^2x_\alpha^{-2j})$ lies in the subspace $W'_{\alpha,\mathrm{nil}}$ of $\mathrm{H}^0(\mathcal{U},\Omega^1)$.

Proof. We have $y^2x_\alpha^{-2j} = yx_\alpha^{-2j} + f(x)x_\alpha^{-2j}$, and therefore $\varphi_\alpha(y^2x_\alpha^{-2j}) = \phi_{\alpha,2j} + \varphi_\alpha(f(x)x_\alpha^{-2j})$.

Suppose $0 \le 2j \le c_\alpha$. If $\beta \neq \alpha$, then $\rho_\beta(\phi_{\alpha,2j}) = R_{\alpha,2j} = -\rho_\beta(\varphi_\alpha(f(x)x_\alpha^{-2j}))$ by Equation (4.9). By Equation (4.10), $\rho_\alpha(\phi_{\alpha,2j}) = S_{\alpha,2j} = -\rho_\alpha(\varphi_\alpha(f(x)x_\alpha^{-2j}))$. Thus, $\rho(\phi_{\alpha,2j})+\rho(\varphi_\alpha(f(x)x_\alpha^{-2j})) = 0$.

Now, suppose that $2j > c_\alpha$. Then yx_α^{-2j} is regular at P_α and therefore $\phi_{\alpha,2j}$ is a coboundary, with $\rho(\phi_{\alpha,2j}) = d\varphi_\alpha(yx_\alpha^{2j})$. Therefore, for $\beta \neq \alpha$,

$$\rho_\beta(\phi_{\alpha,2j}) + \rho_\beta(\varphi_\alpha(f(x)x_\alpha^{-2j})) = -R_{\alpha,2j},$$

and

$$\rho_\alpha(\phi_{\alpha,2j}) + \rho_\alpha(\varphi_\alpha(f(x)x_\alpha^{-2j})) = d(yx_\alpha^{-2j}) + d(f(x)x_\alpha^{-2j}) - R_{\alpha,2j} = -R_{\alpha,2j}.$$

By Remark 4.10, $R_{\alpha,2j} \in \langle \omega_{\alpha,2i-2j+1} \mid j \le i \le c_\alpha \rangle$. If $2j > c_\alpha$ and $j \le i \le c_\alpha$, then $1 \le 2i-2j+1 \le c_\alpha$, and so $R_{\alpha,2j} \in W'_{\alpha,\mathrm{nil}}$. Finally, since $\rho_\beta\varphi_\alpha(y^2x_\alpha^{-2j})$ is independent of the choice of $\beta \in B_\infty$, $\rho\varphi_\alpha(y^2x_\alpha^{-2j})$ lies in the kernel $\mathrm{H}^0(\mathcal{U},\Omega^1)$ of the coboundary map $\delta : C^0(\mathcal{U},\Omega^1) \to C^1(\mathcal{U},\Omega^1)$. □

Lemma 4.16. (i) *If $\phi \in \mathrm{Z}^1(\mathcal{U},\mathcal{O})$, then $\delta\rho(\phi) = d\phi$.*
(ii) *In particular, $\mathscr{C}(\rho_\alpha(\phi)) = \mathscr{C}(\rho_\beta(\phi))$ for all $\alpha, \beta \in B_\infty$.*
(iii) *For all $\alpha \in B$ and $\beta \in B_\infty$, we have $\mathscr{C}(\rho_\beta(\phi_{\alpha,j})) = \mathscr{C}(R_{\alpha,j})$.*

Proof. (i) The definition of ρ_β implies that $\rho_\alpha(\phi) - \rho_\beta(\phi) = d(\phi)_{\alpha\beta}$ for all $\alpha, \beta \in B_\infty$.
(ii) This follows from part (i) since the Cartier operator annihilates exact differential forms.
(iii) This follows from part (ii) and the definition of ρ_β.

□

Remark 4.17. With $a_{\alpha,i}$ defined as in Remark 4.10, one can explicitly compute:

$$\mathscr{C}(R_{\alpha,j}) = \begin{cases} \sum_{i=(j+1)/2}^{c_\alpha} \sqrt{a_{\alpha,i}}\,\omega_{\alpha,i-(j-1)/2} & \text{if } j \text{ is odd}, \\ 0 & \text{if } j \text{ is even}. \end{cases}$$

In particular, $\mathscr{C}(R_{\alpha,j}) \in W'_{\alpha,\mathrm{nil}}$.

4.6. **The $\mathbb{E}$-module structure of the de Rham cohomology.** Consider the exact sequence of $\mathbb{E}$-modules

$$0 \to \mathrm{H}^0(X, \Omega^1) \xrightarrow{\lambda} \mathrm{H}^1_{\mathrm{dR}}(X) \xrightarrow{\gamma} \mathrm{H}^1(X, \mathcal{O}) \to 0,$$

where $\mathbb{E} = k[F, V]$ is the non-commutative ring defined in Section 2.1. Consider the k-linear function

$$\sigma : \mathrm{H}^1(X, \mathcal{O}) \to \mathrm{H}^1_{\mathrm{dR}}(X)$$

defined by $\sigma(\phi) = (\phi, \rho(\phi))$ for $\phi \in \mathrm{Z}^1(\mathcal{U}, \mathcal{O})$.

Lemma 4.18. *The function σ is a section of $\gamma : \mathrm{H}^1_{\mathrm{dR}}(X) \to \mathrm{H}^1(X, \mathcal{O})$.*

Proof. The function σ is well-defined because $\sigma(\mathrm{B}^1(\mathcal{U}, \mathcal{O})) \subset \mathrm{B}^1_{\mathrm{dR}}(\mathcal{U})$ by the definition of ρ_β on $\mathrm{B}^1(\mathcal{U}, \mathcal{O})$. It is clearly a section of γ. □

Note that σ is not a splitting of $\mathbb{E}$-modules.

For $\alpha \in B$, let $\lambda_{\alpha,j} := \lambda(\omega_{\alpha,j})$ and $\sigma_{\alpha,j} := \sigma(\tilde{\phi}_{\alpha,j})$.

Proposition 4.19. *For $0 \leq j \leq c_\alpha$, the action of F and V on $\mathrm{H}^1_{\mathrm{dR}}(X)$ is given by:*

(1) $F\lambda_{\alpha,j} = 0$.

(2) $V\lambda_{\alpha,j} = \begin{cases} \lambda_{\alpha,j/2} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd.} \end{cases}$

(3) $F\sigma_{\alpha,j} = \begin{cases} \sigma_{\alpha,2j} & \text{if } j \leq c_\alpha/2, \\ \lambda(R_{\alpha,2j}) & \text{if } j > c_\alpha/2. \end{cases}$

(4) $V\sigma_{\alpha,j} = \begin{cases} \lambda(\mathscr{C}(R_{\alpha,j})) & \text{if } j \text{ is odd,} \\ 0 & \text{if } j \text{ is even.} \end{cases}$

Proof. (1) This follows from Subsection 2.6.

(2) This follows from Lemma 3.2 after applying λ.

(3) In $\mathrm{Z}^1_{\mathrm{dR}}(\mathcal{U})$,

$$\begin{aligned} F(\sigma_{\alpha,j}) &= \big(F\phi_{\alpha,j}, 0\big) \\ &= \big(\varphi_\alpha(y^2 x_\alpha^{-2j}), \rho\varphi_\alpha(y^2 x_\alpha^{-2j})\big) - \big(0, \rho\varphi_\alpha(y^2 x_\alpha^{-2j})\big) \\ &= \sigma\varphi_\alpha(y^2 x_\alpha^{-2j}) - \big(0, \rho\varphi_\alpha(y^2 x_\alpha^{-2j})\big). \end{aligned}$$

Since $y^2 x_\alpha^{-2j} = y x_\alpha^{-2j} + f(x) x_\alpha^{-2j}$, linearity of σ and φ_α yields that

$$\sigma\varphi_\alpha(y^2 x_\alpha^{-2j}) = \sigma\varphi_\alpha(y x_\alpha^{-2j}) + \sigma\varphi_\alpha\big(f(x) x_\alpha^{-2j}\big).$$

The term $\sigma\varphi_\alpha(f(x)x_\alpha^{-2j})$ is a coboundary by Lemma 4.4(i). The term $\sigma\varphi_\alpha(yx_\alpha^{-2j})$ equals $\sigma_{\alpha,2j}$ if $0 \leq 2j \leq c_\alpha$, and is a coboundary if $2j > c_\alpha$ by Lemma 4.4(ii). By Lemma 4.15,

$$\big(0, \rho\varphi_\alpha(y^2 x_\alpha^{-2j})\big) = \begin{cases} 0 & \text{if } 0 \leq 2j \leq c_\alpha, \\ -\lambda(R_{\alpha,2j}) & \text{if } 2j > c_\alpha. \end{cases}$$

(4) Since $V(\phi, \rho(\phi)) = (0, \mathscr{C}(\rho(\phi)))$, the desired result follows by Lemma 4.16(iii).

□

Consider the subspaces of $\mathrm{H}^1_{\mathrm{dR}}(X)$ given by:

$$\begin{aligned} W_{\alpha,\mathrm{ss}} &:= \big\langle \lambda_{\alpha,0} - \lambda_{0,0}, \sigma_{\alpha,0} \big\rangle, \\ W_{\alpha,\mathrm{nil}} &:= \big\langle \lambda_{\alpha,j}, \sigma_{\alpha,j} \mid 1 \leq j \leq c_\alpha \big\rangle. \end{aligned}$$

Theorem 4.20. *The subspaces $W_{\alpha,\mathrm{ss}}$ and $W_{\alpha,\mathrm{nil}}$ of $\mathrm{H}^1_{\mathrm{dR}}(X)$ are stable under the action of Frobenius and Verschiebung for each $\alpha \in B$. There is an isomorphism of $\mathbb{E}$-modules:*

$$\mathrm{H}^1_{\mathrm{dR}}(X) = \bigoplus_{\alpha\in B'} W_{\alpha,\mathrm{ss}} \oplus \bigoplus_{\alpha\in B} W_{\alpha,\mathrm{nil}}.$$

Proof. The stability is immediate by Proposition 4.19, Remark 4.10, and Lemma 4.15. The decomposition follows from Corollary 4.18 and Lemmas 3.3 and 4.7. □

Theorem 1.2 is immediate from Theorem 4.20.

5. Results on the Ekedahl-Oort type

For a natural number c, let G_c be the unique symmetric BT_1 group scheme of rank p^{2c} with Ekedahl-Oort type $[0, 1, 1, 2, 2, \ldots, \lfloor c/2\rfloor]$. In other words, this means that there is a final filtration $N_1 \subset N_2 \subset \cdots \subset N_{2c}$ of $D(G_c)$ as a k-vector space, which is stable under the action of V and F^{-1} and with $i = \dim(N_i)$, such that $\dim(V(N_i)) = \lfloor i/2\rfloor$. In Section 5.1, we prove that group schemes of the form G_c appear in the decomposition of $J_X[2]$ when X is a hyperelliptic k-curve. In Section 5.2, we describe the Dieudonné module of G_c for arbitrary c and give examples.

5.1. **The final filtration for hyperelliptic curves in characteristic 2.** Suppose X is a hyperelliptic k-curve with affine equation $y^2 - y = f(x)$ as described in Notation 1.1. For $\alpha \in B$, recall that $c_\alpha = (d_\alpha - 1)/2$, where d_α is the ramification invariant of X above α. Recall the subspaces $W_{\alpha,\mathrm{nil}}$ of $\mathrm{H}^1_{\mathrm{dR}}(X)$ from Section 4.6. Define subspaces $N_{\alpha,i}$ of $W_{\alpha,\mathrm{nil}}$ for $0 \le i \le 2c_\alpha$ as follows: $N_{\alpha,0} := \{0\}$ and

$$N_{\alpha,i} := \begin{cases} \langle \lambda_{\alpha,j} \mid 1 \le j \le i\rangle & \text{if } 1 \le i \le c_\alpha, \\ N_{\alpha,c_\alpha} \oplus \langle \sigma_{\alpha,j} \mid 1 \le j \le i\rangle & \text{if } c_\alpha + 1 \le i \le 2c_\alpha. \end{cases}$$

Proposition 5.1. *The filtration $N_{\alpha,0} \subset N_{\alpha,1} \subset N_{\alpha,2} \subset \cdots \subset N_{\alpha,2c_\alpha}$ is a final filtration of $W_{\alpha,\mathrm{nil}}$ for each $\alpha \in B$. Furthermore, $V(N_{\alpha,i}) = N_{\alpha,\lfloor i/2\rfloor}$.*

Proof. Let $0 \le i \le 2c_\alpha$. One sees that $\dim(N_{\alpha,i}) = i$. By Proposition 4.19, $V(N_{\alpha,i}) = N_{\alpha,\lfloor i/2\rfloor}$ and $F^{-1}(N_{\alpha,i}) = N_{\alpha,c_\alpha+\lceil i/2\rceil}$. Thus the filtration $N_{\alpha,0} \subset N_{\alpha,1} \subset N_{\alpha,2} \subset \cdots \subset N_{\alpha,2c_\alpha}$ is stable under the action of V and F^{-1}. □

Theorem 5.2. *Let k be an algebraically closed field of characteristic $p = 2$. Suppose X is a hyperelliptic k-curve with affine equation $y^2 - y = f(x)$ as described in Notation 1.1. Then the 2-torsion group scheme of X decomposes as*

$$J_X[2] \simeq (\mathbb{Z}/2 \oplus \mu_2)^r \oplus \bigoplus_{\alpha\in B} G_{c_\alpha},$$

and the a-number of X is

$$a_X = (g + 1 - \#\{\alpha \in B \mid d_\alpha \equiv 1 \bmod 4\})/2.$$

Proof. By [18, Section 5], there is an isomorphism of $\mathbb{E}$-modules between the Dieudonné module $D(J_X[2])$ and the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR}}(X)$. By Theorem 4.20, there is an isomorphism of $\mathbb{E}$-modules:

$$\mathrm{H}^1_{\mathrm{dR}}(X) = \bigoplus_{\alpha\in B'} W_{\alpha,\mathrm{ss}} \oplus \bigoplus_{\alpha\in B} W_{\alpha,\mathrm{nil}}.$$

If $\alpha \in B'$, then $W_{\alpha,\mathrm{ss}}$ is isomorphic to $\mathbb{E}/\mathbb{E}(F, 1 - V) \oplus \mathbb{E}/\mathbb{E}(V, 1 - F) \simeq D(\mathbb{Z}/2 \oplus \mu_2)$. Finally, Proposition 5.1 shows that $W_{\alpha,\mathrm{nil}} \simeq D(G_{c_\alpha})$, which completes the proof of the statement about $J_X[2]$. The statement about a_X can be found in Proposition 3.4. □

As a corollary, we highlight the special case when $r = 0$ (for example, when $f(x) \in k[x]$). Corollary 5.3 is stated without proof in [26, 3.2].

Corollary 5.3. *Let k be an algebraically closed field of characteristic $p = 2$. Suppose X is a hyperelliptic k-curve of genus g and p-rank $r = 0$. Then the Ekedahl-Oort type of $J_X[2]$ is $[0, 1, 1, 2, 2, \ldots, \lfloor g/2 \rfloor]$ and the a-number is $a_X = \lfloor (g+1)/2 \rfloor$.*

Proof. This is a special case of Theorem 5.2 where $\#B = 1$. □

The next result is included to emphasize that Theorem 5.2 gives a complete classification of the 2-torsion group schemes which occur as $J_X[2]$ when X is a hyperelliptic k-curve.

Corollary 5.4. *Let k be an algebraically closed field of characteristic $p = 2$. Let G be a polarized BT_1 group scheme over k of rank p^{2g}. Let $0 \leq r \leq g$. Then $G \simeq J_X[2]$ for some hyperelliptic k-curve X of genus g and p-rank r if and only if there exist non-negative integers $c_1, \ldots, c_{r+1}$ such that $\sum_{i=1}^{r+1} c_i = g - r$ and such that*

$$G \simeq (\mathbb{Z}/2 \oplus \mu_2)^r \oplus \bigoplus_{\alpha \in B} G_{c_\alpha}.$$

Proof. This is immediate from Theorem 5.2. □

Remark 5.5. For fixed g, the number of isomorphism classes of polarized BT_1 group schemes of rank p^{2g} that occur as $J_X[2]$ for some hyperelliptic k-curve X of genus g equals the number of partitions of $g+1$. To see this, note that the isomorphism class of $J_X[2]$ is determined by the multi-set $\{d_1, \ldots, d_{r+1}\}$ where $d_i = 2c_i + 1$ and $\sum_{i=1}^{r+1}(d_i + 1) = 2g + 2$. So the number of isomorphism classes equals the number of partitions of $2g + 2$ into positive even integers.

Remark 5.6. The examples in Section 5.2 show that the factors G_c appearing in the decomposition of $J_X[2]$ in Theorem 5.2 may not be indecomposable as polarized BT_1 group schemes.

5.2. **Description of a particular Ekedahl-Oort type.** Recall that G_c is the unique polarized BT_1 group scheme over k of rank p^{2c} with Ekedahl-Oort type $[0, 1, 1, 2, 2, \ldots, \lfloor c/2 \rfloor]$. Recall that $\mathbb{E} = k[F, V]$ is the non-commutative ring defined in Section 2.2. In this section, we describe the Dieudonné module $D(G_c)$. We start with some examples to motivate the notation. The examples show that G_c is sometimes indecomposable and sometimes decomposes into polarized BT_1 group schemes of smaller rank. The first four examples were found using pre-existing tables.

Example 5.7.
(1) For $c = 1$, the Ekedahl-Oort type is $[0]$. This Ekedahl-Oort type occurs for the p-torsion group scheme of a supersingular elliptic curve. See [9, Ex. A.3.14] or [20, Ex. 2.3] for a description of G_1. It has Dieudonné module $\mathbb{E}/\mathbb{E}(F + V)$.
(2) For $c = 2$, the Ekedahl-Oort type is $[0, 1]$. This Ekedahl-Oort type occurs for the p-torsion group scheme of a supersingular abelian surface which is not superspecial. See [9, Ex. A.3.15] or [20, Ex. 2.3] for a description of G_2. It has Dieudonné module $\mathbb{E}/\mathbb{E}(F^2 + V^2)$.
(3) For $c = 3$, the Ekedahl-Oort type is $[0, 1, 1]$. This Ekedahl-Oort type occurs for an abelian threefold with p-rank 0 and a-number 2 whose p-torsion is indecomposable as a polarized BT_1 group scheme. By [20, Lemma 3.4], G_3 has Dieudonné module
$$\mathbb{E}/\mathbb{E}(F^2 + V) \oplus \mathbb{E}/\mathbb{E}(V^2 + F).$$
(4) For $c = 4$, the Ekedahl-Oort type is $[0, 1, 1, 2]$. This Ekedahl-Oort type occurs for an abelian fourfold with p-rank 0 and a-number 2 whose p-torsion decomposes as a direct sum of polarized BT_1 group schemes of rank p^2 and p^6. By [20, Table 4.4], G_4 has Dieudonné module
$$\mathbb{E}/\mathbb{E}(F + V) \oplus \mathbb{E}/\mathbb{E}(F^3 + V^3).$$

We now provide an algorithm to determine the Dieudonné module $D(G_c)$ for all positive integers $c \in \mathbb{N}$ following the method of [19, Section 9.1].

Proposition 5.8. *The Dieudonné module $D(G_c)$ is the $\mathbb{E}$-module generated as a k-vector space by $\{X_1, \ldots, X_c, Y_1, \ldots, Y_c\}$ with the actions of F and V given by:*

(1) $F(Y_j) = 0$.

(2) $V(Y_j) = \begin{cases} Y_{2j} & \text{if } j \leq c/2, \\ 0 & \text{if } j > c/2. \end{cases}$

(3) $F(X_i) = \begin{cases} X_{j/2} & \text{if } j \text{ is even}, \\ Y_{c-(j-1)/2} & \text{if } j \text{ is odd}. \end{cases}$

(4) $V(X_j) = \begin{cases} 0 & \text{if } j \leq (c-1)/2, \\ -Y_{2c-2j+1} & \text{if } j > (c-1)/2. \end{cases}$

Proof. By definition of G_c, there is a final filtration $N_1 \subset N_2 \subset \cdots \subset N_{2c}$ of $D(G_c)$ as a k-vector space, which is stable under the action of V and F^{-1} and with $i = \dim(N_i)$, such that $\nu_i := \dim(V(N_i)) = \lfloor i/2 \rfloor$. This implies that $\nu_i = \nu_{i-1}$ if and only if i is odd. In the notation of [19, Section 9.1], this yields $m_i = 2i$ and $n_i = 2g - 2i + 1$ for $1 \leq i \leq g$; also, let

$$Z_i = \begin{cases} X_{i/2} & \text{if } i \text{ is even}, \\ Y_{c-(i-1)/2} & \text{if } i \text{ is odd}. \end{cases}$$

By [19, Section 9.1], for $1 \leq i \leq g$, the action of F is given by $F(Y_i) = 0$ and $F(X_i) = Z_i$; and the action of V is given by $V(Z_i) = 0$ and $V(Z_{2g-i+1}) = (-1)^{i-1} Y_i$. □

More notation is needed to give an explicit description of $D(G_c)$.

Notation 5.9. Let $c \in \mathbb{N}$ be fixed. Let $I = \{j \in \mathbb{N} \mid \lceil (c+1)/2 \rceil \leq j \leq c\}$ which is a set of cardinality $\lfloor (c+1)/2 \rfloor$. For $j \in I$, let $\ell(j)$ be the odd part of j and let $e(j)$ be the non-negative integer such that $j = 2^{e(j)} \ell(j)$. Let $s(j) = c - (\ell(j) - 1)/2$. One can check that $\{s(j) \mid j \in I\} = I$. Also, let $m(j) = 2c - 2j + 1$ and let $\epsilon(j)$ be the non-negative integer such that $t(j) := 2^{\epsilon(j)} m(j) \in I$. One can check that $\{t(j) \mid j \in I\} = I$. Thus there is a unique bijection $\iota : I \to I$ such that $t(\iota(j)) = s(j)$ for each $j \in I$.

Proposition 5.10. *Recall Notation 5.9. For $c \in \mathbb{N}$, the set $\{X_j \mid j \in I\}$ generates the Dieudonné module $D(G_c)$ as an $\mathbb{E}$-module subject to the relations: $F^{e(j)+1}(X_j) + V^{\epsilon(\iota(j))+1}(X_{\iota(j)}) = 0$ for $j \in I$. Also, $\{X_j \mid j \in I\}$ is a basis for the quotient of $D(G_c)$ by the left ideal $D(G_c)(F, V)$.*

Proof. Proposition 5.8 implies that $F^{e(j)}(X_j) = X_{\ell(j)}$ and $F(X_{\ell(j)}) = Y_{s(j)}$. Also, $V(X_j) = -Y_{m(j)}$ and so $V^{\epsilon(j)+1}(X_j) = -Y_{t(j)}$. This yields the stated relations. To complete the first claim, it suffices to show that the span of $\{X_j \mid j \in I\}$ under the action of F and V contains the k-module generators of $D(C_c)$ listed in Proposition 5.8. This follows from the observations that $X_i = F(X_{2i})$ if $1 \leq i \leq \lfloor c/2 \rfloor$, that $Y_i = V(Y_{i/2})$ if i is even and $Y_i = V(-X_{c-(i-1)/2})$ if i is odd. By [14, 5.2.8], the dimension of $D(G_c)$ modulo $D(G_c)(F, V)$ equals the a-number. Since $a = |I|$ by Corollary 5.3, it follows that the set $|I|$ of generators of $D(G_c)$ is linearly independent modulo $D(G_c)(F, V)$. □

Here are some more examples. The columns of the following table list: the value of c; the generators of $D(G_c)$ as an $\mathbb{E}$-module (where $X_{i_1} - X_{i_2}$ denotes $\{X_i \mid i_1 \leq i \leq i_2\}$); and the relations among these generators. The last column is the number of summands of $D(G_c)$ in its decomposition as an $\mathbb{E}$-module (as opposed to as a polarized $\mathbb{E}$-module). The table can be verified in two ways: first, by checking it with Proposition 5.10; second, by computing the action of F and V on a k-basis for $D(G_c)$, using this to construct a final filtration of $D(G_c)$ stable under V and F^{-1}, and then checking that it matches the Ekedahl-Oort type of G_c. In Example 5.11, we illustrate the second method.

c	generators	relations	# summands
5	$X_3 - X_5$	$FX_3 + V^3X_5, F^3X_4 + VX_3, FX_5 + VX_4$	1
6	$X_4 - X_6$	$F^3X_4 + V^2X_5, FX_5 + V^3X_6, F^2X_6 + VX_4$	1
7	$X_4 - X_7$	$F^3X_4 + VX_4, FX_5 + VX_5, F^2X_6 + V^2X_6, FX_7 + V^3X_7$	4
8	$X_5 - X_8$	$FX_5 + V^2X_7, F^2X_6 + VX_5, FX_7 + VX_6, F^4X_8 + V^4X_8$	2
9	$X_5 - X_9$	$FX_5 + VX_6, F^2X_6 + V^4X_9, FX_7 + V^2X_8,$ $F^4X_8 + VX_5, FX_9 + VX_7$	1
10	$X_6 - X_{10}$	$F^2X_6 + VX_6, FX_7 + VX_7, F^4X_8 + V^2X_8,$ $FX_9 + V^2X_9, F^2X_{10} + V^4X_{10}$	5

Example 5.11. For $c = 7$, the group scheme G_7 with Ekedahl-Oort type $[0, 1, 1, 2, 2, 3, 3]$ is isomorphic to a direct sum of polarized BT_1 group schemes of ranks p^2, p^4 and p^8 and has Dieudonné module

$$\mathbf{M} := \mathbb{E}/\mathbb{E}(F + V) \oplus \mathbb{E}/\mathbb{E}(F^2 + V^2) \oplus \mathbb{E}/\mathbb{E}(V + F^3) \oplus \mathbb{E}/\mathbb{E}(F^3 + V).$$

Proof. Let $\{1_A, V_A\}$ be the basis of the submodule $A = \mathbb{E}/\mathbb{E}(F + V)$ of $\mathbf{M}$; let $\{1_B, V_B, V_B^2, F_B^2\}$ be the basis of the submodule $B = \mathbb{E}/\mathbb{E}(F^2 + V^2)$; let $\{1_C, V_C, V_C^2, V_C^3\}$ be the basis of the submodule $C = \mathbb{E}/\mathbb{E}(F + V^3)$; and let $\{1_{C'}, F_{C'}, F_{C'}^2, F_{C'}^3\}$ be the basis of the submodule $C' = \mathbb{E}/\mathbb{E}(F^3 + V)$. The action of Frobenius and Verschiebung on the elements of these bases is:

x	1_A	V_A	1_B	V_B	V_B^2	F_B	1_C	V_C	V_C^2	V_C^3	$1_{C'}$	$F_{C'}$	$F_{C'}^2$	$F_{C'}^3$
Vx	V_A	0	V_B	V_B^2	0	0	V_C	V_C^2	V_C^3	0	$F_{C'}^3$	0	0	0
Fx	V_A	0	F_B	0	0	V_B^2	V_C^3	0	0	0	$F_{C'}$	$F_{C'}^2$	$F_{C'}^3$	0

To verify the proposition, one can repeatedly apply V and F^{-1} to construct a filtration $N_1 \subset N_2 \subset \cdots \subset N_{14}$ of $\mathbf{M}$ as a k-vector space which is stable under the action of V and F^{-1} such that $i = \dim(N_i)$. To save space, we summarize the calculation by listing a generator t_i for N_i/N_{i-1}:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
t_i	V_C^3	V_C^2	V_B^2	V_C	V_A	$F_{C'}^3$	V_B	1_C	$F_{C'}^2$	1_A	F_B	$F_{C'}$	$1_{C'}$	1_B

Then one can check that $V(N_i) = N_{\lfloor i/2 \rfloor}$ and $F^{-1}(N_i) = N_{7+\lceil i/2 \rceil}$, which verifies that the Ekedahl-Oort type of $\mathbf{M}$ is $[0, 1, 1, 2, 2, 3, 3]$. □

Remark 5.12. One could ask when $D(G_c)$ decomposes as much as numerically possible, in other words, when the a-number equals the number of summands of $D(G_c)$ in its decomposition as an $\mathbb{E}$-module. For example, $D(G_c)$ has this property when $c \in \{1, 2, 3, 4, 7, 10\}$ but not when $c \in \{5, 6, 8, 9\}$. This phenomenon occurs if and only if the bijection ι from Notation 5.9 is the identity.

Remark 5.13. The group scheme G_8 decomposes as the direct sum of two indecomposable polarized BT_1 group schemes, one whose Ekedahl-Oort type is $[0, 0, 1, 1]$, and the other whose covariant Dieudonné module is $\mathbb{E}/\mathbb{E}(F^4 + V^4)$. We take this opportunity to note that there is a mistake in [20, Example in Section 3.3]. The covariant Dieudonné module of $I_{4,3} = [0, 0, 1, 1]$ is stated incorrectly. To fix it, consider the method of [19, Section 9.1]. Consider the k-vector space of dimension 8 generated by $X_1, \ldots, X_4$ and $Y_1, \ldots Y_4$. Consider the operation F defined by: $F(Y_i) = 0$ for $1 \leq i \leq 4$ and

$$F(X_1) = Y_4;\ F(X_2) = Y_3;\ F(X_3) = X_1;\ F(X_4) = Y_2.$$

Consider the operation V defined by:

$$V(X_1) = 0;\ V(X_2) = -Y_4;\ V(X_3) = -Y_2;\ V(X_4) = -Y_1;$$

and

$$V(Y_1) = Y_3; V(Y_2) = 0;\ V(Y_3) = 0; V(Y_4) = 0.$$

Thus $D(I_{4,3})$ is generated by X_2, X_3, X_4 modulo the three relations

$$FX_2 + V^2X_4, F^2X_3 + VX_2, VX_3 + FX_4.$$

5.3. **Newton polygons.** There are several results in characteristic 2 about the Newton polygons of hyperelliptic (e.g., Artin-Schreier) curves X of genus g and 2-rank 0. For example, [2, Remark 3.2] states that if $2^{n-1} - 1 \leq g \leq 2^n - 2$, then the generic first slope of the Newton polygon of an Artin-Schreier curve of genus g and 2-rank 0 is $1/n$. This statement is made more precise in [1, Thm. 4.3]. See also earlier work in [21, Thm. 1.1(III)].

The Ekedahl-Oort type of $J_X[2]$ gives information about the Newton polygon of X, but does not determine it completely. Using Corollary 5.3 and [10, Section 3.1, Theorem 4.1], one can show that the first slope of the Newton polygon of X is at least $1/n$. Since this is weaker than [1, Thm. 4.3], we do not include the details.

More generally, one could consider the case that X is a hyperelliptic k-curve of genus g and arbitrary p-rank. One could use Theorem 5.2 to give partial information (namely a lower bound) for the Newton polygon of X.

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