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Semi-direct Galois covers of the affine line

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Abstract. Let k be an algebraically closed field of characteristic $p > 0$. Let G be a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ where b is a positive integer and ℓ is a prime distinct from p. In this paper, we study Galois covers $\psi : Z \to \mathbb{P}^1_k$ ramified only over ∞ with Galois group G. We find the minimal genus of a curve Z which admits a covering map of this form and we give an explicit formula for this genus in terms of ℓ and p. The minimal genus occurs when b equals the order a of ℓ modulo b and we also prove that the number of curves Z of this minimal genus which admit such a covering map is at most $(p-1)/a$ when p is odd.

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1 Introduction

Let k be an algebraically closed field of characteristic $p > 0$. In sharp contrast with the situation in characteristic 0, there exist Galois covers $\psi : Z \to \mathbb{P}^1_k$ ramified only over infinity. By Abhyankar's Conjecture [2], proved by Raynaud and Harbater [9], [4], a finite group G occurs as the Galois group of such a cover ψ if and only if G is quasi-p, i.e., G is generated by p-groups. This result classifies all the finite quotients of the fundamental group $\pi_1(\mathbb{A}^1_k)$. It does not, however, determine the profinite group structure of $\pi_1(\mathbb{A}^1_k)$ because this fundamental group is an infinitely generated profinite group.

There are many open questions about Galois covers $\psi : Z \to \mathbb{P}^1_k$ ramified only over infinity. For example, given a finite quasi-p group G, what is the smallest integer g for which there exists a cover $\psi : Z \to \mathbb{P}^1_k$ ramified only over infinity with Z of genus g? As another example, suppose G and H are two finite quasi-p groups such that H is a quotient of G. Given an unramified Galois cover ϕ of $\mathbb{A}^1_k$ with group H, under what situations can one dominate ϕ with an unramified Galois cover ψ of $\mathbb{A}^1_k$ with Galois group G? Answering these questions will give progress towards understanding how the finite quotients of $\pi_1(\mathbb{A}^1_k)$ fit together in an inverse system. These questions are more tractible for quasi-p groups that are p-groups since the maximal pro-p quotient $\pi_1^p(\mathbb{A}^1_k)$ is free (of infinite rank) [10].

In this paper, we study Galois covers $\psi : Z \to \mathbb{P}^1_k$ ramified only over ∞ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$, where ℓ is a prime distinct from p. Such a cover ψ must be a composition $\psi = \phi \circ \omega$ where $\omega : Z \to Y$ is unramified and $\phi : Y \to \mathbb{P}^1_k$ is an Artin-Schreier cover ramified only over ∞. The cover ϕ has an affine equation $y^p - y = f(x)$ for some $f(x) \in k[x]$ with degree s prime-to-p. The ℓ-torsion $\mathrm{Jac}\,(Y)[\ell]$ of the Jacobian of Y is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^{2g_Y}$. When $f(x) = x^s$, we determine how an automorphism τ of Y of order p acts on $\mathrm{Jac}\,(Y)[\ell]$. This allows us to construct a Galois cover $\psi_a : Z_a \to \mathbb{P}^1_k$ ramified only over ∞ which dominates ϕ, such that the Galois group of ψ_a is $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ where a is the order of ℓ modulo p (Section 3). We prove that the genus of Z_a is minimal among all natural numbers that occur as the genus of a curve Z which admits a covering map $\psi : Z \to \mathbb{P}^1_k$ ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$. We also prove that the number of curves Z of this minimal genus which admit such a covering map is at most $(p-1)/a$ when p is odd (Section 4).

2 Quasi-p semi-direct products

We recall which groups occur as Galois groups of covers of $\mathbb{P}^1_k$ ramified only over ∞.

Definition 2.1 *A finite group is a quasi p-group if it is generated by all of its Sylow p-subgroups.*

It is well-known that there are other equivalent formulations of the quasi-p property, such as the next result.

Lemma 2.2 *A finite group is a quasi p-group if and only if it has no nontrivial quotient group whose order is relatively prime to p.*

The importance of the quasi-p property is that it characterizes which finite groups occur as Galois groups of unramified covers of the affine line.

Theorem 2.3 *A finite group occurs as the Galois group of a Galois cover of the projective line $\mathbb{P}^1_k$ ramified only over infinity if and only if it is a quasi-p group.*

Proof This is a special case of Abhyankar's Conjecture [2] which was jointly proved by Harbater [4] and Raynaud [9]. □

We now restrict our attention to groups G that are semi-direct products of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$. The semi-direct product action is determined by a homomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}\,((\mathbb{Z}/\ell\mathbb{Z})^b)$.

Lemma 2.4 *Suppose a quasi-p group G is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ for a positive integer b.*

1. *Then G is not a direct product.*
2. *Moreover, $b \geq \mathrm{ord}_p(\ell)$ where $\mathrm{ord}_p(\ell)$ is the order of ℓ modulo p.*

Proof Part (1) is true since $(\mathbb{Z}/\ell\mathbb{Z})^b$ cannot be a quotient of the quasi-p group G. For part (2), the structure of a semi-direct product $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ depends on a homomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}\,(\mathbb{Z}/\ell\mathbb{Z})^b$. By part (1), ι is an inclusion. Thus $\mathrm{Aut}\,(\mathbb{Z}/\ell\mathbb{Z})^b \simeq \mathrm{GL}_b(\mathbb{Z}/\ell\mathbb{Z})$ has an element of order p. Now

$$|\mathrm{GL}_b(\mathbb{Z}/\ell\mathbb{Z})| = (\ell^b - 1)(\ell^b - \ell)\cdots(\ell^b - \ell^{b-1}).$$

Thus $\ell^\beta \equiv 1 \bmod p$ for some positive integer $\beta \leq b$ which implies $b \geq \mathrm{ord}_p(\ell)$. □

Lemma 2.5 *If $a = \mathrm{ord}_p(\ell)$, then there exists a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ which is quasi-p. It is unique up to isomorphism.*

Proof If $a = \mathrm{ord}_p(\ell)$, then there is an element of order p in $\mathrm{Aut}\,((\mathbb{Z}/\ell\mathbb{Z})^a)$ and so there is an injective homomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}\,((\mathbb{Z}/\ell\mathbb{Z})^a)$. Thus there exists a non-abelian semi-direct product G of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. To show that G is quasi-p, suppose N is a normal subgroup of G whose index is relatively prime to p. Then N contains an element τ of order p. By [3, 5.4, Thm. 9], since G is not a direct product and $(\mathbb{Z}/\ell\mathbb{Z})^a$ is normal in G, the subgroup $\langle\tau\rangle$ is not normal in G. Thus $\langle\tau\rangle$ is a proper subgroup of N. It follows that ℓ divides $|N|$ and so N contains an element h of order ℓ by Cauchy's theorem. Recall that $\mathrm{Aut}\,((\mathbb{Z}/\ell\mathbb{Z})^\beta)$ contains no element of order p for any positive integer $\beta < a$. Thus the group generated by the conjugates of h under τ has order divisible by ℓ^a. Thus $N = G$ and G has no non-trivial quotient group whose order is relatively prime to p. By Lemma 2.2, G is quasi-p.

The uniqueness follows from [8, Lemma 6.6]. □

3 Explicit construction of $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$-Galois covers of $\mathbb{A}^1_k$

In this section, we give concrete examples of Galois covers $\psi : Z \to \mathbb{P}^1_k$ ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. To compute the genus of the covering curve Z, we will need to determine the higher ramification groups of ψ.

Definition 3.1 *Let L/K be a Galois extension of function fields of curves with Galois group G and let P, P' be primes of K and L such that $P'|P$. Let $\nu_{P'}$ and $\mathcal{O}_{P'}$ be the corresponding valuation function and valuation ring for P'. For any integer $i \geq -1$, the ith ramification group of $P'|P$ is*

$$I_i(P'|P) = \{\sigma \in G \mid \nu_{P'}(\sigma(z) - z) \geq i + 1, \forall z \in \mathcal{O}_{P'}\}.$$

Lemma 3.2 *Suppose $f(x) \in k[x]$ is a polynomial of degree s for a positive integer s prime to p. Let $\phi : Y \to \mathbb{P}^1_k$ be the cover of curves corresponding to the field extension*

$$k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x)).$$

1. *Then $\phi : Y \to \mathbb{P}^1_k$ is a Galois cover with Galois group $\mathbb{Z}/p\mathbb{Z}$ ramified only at the point P_∞ over ∞.*
2. *The ith ramification group at P_∞ satisfies*
$$I_i = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i \le s \\ 0 & \text{if } i > s. \end{cases}$$
3. *The genus g_Y of Y is equal to*
$$g_Y = (p-1)(s-1)/2.$$

Proof For part (1), note that the extension $k(x) \hookrightarrow k(x)[y]/(y^p - y - f(x))$ is cyclic of degree p, with Galois group generated by the automorphism $\tau : y \mapsto y+1$ of order p. Let P be a finite prime of $k(x)$ and let ν_P be the corresponding valuation. Then $\nu_P(f(x)) \ge 0$, hence P is unramified by [12, Prop. III.7.8(b)]. For the infinite prime ∞ with corresponding valuation υ_∞, we have
$$\nu_\infty(f(x) - (z^p - z)) \le 0$$
for all $z \in k[x]$ thus P_∞ is totally ramified by [12, Prop. III.7.8(c)].

To prove part (2), we note that furthermore
$$\upsilon_{P_\infty}(y^p - y) = \upsilon_{P_\infty}(f(x)) = \upsilon_{P_\infty}(x^s) = -sp,$$
which implies that
$$\upsilon_{P_\infty}(y) = -s.$$
Now let $\widehat{\theta}$ be the completion of the valuation ring of $k(x)[y]/(y^p - y - f(x))$ at P_∞, and let π_∞ be a generator of the unique prime in $\widehat{\theta}$. Then write $y = \pi_\infty^{-s}u$, where u is a unit in $\widehat{\theta} \simeq k[[\pi_\infty]]$. Since k is algebraically closed, $\sqrt[s]{u} \in \widehat{\theta}$, and so $\sqrt[s]{y} \in \widehat{\theta}$. After possibly changing π_∞, we can assume without loss of generality that $\sqrt[s]{y} = \pi_\infty^{-1}$. Recalling that τ acts on y by $\tau(y) = y+1$, we have
$$\begin{aligned} \tau(\pi_\infty) &= \tau\,(1/y)^{1/s} = (\pi_\infty^s/(1+\pi_\infty^s))^{1/s} \\ &= \pi_\infty(1 - \pi_\infty^s + \pi_\infty^{2s} - + \ldots)^{\frac{1}{s}} \\ &= \pi_\infty - (1/s)\pi_\infty^{s+1} + a_{2s+1}\pi_\infty^{2s+1} - + \ldots. \end{aligned}$$
Thus $\upsilon_{P_\infty}(\tau(\pi_\infty) - \pi_\infty) = s+1$, which completes the proof of part (2).

To find the genus g_Y of Y for part (3), we make use of the Riemann-Hurwitz formula
$$2g_Y - 2 = p(-2) + \sum_{i=0}^{\infty} (|I_i| - 1)\,,$$
where I_i denotes the ith ramification group at P_∞, [5, Thms. 7.27 & 11.72]). From part (2), we then obtain that $g_Y = (p-1)(s-1)/2$. □

Recall the following facts about the pth cyclotomic polynomial $\Phi_p(t) := t^{p-1} + \cdots + 1$, which is the minimal polynomial over $\mathbb{Q}$ of a primitive pth root of unity ζ_p. Now $\mathbb{Q}(\zeta_p)$ is a Galois extension of $\mathbb{Q}$, unramified over ℓ since $\ell \ne p$, and all primes over ℓ have the same residue field degree. The irreducible factors of $\Phi_p(t)$ modulo ℓ are in one-to-one correspondence with the primes of $\mathbb{Z}[\zeta_p]$ over ℓ, and each of their degrees is equal to the residue field degree of the corresponding prime over ℓ. The latter equals the order $a = \mathrm{ord}_p(\ell)$ of ℓ modulo p [3, Ch. 12.2, Exercise #20].

We shall soon explicitly construct a cover of $\mathbb{P}^1_k$ ramified only over ∞ with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. But before we do so, we start with a specific example.

Example 3.3 Let p be an odd prime. Consider the Artin-Schreier cover $\phi : Y_2 \to \mathbb{P}^1_k$ corresponding to the field extension $k(x) \hookrightarrow k(x)[y]/(y^p - y - x^2)$. By Lemma 3.2(3), the genus of Y_2 is $g_Y = (p-1)/2$.

Let $\mathrm{Jac}\,(Y)$ be the Jacobian of Y. The automorphism τ of Y given by $\tau(y) = y + 1$ defines an automorphism of $\mathrm{Jac}\,(Y)$ of order p.

Now we describe the action of τ on the subgroup $\mathrm{Jac}\,(Y)[2]$ of 2-torsion points of $\mathrm{Jac}\,(Y)$ explicitly. Note that since $2g_Y = (p-1)$, then $\mathrm{Jac}\,(Y)[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{p-1}$ by [7, pg. 64]. Thus we can represent τ as an element of $\mathrm{GL}_{p-1}(\mathbb{Z}/2\mathbb{Z})$.

For $0 \le i \le p-1$, let P_i denote the closed point of Y at which the function $y - i$ vanishes. For each i, the divisors P_i and $D_i = P_i - P_\infty$ on Y can be identified with elements of $\mathrm{Jac}\,(Y)$. Let O be the identity element of $\mathrm{Jac}\,(Y)$, i.e., the linear equivalence class of principal divisors. Then the divisor $2D_i$ is equivalent to O since $\mathrm{div}(y-i) = 2D_i$. Moreover since $\mathrm{div}(x) = D_0 + D_1 + \cdots + D_{p-1}$ is equivalent to 0, we have $D_i \in \mathrm{Jac}\,(Y)[2]$ with the only relation $D_{p-1} = -(D_0 + D_1 + \cdots + D_{p-2})$. In particular, $D_0, \ldots, D_{p-2}$ form a basis of $\mathrm{Jac}\,(Y)[2]$. With respect to this basis, the action of τ can be represented by the $(p-1) \times (p-1)$-matrix

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & 0 & -1 \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}.$$

The characteristic polynomial of τ is $\Phi_p(t) = 1 + t + \ldots t^{p-1} \in (\mathbb{Z}/2\mathbb{Z})[t]$, which factors into irreducible polynomials each of degree equaling the order of 2 modulo p. In particular, τ acts irreducibly on $\mathrm{Jac}\,(Y)[2]$ if and only if 2 is a primitive root modulo p, i.e., if and only if p is an Artin prime.

For example, if $p = 3$, then τ acts irreducibly on $\mathrm{Jac}\,(Y)[2]$ with minimal polynomial $\Phi_3(t) = t^2 + t + 1$. If $p = 7$, then 2 has order 3 modulo 7 and the factorization of $\Phi_7(t)$ into irreducible polynomials is $\Phi_7(t) \equiv (x^3 + x^2 + 1)(x^3 + x + 1)$ modulo 2. Thus the action of τ on $\mathrm{Jac}\,(Y)[2]$ can be represented by the 6×6-matrix

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 and A_2 are the irreducible 3-dimensional companion matrices of $x^3 + x^2 + 1$ and $x^3 + x + 1$ respectively.

For the rest of the paper, let $\phi_s : Y_s \to \mathbb{P}^1_k$ be the Artin-Schreier cover corresponding to the field extension

$$k(x) \hookrightarrow k(x)[y]/(y^p - y - x^s).$$

We show that ϕ_s can be dominated by a Galois cover of $\mathbb{P}^1_k$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ for a equal to the order of ℓ modulo p.

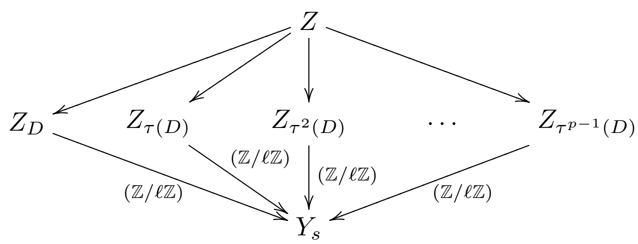
Proposition 3.4 *Let s and ℓ be primes distinct from p. Let $\phi_s : Y_s \to \mathbb{P}^1_k$ be the Artin-Schreier cover with affine equation $y^p - y = x^s$. Let $a = \mathrm{ord}_p(\ell)$ be the order of ℓ modulo p. Then there exists an unramified Galois cover $\omega : Z_a \to Y_s$ with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^a$ such that $\psi_a = \phi_s \circ \omega : Z_a \to \mathbb{P}^1_k$ is a Galois cover of $\mathbb{P}^1_k$ ramified only over ∞ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$.*

Proof By Lemma 3.2(1), $\phi_s : Y_s \to \mathbb{P}^1_k$ is a Galois cover with Galois group $\mathbb{Z}/p\mathbb{Z}$ ramified only at the point P_∞ over ∞. The genus g_s of Y_s is $(p-1)(s-1)/2$. Consider two commuting automorphisms of Y_s defined by

$$\tau : \begin{cases} x \mapsto x, \\ y \mapsto y+1, \end{cases} \qquad \sigma : \begin{cases} x \mapsto \zeta_s x, & \text{where } \zeta_s \text{ is a primitive } s\text{th root of unity,} \\ y \mapsto y. \end{cases}$$

Let $\mathrm{Jac}\,(Y_s)$ be the Jacobian of Y_s. Then τ and σ define commuting automorphisms of $\mathrm{Jac}\,(Y_s)$ of orders p and s respectively. Therefore, $\mathrm{End}\,(\mathrm{Jac}\,(Y_s))$ contains a ring isomorphic to $\mathbb{Z}[\zeta_p, \zeta_s] \cong \mathbb{Z}[\zeta_{ps}]$, which is a $\mathbb{Z}$-module of rank $\phi(ps) = (p-1)(s-1) = 2g_s$. Then $\mathbb{Q}(\zeta_{ps})$ is contained in $\mathrm{End}\,(\mathrm{Jac}\,(Y_s)) \otimes \mathbb{Q}$. In other words, $\mathrm{Jac}\,(Y_s)$ has complex multiplication by $\mathbb{Q}(\zeta_{ps})$.

For a prime ℓ distinct from p, the automorphism τ induces an action on the subgroup $\mathrm{Jac}\,(Y_s)[\ell]$ of ℓ-torsion points of $\mathrm{Jac}\,(Y_s)$. Recall that there is a bijection between ℓ-torsion points D of $\mathrm{Jac}\,(Y_s)$ and unramified $(\mathbb{Z}/\ell\mathbb{Z})$-Galois covers $\omega_D : Z_D \to Y_s$ [6, Prop. 4.11]. Also D has order ℓ if and only if Z_D is connected. This induces a bijection between orbits of τ on the set of unramified $(\mathbb{Z}/\ell\mathbb{Z})$-Galois covers $\omega_D : Z_D \to Y_s$ and on the set of ℓ-torsion points of $\mathrm{Jac}\,(Y_s)$. For a point D of order ℓ of $\mathrm{Jac}\,(Y_s)$, consider the compositum $\omega : Z \to Y_s$ of all of the conjugates $\omega_{\tau^j(D)} : Z_{\tau^i(D)} \to Y_s$ for $0 \le j \le p-1$:



Then Z is invariant under τ and so $\phi_s \circ \omega : Z \to \mathbb{P}^1_k$ is Galois. Moreover, $\phi_s \circ \omega$ is the Galois closure of $\phi_s \circ \omega_D : Z_D \to \mathbb{P}^1_k$.

Suppose there is a non-trivial one-dimensional τ-invariant subspace of $\mathrm{Jac}\,(Y_s)[\ell]$ with eigenvalue 1; i.e. τ acts trivially on this subgroup of order ℓ. This yields a cover $\psi_s \circ \omega_1 : Z_1 \to Y_s \to \mathbb{P}^1_k$. Since the action of τ is trivial, $\psi_s \circ \omega_1$ is Galois, ramified only over ∞, with abelian Galois group $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. This contradicts Lemma 2.4.

Since τ has order p, the minimal polynomial $m_\tau(t)$ of τ divides $t^p - 1 = (t-1)(t^{p-1} + \cdots + 1)$ in $(\mathbb{Z}/\ell\mathbb{Z})[t]$. From the preceding paragraph, there is no non-trivial one-dimensional τ-invariant subspace of $\mathrm{Jac}\,(Y_s)[\ell]$ with eigenvalue 1. This implies that $m_\tau(t)$ divides the pth cyclotomic polynomial $\Phi_p(t) = t^{p-1} + \cdots + 1$ in $(\mathbb{Z}/\ell\mathbb{Z})[t]$. The irreducible factors of $\Phi_p(t)$ in $(\mathbb{Z}/\ell\mathbb{Z})[t]$ all have degree a. Thus the degree of $m_\tau(t)$ equals a.

Since $2g_s = (p-1)(s-1)$, we have $\mathrm{Jac}\,(Y_s)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{(p-1)(s-1)}$, so we can represent τ as an element of $\mathrm{GL}_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})$. We can choose a basis of $\mathrm{Jac}\,(Y_s)[\ell]$ such that τ is represented as an element of $\mathrm{GL}_{(p-1)(s-1)}(\mathbb{Z}/\ell\mathbb{Z})$ in block form. The first irreducible subrepresentation of τ has dimension a. Moreover, since $\mathbb{Q}(\zeta_{ps})$ is a Galois extension of $\mathbb{Q}$, the block form of τ consists entirely of irreducible blocks of the same size. In particular, the number of irreducible blocks is $(p-1)(s-1)/a$. In other words, τ can be represented by an element of $\mathrm{GL}_{(s-1)(p-1)}(\mathbb{Z}/\ell\mathbb{Z})$ of the form

$$\begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ & \ddots & \\ 0 & & A_{(p-1)(s-1)/a} \end{pmatrix},$$

where A_i is an $a \times a$ matrix representing an a-dimensional irreducible subrepresentation of τ on $\mathrm{Jac}\,(Y_s)[\ell]$.

Using the bijection between orbits of $\mathrm{Jac}\,(Y_s)[\ell]$ and orbits of $(\mathbb{Z}/\ell\mathbb{Z})$-covers of Y_s under τ and the above observation for the action of τ on $\mathrm{Jac}\,(Y_s)[\ell]$, there exists an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois cover $\omega : Z_a \to Y_s$ such that $\psi_a = \phi_s \circ \omega : Z_a \to \mathbb{P}^1_k$ is a Galois cover of $\mathbb{P}^1_k$ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. Also ψ_a is ramified only over infinity since ϕ_s is ramified only over ∞ and since ω is unramified. □

4 Minimal genus of $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$-Galois covers of $\mathbb{A}^1_k$

In this section, we find the minimal genus of a curve Z that admits a covering map $\psi : Z \to \mathbb{P}^1_k$ ramified only over ∞, with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$. The minimal genus depends only on ℓ and p. We consider the cases p odd and $p = 2$ separately. We also prove that the number of curves Z of this minimal genus which admit such a covering map is at most $(p-1)/a$ when p is odd and at most $\ell + 1$ when $p = 2$. The following lemma will be useful.

Lemma 4.1 *Let G be a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ where ℓ is a prime distinct from p. If $\psi : Z \to \mathbb{P}^1_k$ is a Galois cover ramified only over ∞ with Galois group G, then the subcover $\omega : Z \to Y$ with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^b$ is unramified.*

Proof The quotient of G by the normal subgroup $N = (\mathbb{Z}/\ell\mathbb{Z})^b$ is $\mathbb{Z}/p\mathbb{Z}$. Thus the cover ψ is a composition $\psi = \phi \circ \omega$ where $\phi : Y \to \mathbb{P}^1_k$ has Galois group $\mathbb{Z}/p\mathbb{Z}$ and is totally ramified at the unique point P_∞ over ∞ and where $\omega : Z \to Y$ has Galois group N and is branched only over P_∞. Then ω is a prime-to-p abelian cover of Y. Let g be the genus of Y. Then by [1, XIII, Cor. 2.12], the prime-to-p fundamental group of $Y - \{P_\infty\}$ is isomorphic to the prime-to-p quotient Γ of the free group on generators $\{a_1, b_1, \ldots, a_g, b_g, c\}$ subject to the relation $\prod_{i=1}^{g}[a_i, b_i] = c^{-1}$. The cover ω corresponds to a surjection of Γ onto N where c maps to the canonical generator of inertia γ of a point of Z over P_∞. Thus N is generated by elements $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma\}$ subject to the relation $\prod_{i=1}^{g}[\alpha_i, \beta_i] = \gamma^{-1}$. Then $\gamma = 1$ since N is abelian and so ω is unramified. □

Theorem 4.2 *Let p be an odd prime. Let ℓ be a prime distinct from p and let a be the order of ℓ modulo p. Then:*

1. *There exists a Galois cover $\psi_a : Z_a \to \mathbb{P}^1_k$ ramified only over ∞ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ such that $g_{Z_a} = 1 + \ell^a(p-3)/2$.*
2. *The integer g_{Z_a} is the minimal genus of a curve Z which admits a covering map $\psi : Z \to \mathbb{P}^1_k$ ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$ for any positive integer b.*
3. *There are at most $(p-1)/a$ isomorphism classes of curves Z which admit a Galois covering map as in part (1) with minimal genus g_{Z_a}.*

Proof By the construction in Proposition 3.4, there exists a Galois cover $\psi_a : Z_a \to \mathbb{P}^1_k$ ramified only over ∞ whose Galois group is a semi-direct product of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$. We compute the genus of the curve Z_a. Recall that ψ_a is a composition $\psi = \phi_2 \circ \omega$ where $\omega : Z \to Y_2$ is an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois cover and $\phi_2 : Y_2 \to \mathbb{P}^1_k$ has Artin-Schreier equation $y^p - y = x^2$. Then Y_2 has genus $g_{Y_2} = (p-1)/2$ by Lemma 3.2(3). By the Riemann-Hurwitz formula, $2g_{Z_a} - 2 = \ell^a(2g_{Y_2} - 2) = \ell^a(p-3)$, i.e., $g_{Z_a} = 1 + \ell^a(p-3)/2$. This completes part (1).

For part (2), suppose $\psi : Z \to \mathbb{P}^1_k$ is a Galois cover ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^b \rtimes \mathbb{Z}/p\mathbb{Z}$. If g is the genus of Z, we will show that $g \geq g_{Z_a}$. As described in the proof of Lemma 4.1, the cover ψ is a composition $\psi = \phi \circ \omega$ where $\phi : Y \to \mathbb{P}^1_k$ has Galois group $\mathbb{Z}/p\mathbb{Z}$ and is ramified only over ∞ and where ω is unramified with group $(\mathbb{Z}/\ell\mathbb{Z})^b$. By the Riemann-Hurwitz formula, $2g-2 = \ell^b(2g_Y - 2)$.

By Artin-Schreier theory, ϕ is given by an equation $y^p - y = f(x)$ where $f \in k[x]$ has degree s for some integer s relatively prime to p. Since the genus g_Y of Y is $(p-1)(s-1)/2$ by Lemma 3.2 (3), we should make s as small as possible. The value $s=1$ is impossible since then Y is a projective line and there do not exist Galois covers of the projective line ramified only over one point with Galois group $\mathbb{Z}/\ell\mathbb{Z}$. Thus $s=2$ yields the smallest possible value for g_Y, namely $(p-1)/2$. Recall that $b \geq a$ by Lemma 2.4. Thus $g \geq 1 + \ell^a(p-3)/2 = g_{Z_a}$.

For part (3), suppose $\psi : Z \to \mathbb{P}^1_k$ is a Galois cover ramified only over ∞ with Galois group of the form $(\mathbb{Z}/\ell\mathbb{Z})^a \rtimes \mathbb{Z}/p\mathbb{Z}$ and the genus of Z satisfies $g_Z = 1 + \ell^a(p-3)/2$. As in part (2), ψ factors as $\phi \circ \omega$ where $\omega : Z \to Y$ is an unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois cover, where $\phi : Y \to \mathbb{P}^1_k$ is an Artin-Schreier cover ramified only over ∞, and where Y has genus $(p-1)/2$. By Lemma 3.2(3), Y has an affine equation $y^p - y = a_2x^2 + a_1x + a_0$ for some $a_0, a_1 \in k$ and $a_2 \in k^*$. Since p is odd and k is algebraically closed, it is possible to complete the square and write $a_2x^2 + a_1x + a_0 = x_1^2 + \epsilon$ where $x_1 = \sqrt{a_2}x + a_1/2\sqrt{a_2}$. After modifying by an automorphism of the projective line, specifically by the affine linear transformation $x \mapsto x_1$, the equation for Y can be rewritten as $y^p - y = x_1^2 + \epsilon$. Since k is algebraically closed, there exists $\delta \in k$ such that $\delta^p - \delta = \epsilon$. Let $y_1 = y - \delta$. After the change of variables $y \mapsto y_1$, the curve Y is isomorphic to the curve Y_2 with affine equation $y_1^p - y_1 = x_1^2$. Thus there is a unique possibility for the isomorphism class of the curve Y.

From the proof of Proposition 3.4, there is a bijection between τ-invariant connected unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois covers of Y_2 and orbits of τ on points D of order ℓ on $\mathrm{Jac}\,(Y_2)$. The action of τ on $\mathrm{Jac}\,(Y_2)[\ell]$ decomposes into $(p-1)/a$ irreducible subrepresentations. Each of these is distinct, because the irreducible factors of $\Phi_p(t) \in (\mathbb{Z}/\ell\mathbb{Z})[t]$ are distinct. Thus there are $(p-1)/a$ choices for a τ-invariant unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois cover of Y_2. Thus there are at most $(p-1)/a$ isomorphism classes of curves Z which admit a Galois covering map as in part (1) with minimal genus g_{Z_a}. □

We note that the set of curves which are unramified $(\mathbb{Z}/\ell\mathbb{Z})^a$-Galois covers of Y_2 may contain fewer than $(p-1)/a$ isomorphism classes of curves.

Theorem 4.3 *Let $p = 2$ and let ℓ be an odd prime. Then:*

1. *There exists a Galois cover $\psi : Z \to \mathbb{P}^1_k$ ramified only over ∞ with Galois group of the form $\mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.*
2. *The minimal genus of a curve Z which admits a covering map as in part (1) is $g_Z = 1$.*
3. *There are at most $\ell + 1$ isomorphism classes of curves Z which admit a Galois covering map as in part (1) with minimal genus $g_Z = 1$.*

Proof Note that the order of ℓ modulo 2 is $a = 1$. For part (1), Lemma 2.5 shows that there exists a semi-direct product of the form $\mathbb{Z}/\ell\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ which is quasi-2. The result is then immediate from Theorem 2.3.

Suppose $\psi : Z \to \mathbb{P}^1_k$ is a Galois cover ramified only over ∞ with Galois group as in part (1). As before, ψ factors as a composition $\phi \circ \omega$. where $\omega : Z \to Y$ has Galois group $\mathbb{Z}/\ell\mathbb{Z}$ and $\phi : Y \to \mathbb{P}^1_k$ is an Artin-Schreier extension with affine equation $y^2 - y = f(x)$ for some $f(x) \in k[x]$ of odd degree s. By Lemma 4.1, ω is unramified.

The minimal genus for Z will thus occur when s is as small as possible. As before, $s = 1$ is impossible, and so $s = 3$ is the smallest choice. In this case, by Lemma 3.2(3), $g_Y = 1$, i.e., Y is an elliptic curve. By the Riemann-Hurwitz formula, the minimal genus for Z is $g_Z = 1 + \ell(g_Y - 1) = 1$, which completes part (2).

For part (3), since k is algebraically closed, we can complete the cube of $f(x)$ and make the corresponding change of variables, which is a scaling and translation of x. So we can assume that Y has affine equation $y^2 - y = x^3 + a_1 x + a_0$ for some $a_0, a_1 \in k$. Then it follows from [11, Appendix A, Prop. 1.1c] that the j-invariant of Y is $j(Y) = 0$ and that the discriminant is $\Delta(Y) = (-1)^4 = 1$. Since k is algebraically closed, by [11, Appendix A, Prop. 1.2b], all elliptic curves Y with $j(Y) = 0$ are isomorphic over k. Thus there is a unique choice for Y up to isomorphism. Without loss of generality, we may assume that $Y = Y_3$ has affine equation $y^2 - y = x^3$.

From the proof of Proposition 3.4, the action of τ on $\mathrm{Jac}\,(Y_3)[\ell]$ decomposes into the direct sum of two 1-dimensional subrepresentations. In other words, the action of τ is diagonal with both eigenvalues equal to -1. The number of non-trivial τ-invariant subgroups of $\mathrm{Jac}\,(Y_3)[\ell]$ is the number of subgroups of order ℓ in $(\mathbb{Z}/\ell\mathbb{Z})^2$, which is $\ell + 1$. As in Theorem 4.2, this implies that there are at most $\ell + 1$ isomorphism classes of curves Z which admit a Galois covering map as in part (1) with minimal genus $g_Z = 1$. □

We note that the set of curves which are unramified $\mathbb{Z}/\ell\mathbb{Z}$-Galois covers of Y_3 may contain fewer than $\ell + 1$ isomorphism classes of curves.

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