THE p-RANK STRATA OF THE MODULI SPACE OF HYPERELLIPTIC CURVES

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ABSTRACT. We prove results about the intersection of the p-rank strata and the boundary of the moduli space of hyperelliptic curves in characteristic $p \geq 3$. Using this, we prove that the $\mathbb{Z}/\ell$-monodromy of every irreducible component of the stratum $\mathcal{H}_g^f$ of hyperelliptic curves of genus g and p-rank f is the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ if $g \geq 3$, $f \geq 1$ and $\ell \neq p$ is an odd prime. These results yield applications about the generic behavior of hyperelliptic curves of given genus and p-rank. The first application is that a generic hyperelliptic curve of genus $g \geq 3$ and p-rank 0 is not supersingular. Other applications are about absolutely simple Jacobians and the generic behavior of class groups and zeta functions of hyperelliptic curves of given genus and p-rank over finite fields.

1. INTRODUCTION

Suppose C is a smooth connected projective hyperelliptic curve of genus $g \geq 1$ over an algebraically closed field k of characteristic $p \geq 3$. The Jacobian $\mathrm{Pic}^0(C)$ is a principally polarized abelian variety of dimension g. The number of physical p-torsion points of $\mathrm{Pic}^0(C)$ is p^f for some integer f, called the p-rank of C, with $0 \leq f \leq g$.

Let $\mathcal{H}_g$ be the moduli space over k of smooth connected projective hyperelliptic curves of genus g; it is a smooth Deligne-Mumford stack over k. The p-rank induces a stratification $\mathcal{H}_g = \cup \mathcal{H}_g^f$ by locally closed reduced substacks $\mathcal{H}_g^f$, whose geometric points correspond to hyperelliptic curves of genus g and p-rank f.

In this paper, we prove two results about $\mathcal{H}_g^f$ for $g \geq 3$. The first is about the boundary of $\mathcal{H}_g^f$, specifically that the boundary of every irreducible component of $\mathcal{H}_g^f$ contains the moduli point of a singular curve which is a tree of elliptic curves and which has p-rank f. The second is that, for an odd prime number ℓ distinct from p, the ℓ-adic monodromy group of every irreducible component of $\mathcal{H}_g^f$ is the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. Heuristically, this means that p-rank constraints alone do not force the existence of extra automorphisms (or other algebraic cycles) on a family of hyperelliptic curves.

This paper is a natural generalization of our paper [AP08], which is about $\mathcal{M}_g^f$, the p-rank strata of the moduli space of curves. The two papers share essential similarities, but there are several new phenomena for hyperelliptic p-rank strata which increase the difficulty of the proofs and influence the final results in this paper. First, the boundary component Δ_0 is more complicated for $\overline{\mathcal{H}}_g$ than for $\overline{\mathcal{M}}_g$. Second, for a singular hyperelliptic curve which is formed as a chain of two hyperelliptic curves of smaller genera, the set of possibilities for the location of the ordinary

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double point is discrete. These two facts play a key role in the degeneration arguments which are part of the inductive step of the proofs. The third issue, which arises in a base case for the monodromy result, is that the stratum $\mathcal{H}_3^0$ is not nearly as well understood as $\mathcal{M}_3^0$.

We now state the results of this paper more precisely.

Theorem 3.11(c). *Suppose p is an odd prime, $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$, the p-rank f stratum in $\mathcal{H}_g$. Then the closure $\overline{S}$ of S in $\overline{\mathcal{H}}_g$ contains the moduli point of a tree of g elliptic curves, of which f are ordinary and $g - f$ are supersingular.*

Our proof does not yield much information on the structure of the tree in Theorem 3.11; however, once the tree's structure is fixed, we prove that any choice of labeling of f components as ordinary and $g - f$ components as supersingular will occur for some moduli point in $\overline{S}$.

As a consequence of Theorem 3.11, we give an application to Newton polygons of hyperelliptic curves. This relies on and generalizes [Oor91, Thm. 1.12], which is the case $g = 3$.

Corollary 3.16. *Suppose p is an odd prime and $g \geq 3$. Let S be an irreducible component of $\mathcal{H}_g^0$, the p-rank 0 stratum in $\mathcal{H}_g$. Then S contains the moduli point of a curve whose Jacobian is not supersingular.*

The second theorem requires some notation. Let S be a connected stack over k, and let s be a geometric point of S. Let $C \to S$ be a relative smooth proper curve of genus g over S. Then $\mathrm{Pic}^0(C)[\ell]$ is an étale cover of S with geometric fibers isomorphic to $(\mathbb{Z}/\ell)^{2g}$. The fundamental group $\pi_1(S, s)$ acts linearly on the fiber $\mathrm{Pic}^0(C)[\ell]_s$, and the monodromy group $\mathsf{M}_\ell(C \to S, s)$ is the image of $\pi_1(S, s)$ in $\mathrm{Aut}(\mathrm{Pic}^0(C)[\ell]_s)$. For the second result, we determine $\mathsf{M}_\ell(S) := \mathsf{M}_\ell(C \to S, s)$, where S is an irreducible component of $\mathcal{H}_g^f$ and $C \to S$ is the tautological curve.

Theorem 4.10-4.11. *Suppose p is an odd prime, $g \geq 1$, and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$, the p-rank f stratum in $\mathcal{H}_g$.*

(i) *If $1 \leq f \leq g$ and ℓ is an odd prime distinct from p, then $\mathsf{M}_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$*

(ii) *If $f = 0$ and if $g \geq 5$, then $\mathsf{M}_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ for all primes ℓ outside a finite set which depends only on p.*

We also prove that the ℓ-adic monodromy group is $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$ in the situation of Theorem 4.10-4.11. (Note that the case of ordinary hyperelliptic curves, i.e., when $f = g$, follows directly from previous work, see J.K.Yu [unpublished], [AP07, Thm. 3.4], or [Hal08, Thm. 5.1].) In addition, we determine the p-adic monodromy of components of $\mathcal{H}_g^f$ when $f \geq 1$ (Proposition 4.14).

We give applications to class groups and zeta functions of hyperelliptic curves of given genus and p-rank over finite fields. These build upon [KS99, Thm. 9.7.13] and [Kow06, Thm. 6.1].
Applications: Let $\mathbb{F}$ be a finite field of characteristic p. Under the hypotheses of Theorem 4.10-4.11:

(i) there is an $\overline{\mathbb{F}}$-curve C of genus g and p-rank f whose Jacobian is absolutely simple (Application 4.15);

(ii) if $|\mathbb{F}| \equiv 1 \bmod \ell$, about $\ell/(\ell^2 - 1)$ of the $\mathbb{F}$-curves of genus g and p-rank f have a point of order ℓ on their Jacobian (Application 4.17);

(iii) for most $\mathbb{F}$-curves C of genus g and p-rank f, the splitting field of the numerator of the zeta function of C has degree $2^g g!$ over $\mathbb{Q}$ (Application 4.19).

The proof of Theorem 4.10-4.11 relies on Theorem 3.11 because it uses degeneration in order to proceed by induction on the genus. For the inductive step, Theorem 3.11 implies that the closure of every component S of $\mathcal{H}_g^f$ in $\overline{\mathcal{H}}_g$ intersects the stratum $\Delta_{1,1}$ of the boundary of $\mathcal{H}_g$ (Corollary 3.14). As in [AP07], this implies that the monodromy group of S contains two non-identical copies of $\mathrm{Sp}_{2g-2}(\mathbb{Z}/\ell)$. The base cases use [Cha05, Prop. 4.4] for $\mathcal{H}_2^f$ and an analysis of endomorphism algebras for $\mathcal{H}_5^0$.

Here is an outline of the paper. Notation and background are found in Section 2. Section 3 contains the results about the boundary of the p-rank f stratum $\mathcal{H}_g^f$ and the application to Newton polygons. This section ends with some open questions about the geometry of $\mathcal{H}_g^f$. For example, the number of irreducible components of $\mathcal{H}_g^f$ is known only in special cases. The results about monodromy and the applications to Jacobians, class groups and zeta functions are in Section 4.

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2. BACKGROUND

Let k be an algebraically closed field of characteristic $p \geq 3$. With the exception of Section 4.5, where we work over a finite field, all objects are defined on the category of k-schemes. Let ℓ be an odd prime distinct from p. We fix an isomorphism $\boldsymbol{\mu}_\ell \cong \mathbb{Z}/\ell$.

2.1. **Moduli spaces.** For a natural number g consider the following well-known categories, each of which is fibered in groupoids over the category of k-schemes in the étale topology:

- $\mathcal{A}_g$ principally polarized abelian schemes of dimension g;
- $\mathcal{M}_g$ smooth connected proper relative curves of genus g;
- $\mathcal{H}_g$ smooth connected proper relative hyperelliptic curves of genus g;
- $\overline{\mathcal{M}}_g$ stable curves of genus g.

Each of these is a smooth Deligne-Mumford stack, and $\overline{\mathcal{M}}_g$ is proper [DM69, Thm. 5.2]. There is a natural inclusion $\mathcal{H}_g \to \mathcal{M}_g$; let $\overline{\mathcal{H}}_g$ be the closure of $\mathcal{H}_g$ in $\overline{\mathcal{M}}_g$. Thus there are the following categories:

- $\overline{\mathcal{H}}_g$ stable hyperelliptic curves of genus g;
- $\widetilde{\mathcal{H}}_g$ stable hyperelliptic curves of genus g, along with a labeling of the smooth ramification locus (see [AP07, Section 2.2]).

Again, these are smooth proper Deligne-Mumford stacks [Eke95, Thm. 3.2], see e.g. [AP07, Lemma 2.2] for $\widetilde{\mathcal{H}}_g$. The forgetful map $\varpi_g : \widetilde{\mathcal{H}}_g \to \overline{\mathcal{H}}_g$ is étale and Galois, with covering group $\Gamma_g \cong \mathrm{Sym}(2g+2)$. If $S \subset \overline{\mathcal{H}}_g$, let $\overline{S}$ be the closure of S in $\overline{\mathcal{H}}_g$. Let $\mathcal{C}_g$ be the tautological curve over $\overline{\mathcal{H}}_g$.

For a natural number r, let $\overline{\mathcal{M}}_{g;r}$ be the Deligne-Mumford stack of stable curves of genus g with r marked points. Let $\overline{\mathcal{H}}_{g;r} = \overline{\mathcal{H}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g;r}$ and let $\widetilde{\mathcal{H}}_{g;r} = \widetilde{\mathcal{H}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g;r}$. The forgetful map $\phi_{g;r} : \overline{\mathcal{H}}_{g;r} \to \overline{\mathcal{H}}_g$ is proper, flat and surjective with connected fibers, and thus is a fibration. Let $\mathcal{H}_{g;r} = \overline{\mathcal{H}}_{g;r} \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$.

2.2. **Stratifications.** Let X be a principally polarized abelian variety of dimension g defined over k. There is an integer $f(X)$ between 0 and g, called the p-rank of X, such that $X[p](k) \cong (\mathbb{Z}/p)^{f(X)}$. More generally, if X/k is a semiabelian variety, then its p-rank is $\dim_{\mathbb{F}_p} \mathrm{Hom}(\boldsymbol{\mu}_p, X)$. The p-rank of a curve is that of its Jacobian. If $X \to S$ is a semiabelian scheme over a Deligne-Mumford stack, then there is a stratification $S = \cup S^f$ by locally closed substacks such that $s \in S^f(k)$ if and only if $f(X_s) = f$ (this follows from [Kat79, Thm. 2.3.1], see, e.g., [AP08, Lemma 2.1]). Thus, $\mathcal{H}_g^f$ is the locus in $\mathcal{H}_g$ parametrizing hyperelliptic curves of p-rank f. Every component of $\mathcal{H}_g^f$ has dimension $g - 1 + f$ [GP05, Thm. 1].

Here is the definition of the Newton polygon of an abelian variety; see [Dem72, Chap. IV] for details. The isogeny class of a p-divisible group G/k is determined by $\nu(G)$, a lower-convex polygon in $\mathbb{R}^2$ connecting $(0,0)$ to $(\mathrm{height}(G), \dim(G))$ with slopes $\lambda \in \mathbb{Q} \cap [0,1]$ and integral breakpoints. If X/k is an abelian variety, its Newton polygon is that of its p-divisible group $X[p^\infty]$. The Newton polygon is a finer invariant than the p-rank; indeed, the p-rank of X is exactly the length of the slope 0 part of the Newton polygon. For example, X is ordinary exactly when its Newton polygon only has slopes 0 and 1. By definition, X is supersingular if its Newton polygon only has slope $1/2$.

2.3. **The boundary of the moduli space of hyperelliptic curves.** The boundary of $\overline{\mathcal{H}}_g$ is $\partial\overline{\mathcal{H}}_g = \overline{\mathcal{H}}_g - \mathcal{H}_g$. The following description of $\partial\overline{\mathcal{H}}_g$ follows [CH88, Sec. 4(b)] closely. In fact, while [CH88] is written for the base field $\mathbb{C}$, the description of $\mathcal{H}_g$ and $\overline{\mathcal{H}}_g$ is valid in any characteristic [Yam04]. Briefly, the irreducible components of $\partial\overline{\mathcal{H}}_g$ come from restriction of the components of the boundary of $\overline{\mathcal{M}}_g$, except that Δ_0 breaks into several components.

If $g \geq 2$, the boundary $\partial\overline{\mathcal{H}}_g$ is the union of components $\Delta_i = \Delta_i[\overline{\mathcal{H}}_g]$ for $1 \leq i \leq g-1$ and $\Xi_i = \Xi_i[\overline{\mathcal{H}}_g]$ for $0 \leq i \leq g-2$ by [Yam04, p.410]. Here Δ_i and Δ_{g-i} are the same substack of $\overline{\mathcal{H}}_g$ and Ξ_i and Ξ_{g-i-1} are the same substack of $\overline{\mathcal{H}}_g$. Each Δ_i and Ξ_i is an irreducible divisor in $\overline{\mathcal{H}}_g$.

For $1 \leq i \leq g-1$, if η is the generic point of Δ_i, then the curve $\mathcal{C}_{g,\eta}$ is a chain of two smooth irreducible hyperelliptic curves Y_1 and Y_2, of genera i and $g-i$, intersecting in one ordinary double point P. The hyperelliptic involution ι stabilizes each of Y_1 and Y_2. The point P is a ramification point for the restriction of ι to each of Y_1 and Y_2 but is not part of the smooth ramification locus.

If η is the generic point of Ξ_0, then the curve $\mathcal{C}_{g,\eta}$ is an irreducible hyperelliptic curve self-intersecting in an ordinary double point P. The normalization of $\mathcal{C}_{g,\eta}$ is a smooth hyperelliptic curve Y_1 of genus $g-1$ and the inverse image of P in the normalization consists of an orbit under the hyperelliptic involution.

For $1 \leq i \leq g-2$, if η is the generic point of Ξ_i, then the curve $\mathcal{C}_{g,\eta}$ has two components Y_1 and Y_2, which are smooth irreducible hyperelliptic curves, of genera i and $g-1-i$, intersecting in two ordinary double points P and Q. The hyperelliptic involution ι stabilizes each of Y_1 and Y_2. The points P and Q form an orbit of the restriction of ι to each of Y_1 and Y_2.

One can associate to a stable curve C its dual graph, in which the vertices are in bijection with the irreducible components of C and in which there is an edge between two vertices exactly when the corresponding components intersect. A component of C is called terminal if the corresponding vertex is a leaf of the dual graph. A curve is called a tree if its dual graph is a tree. A curve is called a tree of elliptic curves if it is a tree and if each of its irreducible components is an elliptic curve.

A stable curve is a tree if and only if its Picard variety is represented by an abelian scheme; such a curve is also said to be of compact type. Let $\Delta_0 = \Delta_0[\overline{\mathcal{H}}_g]$ be the union of Ξ_i for $0 \leq i \leq \lfloor (g-1)/2 \rfloor$. The moduli points of stable hyperelliptic curves which are not of compact type are exactly the points of $\Delta_0[\overline{\mathcal{H}}_g]$.

If S is a stack equipped with a map $S \to \overline{\mathcal{H}}_g$, let $\Delta_i[S]$ denote $S \times_{\overline{\mathcal{H}}_g} \Delta_i[\overline{\mathcal{H}}_g]$. In particular, $\Delta_i[\widetilde{\mathcal{H}}_g] = \widetilde{\mathcal{H}}_g \times_{\overline{\mathcal{H}}_g} \Delta_i$. Also define $\Delta_i[\overline{\mathcal{H}}_g]^f := (\Delta_i[\overline{\mathcal{H}}_g])^f$. Similar conventions are employed for Ξ_i.

2.4. **Clutching maps.** Here are three types of clutching maps for positive integers g_1 and g_2:

$$\kappa_{g_1,g_2} : \widetilde{\mathcal{H}}_{g_1} \times \widetilde{\mathcal{H}}_{g_2} \xrightarrow{\widetilde{\kappa}_{g_1,g_2}} \widetilde{\mathcal{H}}_{g_1+g_2} \xrightarrow{\varpi_{g_1+g_2}} \overline{\mathcal{H}}_{g_1+g_2};$$

$$\kappa_{g_1} : \overline{\mathcal{H}}_{g_1;1} \longrightarrow \overline{\mathcal{H}}_{g_1+1};$$

$$\lambda_{g_1,g_2} : \overline{\mathcal{H}}_{g_1;1} \times \overline{\mathcal{H}}_{g_2;1} \longrightarrow \overline{\mathcal{H}}_{g_1+g_2+1}.$$

Each of these clutching maps is the restriction of a finite, unramified morphism between moduli spaces of labeled curves [Knu83, Cor. 3.9]. These maps can be described in terms of their images on T-points for an arbitrary k-scheme T.

2.4.1. *Information about* κ_{g_1,g_2}. For $i = 1, 2$, suppose $s_i \in \widetilde{\mathcal{H}}_{g_i}(T)$ is the moduli point of a hyperelliptic curve Y_i with labeled smooth ramification locus. Then $\widetilde{\kappa}_{g_1,g_2}(s_1, s_2)$ is the moduli point of the labeled T-curve Y where Y has components Y_1 and Y_2, and where the last ramification point of Y_1 and the first ramification point of Y_2 are identified in an ordinary double point. This nodal section is dropped from the labeling of the ramification points; the remaining smooth ramification sections of Y_1 maintain labels $\{1, \ldots, 2g_1 + 1\}$ and the remaining ramification sections of Y_2 are relabeled $\{2g_1 + 2, \ldots, 2(g_1 + g_2) + 2\}$. There is a unique hyperelliptic involution on Y which restricts to the hyperelliptic involution on Y_1 and Y_2. Moreover, $\kappa_{g_1,g_2}(s_1, s_2)$ is the moduli point of the (unlabeled) hyperelliptic curve Y. The image of κ_{g_1,g_2} is $\Delta_{g_1}[\overline{\mathcal{H}}_{g_1+g_2}]$.

By [BLR90, Ex. 9.2.8],

$$\mathrm{Pic}^0(Y) \cong \mathrm{Pic}^0(Y_1) \times \mathrm{Pic}^0(Y_2). \tag{2.4.1}$$

Then the p-rank of Y is

$$f(Y) = f(Y_1) + f(Y_2). \tag{2.4.2}$$

2.4.2. *Information about* κ_{g_1}. Suppose $s_1 \in \overline{\mathcal{H}}_{g_1;1}(T)$ is the moduli point of $(Y_1; P)$, a hyperelliptic curve with a marked section. Then $\kappa_{g_1}(s_1)$ is the moduli point of the T-curve Y, where Y is the stable model of the curve obtained by identifying the sections P and $\iota(P)$ in an ordinary double point P'. The hyperelliptic involution on Y_1 descends to a unique hyperelliptic involution on Y. The image of κ_{g_1} is $\Xi_0[\overline{\mathcal{H}}_{g_1+1}]$.

By [BLR90, Ex. 9.2.8], $\mathrm{Pic}^0(D)$ is an extension

$$0 \longrightarrow Z \longrightarrow \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(Y_1) \longrightarrow 0\ , \tag{2.4.3}$$

where Z is a one-dimensional torus. In particular, the toric rank of $\mathrm{Pic}^0(Y)$ is one greater than that of $\mathrm{Pic}^0(Y_1)$, and their maximal projective quotients are isomorphic, so that

$$f(Y) = f(Y_1) + 1. \tag{2.4.4}$$

2.4.3. *Information about λ_{g_1,g_2}.* For $i = 1, 2$, suppose $s_i \in \overline{\mathcal{H}}_{g_i;1}(T)$ is the moduli point of $(Y_i; P_i)$, a hyperelliptic curve with a marked section. Then $\lambda_{g_1,g_2}(s_1, s_2)$ is the moduli point of the stable model Y of the T-curve obtained by identifying P_1 and P_2 in an ordinary double point P and by identifying $\iota(P_1)$ and $\iota(P_2)$ in an ordinary double point Q. There is a unique hyperelliptic involution on Y that restricts to the hyperelliptic involution on Y_1 and Y_2. The image of λ_{g_1,g_2} is $\Xi_{g_1}[\overline{\mathcal{H}}_{g_1+g_2+1}]$.

By [BLR90, Ex. 9.2.8], $\mathrm{Pic}^0(Y)$ is an extension

$$0 \longrightarrow Z \longrightarrow \mathrm{Pic}^0(Y) \longrightarrow \mathrm{Pic}^0(Y_1) \times \mathrm{Pic}^0(Y_2) \longrightarrow 0\ , \tag{2.4.5}$$

where Z is a one-dimensional torus. In particular,

$$f(D) = f(Y_1) + f(Y_2) + 1. \tag{2.4.6}$$

For later use, here is a description of the stable model Y when P_1 is a ramification point of Y_1, but P_2 is not a ramification point of Y_2. Then Y consists of three components, namely the strict transforms of Y_1 and Y_2 and an exceptional component W which is a projective line. Also Y_1 intersects W in an ordinary double point and Y_2 intersects W in two other points, which are also ordinary double points.

2.4.4. *Clutching along trees.* The definition of $\widetilde{\kappa}_{g_1,g_2}$ above relies on an arbitrary, although convenient, choice of sections along which to glue, and a labeling of the smooth ramification locus of the resulting curve. By considering morphisms of the form $\gamma_{g_1+g_2} \circ \widetilde{\kappa}_{g_1,g_2} \circ (\gamma_{g_1} \times \gamma_{g_2})$, where $\gamma_{g_1+g_2} \in \Gamma_{g_1+g_2}$ and $\gamma_{g_1} \in \Gamma_{g_1}$ and $\gamma_{g_2} \in \Gamma_{g_2}$, it is possible to clutch along arbitrary sections, with complete control over the subsequent labeling.

This can be used to describe configurations of curves of compact type, as follows. A clutching tree is a finite tree Λ along with a choice of natural number g_v for each vertex $v \in \Lambda$ such that $\deg(v) \le 2g_v + 2$. Such a tree is called a clutching tree of elliptic curves if $g_v = 1$ for all $v \in \Lambda$. Let $|\Lambda|$ be the number of vertices in Λ, and let $g(\Lambda) = \sum_{v \in \Lambda} g_v$.

Using a product of the clutching maps defined above, one can define a morphism

$$\kappa_\Lambda : \times_{v \in V} \widetilde{\mathcal{H}}_{g_v} \xrightarrow{\widetilde{\kappa}_\Lambda} \widetilde{\mathcal{H}}_{g(\Lambda)} \xrightarrow{\varpi_{g(\Lambda)}} \overline{\mathcal{H}}_{g(\Lambda)}.$$

Let Δ_Λ be the image of κ_Λ. If η is the generic point of Δ_Λ, then $\mathcal{C}_{g,\eta}$ is a hyperelliptic curve of compact type with dual graph isomorphic to Λ, such that the irreducible component of $\mathcal{C}_{g,\eta}$ corresponding to the vertex v has genus g_v.

Suppose v_1 and v_2 are adjacent vertices in a clutching tree Λ. Consider the tree Λ' obtained by identifying v_1 and v_2 in a new vertex, v, which is adjacent to all neighbors of v_1 or v_2 in Λ, with label $g_v = g_{v_1} + g_{v_2}$. Then κ_Λ factors through $\kappa_{\Lambda'}$ and $\Delta_\Lambda \subset \Delta_{\Lambda'}$. A tree Λ refines a tree Λ' if Λ' can be obtained from Λ through iterations of this construction.

2.4.5. *One more clutching map.* In the special case where Λ is a tree on three vertices, where the leaves v_1 and v_3 have $g_{v_1} = g_{v_3} = 1$, and where $g_{v_2} = g - 2$, one obtains the following diagram:

(2.4.7)
$$\begin{array}{ccc} \widetilde{\mathcal{H}}_1 \times \widetilde{\mathcal{H}}_{g-2} \times \widetilde{\mathcal{H}}_1 & \longrightarrow & \widetilde{\mathcal{H}}_{g-1} \times \widetilde{\mathcal{H}}_1 \\ \downarrow & \searrow^{\widetilde{\kappa}_{1,g-2,1}} & \downarrow \\ \widetilde{\mathcal{H}}_1 \times \widetilde{\mathcal{H}}_{g-1} & \longrightarrow & \widetilde{\mathcal{H}}_g. \end{array}$$

Let $\Delta_{1,1}[\overline{\mathcal{H}}_g]$ be the image of $\kappa_{1,g-2,1} = \varpi_g \circ \widetilde{\kappa}_{1,g-2,1}$; it is an irreducible component of the self-intersection locus of $\Delta_1[\overline{\mathcal{H}}_g]$. If η is the generic point of $\Delta_{1,1}[\overline{\mathcal{H}}_g]$, then the curve $\mathcal{C}_{g,\eta}$ is a chain of three smooth irreducible hyperelliptic curves Y_1, Y_2, Y_3 with $g_{Y_1} = g_{Y_3} = 1$ and $g_{Y_2} = g - 2$. For $i \in \{1, 3\}$, the curves Y_i and Y_2 intersect in a point P_i which is an ordinary double point.

3. Boundary of the hyperelliptic p-rank strata

3.1. **Preliminary intersection results.** Suppose $p \geq 3$, and $g \geq 1$ and $0 \leq f \leq g$. The p-rank strata of the boundary of $\overline{\mathcal{H}}_g$ are easy to describe using the clutching maps. First, if $1 \leq i \leq g-1$, then (2.4.2) implies that $\Delta_i[\overline{\mathcal{H}}_g]^f$ is the union of the images of $\widetilde{\mathcal{H}}_i^{f_1} \times \widetilde{\mathcal{H}}_{g-i}^{f_2}$ under $\kappa_{i,g-i}$ as (f_1, f_2) ranges over all pairs such that

(3.1.1)
$$0 \leq f_1 \leq i,\ 0 \leq f_2 \leq g - i \text{ and } f_1 + f_2 = f.$$

Second, if $f \geq 1$, then $\Xi_0[\overline{\mathcal{H}}_g]^f$ is the image of $\overline{\mathcal{H}}_{g-1;1}^{f-1}$ under κ_{g-1} by (2.4.4). Third, if $f \geq 1$ and $1 \leq i \leq g-2$, then (2.4.6) implies that $\Xi_i[\overline{\mathcal{H}}_g]^f$ is the image of $\overline{\mathcal{H}}_{i;1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i;1}^{f_2}$ under $\lambda_{i,g-1-i}$ as (f_1, f_2) ranges over all pairs such that

(3.1.2)
$$0 \leq f_1 \leq i,\ 0 \leq f_2 \leq g - 1 - i \text{ and } f_1 + f_2 = f - 1.$$

Lemma 3.1. *Suppose $g \geq 2$ and $0 \leq f \leq g$.*

(a) *If $1 \leq i \leq g-1$, then every component of $\Delta_i[\overline{\mathcal{H}}_g]^f$ and of $\Delta_i[\widetilde{\mathcal{H}}_g]^f$ has dimension $g - 2 + f$.*
(b) *If $f \geq 1$ and $0 \leq i \leq g-2$, then every component of $\Xi_i[\overline{\mathcal{H}}_g]^f$ and of $\Xi_i[\widetilde{\mathcal{H}}_g]^f$ has dimension $g - 2 + f$.*

Proof. For parts (a) and (b), the claims for $\overline{\mathcal{H}}_g$ and for $\widetilde{\mathcal{H}}_g$ are equivalent, since ϖ_g is a finite map which preserves the p-rank stratification. Recall that $\overline{\mathcal{H}}_g^f$ and $\widetilde{\mathcal{H}}_g^f$ are pure of dimension $g - 1 + f$ [GP05, Thm. 1].

For part (a), suppose $0 \leq f \leq g$, $1 \leq i \leq g-1$, and that (f_1, f_2) is a pair which satisfies (3.1.1). Then $\widetilde{\mathcal{H}}_i^{f_1} \times \widetilde{\mathcal{H}}_{g-i}^{f_2}$ is pure of dimension $\dim(\widetilde{\mathcal{H}}_i^{f_1}) + \dim(\widetilde{\mathcal{H}}_{g-i}^{f_2}) = g - 2 + f$. Since $\kappa_{i,g-i}$ is finite, $\Delta_i[\overline{\mathcal{H}}_g]^f$ is pure of dimension $g - 2 + f$ as well.

For part (b), first suppose $i = 0$. Then $\overline{\mathcal{H}}_{g-1;1}^{f-1}$ is pure of dimension $\dim(\overline{\mathcal{H}}_{g-1}^{f-1}) + 1 = g - 2 + f$. Since κ_{g-1} is finite, $\Xi_0[\overline{\mathcal{H}}_g]^f$ is pure of dimension $g - 2 + f$ as well.

To finish part (b), suppose $1 \leq i \leq g-2$, and that (f_1, f_2) is a pair which satisfies (3.1.2). Then $\widetilde{\mathcal{H}}_{i;1}^{f_1} \times \widetilde{\mathcal{H}}_{g-1-i;1}^{f_2}$ is pure of dimension $\dim(\widetilde{\mathcal{H}}_i^{f_1}) + \dim(\widetilde{\mathcal{H}}_{g-i-1}^{f_2}) + 2 = g - 2 + f$. Since $\lambda_{i,g-1-i}$ is finite, $\Xi_i[\overline{\mathcal{H}}_g]^f$ is pure of dimension $g-2+f$ as well. □

The next lemma shows that if η is a generic point of $\overline{\mathcal{H}}_g^f$, then the curve $\mathcal{C}_{g,\eta}$ is smooth. Thus no component of $\overline{\mathcal{H}}_g^f$ is contained in the boundary $\partial\overline{\mathcal{H}}_g$.

Lemma 3.2. *Suppose $g \geq 1$ and $0 \leq f \leq g$.*

(a) *Then $\mathcal{H}_g^f$ is open and dense in $\overline{\mathcal{H}}_g^f$ and $\widetilde{\mathcal{H}}_g^f \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$ is open and dense in $\widetilde{\mathcal{H}}_g^f$.*

(b) *If $r \geq 1$, then $\mathcal{H}_{g;r}^f$ is open and dense in $\overline{\mathcal{H}}_{g;r}^f$ and $\widetilde{\mathcal{H}}_g^f \times_{\overline{\mathcal{H}}_g} \mathcal{H}_g$ is open and dense in $\widetilde{\mathcal{H}}_g^f$.*

Proof. Part (a) is well-known for $g = 1$. For $g \geq 2$, part (a) follows from Lemma 3.1 since $\overline{\mathcal{H}}_g^f$ and $\widetilde{\mathcal{H}}_g^f$ are pure of dimension $g - 1 + f$ [GP05, Thm. 1]. Part (b) follows from part (a) since the p-rank of a labeled curve depends only on the underlying curve, so that $\overline{\mathcal{H}}_{g;r}^f = \overline{\mathcal{H}}_{g;r} \times_{\overline{\mathcal{H}}_g} \overline{\mathcal{H}}_g^f$. □

Lemma 3.3. *Let S be an irreducible component of $\mathcal{H}_g^f$. If $\overline{S}$ intersects a component Γ of $\Delta_i[\overline{\mathcal{H}}_g]^f$ then $\overline{S}$ contains Γ.*

Proof. A smooth proper stack has the same intersection-theoretic properties as a smooth proper scheme [Vis89, p. 614]. In particular, if two closed substacks of $\overline{\mathcal{H}}_g$ intersect then the codimension of their intersection is at most the sum of their codimensions. Now $\overline{S}$ and $\Delta_i[\overline{\mathcal{H}}_g]$ are closed substacks of $\overline{\mathcal{H}}_g$ and $\overline{S} \not\subset \Delta_i[\overline{\mathcal{H}}_g]$. Thus the intersection of $\overline{S}$ with the divisor $\Delta_i[\overline{\mathcal{H}}_g]$ has pure dimension $\dim \overline{S} - 1$, which equals $\dim(\Delta_i[\overline{\mathcal{H}}_g]^f)$ by Lemma 3.1. Thus if $\overline{S}$ intersects a component Γ of $\Delta_i[\overline{\mathcal{H}}_g]^f$ then it must contain the full component Γ. □

Lemma 3.4. *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$.*

(a) *Then $\overline{S}$ intersects $\Delta_0[\overline{\mathcal{H}}_g]$ if and only if $f \geq 1$.*

(b) *If $f \geq 1$, then each irreducible component of $\Delta_0[\overline{S}]$ contains either (i) the image of a component of $\overline{\mathcal{H}}_{g-1;1}^{f-1}$ under κ_{g-1} or (ii) the image of a component of $\overline{\mathcal{H}}_{i;1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i;1}^{f_2}$ under $\lambda_{i,g-1-i}$ for some $1 \leq i \leq g-2$ and some pair (f_1, f_2) which satisfies* (3.1.2).

(c) *If $f = 0$, then $\overline{S}$ contains the image of a component of $\widetilde{\mathcal{H}}_i^0 \times \widetilde{\mathcal{H}}_{g-i}^0$ under $\kappa_{i,g-i}$ for some $1 \leq i \leq g-1$.*

Proof. If $f = 0$, then (2.4.4) implies that $\overline{S}$ does not intersect $\Delta_0[\overline{\mathcal{H}}_g]$. If $f \geq 1$, then $\overline{S}$ is a complete substack of dimension greater than $g-1$. By [FvdG04, Lemma 2.6], a complete substack of $\overline{\mathcal{H}}_g - \Delta_0[\overline{\mathcal{H}}_g]$ has dimension at most $g-1$. Therefore, $\overline{S}$ intersects $\Delta_0[\overline{\mathcal{H}}_g]$. This completes part (a).

For part (b), each irreducible component of $\Delta_0[\overline{S}]$ intersects either $\kappa_{g-1}(\overline{\mathcal{H}}_{g-1;1}^{f-1}) \subset \Xi_0[\overline{\mathcal{H}}_g]$ or $\lambda_{i,g-1-i}(\overline{\mathcal{H}}_{i;1}^{f_1} \times \overline{\mathcal{H}}_{g-1-i;1}^{f_2}) \subset \Xi_i[\overline{\mathcal{H}}_g]$ for some $1 \leq i \leq g-2$ and some pair (f_1, f_2) which satisfies (3.1.2). The result then follows from Lemma 3.3.

For part (c), recall that $\mathcal{H}_g$ contains no complete substacks of positive dimension (e.g., [Yam04, Cor. 1.9]). Thus $\overline{S}$ intersects Δ_i for some $0 \le i \le g-1$. By part (a), $i \neq 0$. The result follows from Lemma 3.3. □

3.2. **Complements on trees.** Many of the results for Δ_i for positive i have analogues for Δ_Λ. For a clutching tree Λ and a nonnegative integer f, define an index set by

$$\mathcal{F}(\Lambda, f) = \left\{ \{f_v : v \in \Lambda\} : 0 \le f_v \le g_v, \sum_v f_v = f \right\}. \tag{3.2.1}$$

Lemma 3.5. *Let Λ be a clutching tree with $g(\Lambda) = g$.*

(a) *The p-rank strata of $\Delta_\Lambda[\overline{\mathcal{H}}_g]$ are given by*

$$\Delta_\Lambda[\overline{\mathcal{H}}_g]^f = \bigcup_{\{f_v\} \in \mathcal{F}(\Lambda, f)} \kappa_\Lambda(\times_{v \in \Lambda} \widetilde{\mathcal{H}}_{g_v}^{f_v}). \tag{3.2.2}$$

(b) *Every component of $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$ has dimension $g + f - |\Lambda|$.*

Proof. Part (a) follows from (3.1.1) and induction on $|\Lambda|$. Part (b) follows from this and the calculation that, for $\{f_v\} \in \mathcal{F}(\Lambda, f)$,

$$\dim(\times_{v \in \Lambda} \widetilde{\mathcal{H}}_{g_v}^{f_v}) = \sum_{v \in \Lambda} (g_v + f_v - 1) = g + f - |\Lambda|.$$

□

Lemma 3.6. *Let S be an irreducible component of $\mathcal{H}_g^f$. Let Λ be a clutching tree with $g(\Lambda) = g$. If $\overline{S}$ intersects a component Γ of $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$, then $\overline{S}$ contains Γ.*

Proof. The proof is similar to that of Lemma 3.3. Note that $\dim \Gamma \ge \dim \overline{S} + \dim \Delta_\Lambda - \dim \overline{\mathcal{H}}_g$. By Lemma 3.5(b), this equals $g + f - |\Lambda| = \dim \Delta_\Lambda[\overline{\mathcal{H}}_g]^f$. □

3.3. **Adjusting marked points and trees.** The next lemma shows that one can adjust the marked points of an r-marked hyperelliptic curve of genus g and p-rank f without leaving the irreducible component of $\overline{\mathcal{H}}_{g;r}^f$ to which its moduli point belongs.

Lemma 3.7. *Let S be an irreducible component of $\mathcal{H}_{g;r}^f$, and let $\overline{S}$ be the closure of S in $\overline{\mathcal{H}}_{g;r}^f$. Then $\overline{S} = \phi_{g;r}^{-1}(\phi_{g;r}(\overline{S}))$. Equivalently, if T is a k-scheme, if $(C; P_1, \ldots, P_r) \in \overline{S}(T)$, and if $(Q_1, \ldots, Q_r)$ is any other marking of C, then $(C; Q_1, \ldots, Q_r) \in \overline{S}(T)$.*

Proof. It suffices to show that $\phi_{g;r}^{-1}(\phi_{g;r}(\overline{S})) \subseteq \overline{S}$. Note that $\overline{S}$ is the largest irreducible substack of $\overline{\mathcal{H}}_{g;r}^f$ which contains S. The fibers of $\phi_{g;r}|_S$ are irreducible, so $\phi_{g;r}^{-1}(\phi_{g;r}(S))$ is also an irreducible substack of $\overline{\mathcal{H}}_{g;r}^f$ which contains S. Thus $\phi_{g;r}^{-1}(\phi_{g;r}(S)) \subset \overline{S}$. This shows that $\phi_{g;r}^{-1}(\phi_{g;r}(S)) = S$.

To finish the proof, it suffices to show that the T-points of $\overline{S}$ and $\phi_{g;r}^{-1}(\phi_{g;r}(\overline{S}))$ coincide for an arbitrary k-scheme T. To this end, let $\alpha = (C; P_1, \ldots, P_r) \in \overline{S}(T)$, and let $\beta = (C; Q_1, \ldots, Q_r) \in \overline{\mathcal{H}}_{g;r}^f(T)$. Note that $\phi_{g;r}(\beta) = \phi_{g;r}(\alpha)$, and $\phi_{g;r}(\alpha)$ is supported in the closure of $\phi_{g;r}(S)$ in $\overline{\mathcal{H}}_g^f$. Because $\mathcal{H}_{g;r}$ is dense in $\overline{\mathcal{H}}_{g;r}$, it follows that β is supported in the closure of $\phi_{g;r}^{-1}(\phi_{g;r}(S))$ in $\overline{\mathcal{H}}_{g;r}^f$, which is $\overline{S}$. □

It is not clear whether one can change the labeling of the smooth ramification locus of a hyperelliptic curve without changing the irreducible component of $\widetilde{\mathcal{H}}_g^f$ to which its moduli point belongs. To circumvent this issue, the following lemma about hyperelliptic curves of genus 2 and p-rank 1 will be useful.

Lemma 3.8. (a) *First, $\overline{\mathcal{H}}_2^1$ is irreducible and intersects $\kappa_{1,1}(\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_1^0)$.*

(b) *Second, let $\widetilde{S}$ be an irreducible component of $\widetilde{\mathcal{H}}_2^1$. If $\widetilde{S}$ intersects $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_1^0)$, then $\widetilde{S}$ also intersects $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^0 \times \widetilde{\mathcal{H}}_1^1)$.*

Proof. For part (a), recall that the Torelli morphism $\mathcal{H}_2 \to \mathcal{A}_2$ is an inclusion [OS80, Lemma 1.11]. Since $\dim(\mathcal{H}_2^1) = \dim(\mathcal{A}_2^1)$, and since $\mathcal{A}_2^1$ is irreducible (e.g., [EvdG09, Ex. 11.6]), it follows that $\overline{\mathcal{H}}_2^1$ is irreducible. Consider a chain Y of two elliptic curves, one ordinary and one supersingular, intersecting in an ordinary double point, which is a fixed point of the hyperelliptic involution on each elliptic curve. The moduli point of Y is in the intersection of $\kappa_{1,1}(\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_1^0)$ and $\overline{\mathcal{H}}_2^1$.

For part (b), let S^* be the closure of $\widetilde{S}$ in $\widetilde{\mathcal{H}}_2$. The reason for working with S^* is that it can contain the moduli points of curves with p-rank 0. By hypothesis and Lemma 3.3, $\widetilde{S}$ contains a component of $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_1^0)$. Every component of $\widetilde{H}_1^1$ contains a component of $\widetilde{\mathcal{H}}_1^0$ in its closure since any nonisotrivial proper family of curves of genus one has supersingular fibers. It follows that S^* contains a component of $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^0 \times \widetilde{\mathcal{H}}_1^0)$, and thus intersects the closure of a component of $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^0 \times \widetilde{\mathcal{H}}_1^1)$. By Lemma 3.3, S^* contains a component of $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^0 \times \widetilde{\mathcal{H}}_1^1)$, which then implies the same for $\widetilde{S}$. □

Remark 3.9. Note that a genus two curve has six ramification points and thus there are potentially up to $6! = |\Gamma_2|$ irreducible components of $\widetilde{\mathcal{H}}_2^1$. In particular, the fact from Lemma 3.8(a) that $\overline{\mathcal{H}}_2^1$ intersects $\kappa_{1,1}(\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_1^0)$ does not imply the hypothesis in part (b) that $\widetilde{S}$ intersects $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_1^0)$.

Lemma 3.10. *Let S be an irreducible component of $\mathcal{H}_g^f$. Suppose Λ is a clutching tree of elliptic curves with $g(\Lambda) = g$. If $\overline{S}$ intersects $\Delta_\Lambda[\overline{\mathcal{H}}_g]$, then for any choice of $\{f_v\} \in \mathcal{F}(\Lambda, f)$, $\overline{S}$ contains an irreducible component of $\kappa_\Lambda(\times_{v\in\Lambda}\widetilde{\mathcal{H}}_{g_v}^{f_v})$.*

Proof. By Lemma 3.6, there exist a choice of data $\{f_v^*\} \in \mathcal{F}(\Lambda, f)$ and components $\widetilde{T}_v \subset \widetilde{\mathcal{H}}_{g_v}^{f_v^*}$ such that $\overline{S}$ contains $\kappa_\Lambda(\times_{v\in\Lambda}\widetilde{T}_v)$. One immediately reduces to the case in which v_1 and v_2 are adjacent vertices in Λ with $f_{v_1}^* = 1$ and $f_{v_2}^* = 0$, and $\{f_v\} \in \mathcal{F}(\Lambda, f)$ is given by

$$f_v = \begin{cases} f_v^* & v \notin \{v_1, v_2\} \\ 1 - f_v^* & v \in \{v_1, v_2\}. \end{cases}$$

Let Λ' be the tree obtained by identifying v_1 and v_2 in a new vertex v_{12} with $g_{v_{12}} = 2$. By Lemma 3.6, there is a component $\widetilde{T}_{v_{12}} \subset \widetilde{\mathcal{H}}_2^1$ such that $\overline{S}$ contains

$$\kappa_{\Lambda'}(\widetilde{T}_{v_{12}} \times (\times_{v\in\Lambda', v\neq v_{12}}\widetilde{T}_v)).$$

Now, $\widetilde{T}_{v_{12}}$ contains a component of $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^1 \times \mathcal{H}_1^0)$. By Lemma 3.8(b), $\widetilde{T}_{v_{12}}$ contains a component of $\widetilde{\kappa}_{1,1}(\widetilde{\mathcal{H}}_1^0 \times \widetilde{\mathcal{H}}_1^1)$ as well. Then $\overline{S}$ contains a component of $\kappa_\Lambda(\times_{v\in\Lambda}\widetilde{\mathcal{H}}_{g_v}^{f_v})$. □

3.4. **Main intersection results.** In this section, we prove that the closure of each irreducible component S of $\mathcal{H}_g^f$ contains the moduli point of a singular curve which is a tree of elliptic curves and has p-rank f.

Theorem 3.11. *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$.*

(a) *There exists a clutching tree of elliptic curves Λ with $g(\Lambda) = g$ such that $\overline{S}$ contains an irreducible component of $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$.*
(b) *For any choice of $\{f_v\} \in \mathcal{F}(\Lambda, f)$, $\overline{S}$ contains an irreducible component of $\kappa_\Lambda(\times_{v\in\Lambda}\widetilde{\mathcal{H}}_{g_v}^{f_v})$.*
(c) *In particular, $\overline{S}$ contains the moduli point of a tree of elliptic curves, of which f are ordinary and $g - f$ are supersingular.*

Proof. It suffices to prove part (c) since parts (a) and (c) are equivalent by Lemma 3.6 and since parts (a) and (b) are equivalent by Lemma 3.10.

First suppose $g = 2$. If $f = 2$, then $\mathcal{H}_2^2$ is irreducible and affine and $\overline{\mathcal{H}}_2^2$ contains the moduli point of a tree of 2 ordinary elliptic curves. If $f = 1$ (resp. $f = 0$), the result is true by Lemma 3.8(a) (resp. Lemma 3.4(c)). Now suppose $g \geq 3$ and $0 \leq f \leq g$ and suppose as an inductive hypothesis that the result is true when $2 \leq g' < g$. Let S be an irreducible component of $\mathcal{H}_g^f$.

Claim 3.12. *To complete the proof, it suffices to show that $\overline{S}$ intersects $\Delta_i[\overline{\mathcal{H}}_g]^f$ for some $1 \leq i \leq g - 1$.*

Proof of claim. Suppose $\overline{S}$ intersects $\Delta_i[\overline{\mathcal{H}}_g]^f$ for some $1 \leq i \leq g-1$. Let $g_1 = i$ and $g_2 = g - i$. By Lemma 3.3, $\overline{S}$ contains a component of $\Delta_{g_1}[\overline{\mathcal{H}}_g]^f$. In other words, there exist a pair (f_1, f_2) satisfying (3.1.1) and, for $j = 1, 2$, components $\widetilde{V}_j$ of $\widetilde{\mathcal{H}}_{g_j}^{f_j}$ such that $\overline{S}$ contains $\kappa_{g_1,g_2}(\widetilde{V}_1 \times \widetilde{V}_2)$. Then $\overline{V}_j = \varpi_{g_j}(\widetilde{V}_j)$ is a component of $\overline{\mathcal{H}}_{g_j}^{f_j}$.

By the inductive hypothesis, $\overline{V}_j$ contains the moduli point s_j of a tree Y_j of g_j elliptic curves, of which f_j are ordinary and $g_j - f_j$ are supersingular. Let Λ_j be the dual graph of Y_j. Let $\widetilde{s}_j \in \widetilde{V}_j$ be such that $\varpi_{g_j}(\widetilde{s}_j) = s_j$. In other words, $\widetilde{s}_j$ is the moduli point of Y_j along with the data of a choice of labeling of the smooth ramification locus. Then $\kappa_{g_1,g_2}(\widetilde{s}_1, \widetilde{s}_2)$ is the moduli point of a curve C whose dual graph is obtained by connecting a vertex of Λ_1 with a vertex of Λ_2. Since C is a tree, $\kappa_{g_1,g_2}(\widetilde{s}_1, \widetilde{s}_2)$ is the moduli point of a tree of g elliptic curves, of which $f = f_1 + f_2$ are ordinary and $g - f$ are supersingular. This completes the proof of the claim since $\kappa_{g_1,g_2}(\widetilde{s}_1, \widetilde{s}_2)$ is in $\overline{S}$. □

Continuing the proof of Theorem 3.11, first suppose $f = 0$. By Lemma 3.4(c), $\overline{S}$ intersects $\Delta_i[\overline{\mathcal{H}}_g]^f$ for some $1 \leq i \leq g - 1$. By Claim 3.12, this completes the proof when $f = 0$.

Now suppose $f > 0$. By Lemma 3.4(a), $\overline{S}$ intersects $\Delta_0[\overline{\mathcal{H}}_g]^f$.

Case (i): $\overline{S}$ intersects Ξ_0.

By Lemma 3.4(b), $\overline{S}$ contains the image of a component $\overline{V}'$ of $\overline{\mathcal{H}}_{g-1;1}^{f-1}$ under κ_{g-1}. Consider $\overline{V} = \phi_{g-1;1}(\overline{V}')$ which is a component of $\overline{\mathcal{H}}_{g-1}^{f-1}$. By the inductive hypothesis, $\overline{V}$ contains the moduli point of a curve Y_1 which is a tree of $g - 1$ elliptic curves, of which $f - 1$ are ordinary and $g - f$ are supersingular. Let E be a terminal component of Y_1 and let Y_1' be the closure of $Y_1 - E$ in Y_1. Let R be the point of intersection of E and Y_1'. Since the quotient of Y_1 by the hyperelliptic involution ι has genus 0, the elliptic curve E is stabilized by ι. Let $P \neq R$ be a point of E which is not a ramification point of ι. By Lemma 3.7, the moduli point t of $(Y_1; P)$ is in $\overline{V}'$.

Let Z be the singular irreducible hyperelliptic curve of genus two with exactly one ordinary double point P' such that the normalization of Z is the elliptic curve E and the pre-image of P' consists of the points P and $\iota(P)$. In other words, the moduli point of Z is the image of the moduli point of $(E; P)$ under κ_1.

Consider the point $s = \kappa_{g-1}(t)$ of $\overline{S}$. The curve $\mathcal{C}_{g,s}$ has components Z and Y_1' which intersect in exactly one ordinary double point R. The p-rank of $\mathcal{C}_{g,s}$ is $f(E) + f(Y_1') + 1 = f$. Since $g \geq 3$, there is a terminal component of Y_1' not containing R which is an elliptic curve. Thus s is in $\Delta_1[\overline{\mathcal{H}}_g]^f$. (In fact, s is also in $\Delta_2[\overline{\mathcal{H}}_g]^f$ because of the component Z.) By Claim 3.12, this completes Case (i).

Case (ii): $\overline{S}$ intersects Ξ_i for some $1 \leq i \leq g-2$.

Let $g_1 = i$ and $g_2 = g - 1 - i$. By Lemma 3.4(b), $\overline{S}$ contains a component of $\Xi_i[\overline{\mathcal{H}}_g]^f$. In other words, there exists a pair (f_1, f_2) satisfying (3.1.2) and, for $j = 1, 2$, there exist components $\overline{V}_j'$ of $\overline{\mathcal{H}}_{g_j;1}^{f_j}$ such that $\overline{S}$ contains $\lambda_{g_1,g_2}(\overline{V}_1' \times \overline{V}_2')$. Then $\overline{V}_j = \phi_{g_j;1}(\overline{V}_j')$ is a component of $\overline{\mathcal{H}}_{g_j}^{f_j}$.

By the inductive hypothesis, $\overline{V}_j$ contains the moduli point s_j of a tree Y_j of g_j elliptic curves, of which f_j are ordinary and $g_j - f_j$ are supersingular. Let E_j be a terminal component of Y_j and let Y_j' be the closure of $Y_j - E_j$ in Y_j. Let R_j be the point of intersection of E_j and Y_j'. Since the quotient of Y_j by the hyperelliptic involution ι_j has genus 0, the elliptic curve E_j is stabilized by ι_j. Let $P_1 \neq R_1$ be a point of E_1 which is a ramification point of ι_1. Let $P_2 \neq R_2$ be a point of E_2 which is not a ramification point of ι_2.

By Lemma 3.7, the moduli point s_j' of $(Y_j; P_j)$ is in $\overline{V}_j'$. Consider $s = \lambda_{g_1,g_2}(s_1', s_2')$ which is a point of $\overline{S}$. By Section 2.4.3, the components of the stable model $\mathcal{C}_{g,s}$ are the strict transforms of Y_1 and Y_2 and an exceptional component W which is a projective line. Moreover, Y_1 intersects W in an ordinary double point and Y_2 intersects W in two other points, which are also ordinary double points. The p-rank of $\mathcal{C}_{g,s}$ is $f(Y_1) + f(Y_2) + 1 = f$.

The curve $\mathcal{C}_{g,s}$ has a terminal component E_1' of genus 1. To see this, when $i = 1$ let $E_1' = E_1$, and when $i > 1$ let $E_1' \neq E_1$ be another terminal component of Y_1. It follows that s is in $\Delta_1[\overline{\mathcal{H}}_g]^f$. (Also s is in $\Delta_i[\mathcal{H}_g]^f$ because of the component Y_1.) By Claim 3.12, this completes Case (ii). □

Here are several consequences of Theorem 3.11 which will be used later in the paper.

In the setting of Theorem 3.11, one can deduce that $\overline{S}$ intersects Δ_i nontrivially only when Λ has an edge whose removal yields two trees of size i and $g - i$. This is only guaranteed when $i = 1$. Luckily, the following information on degeneration to Δ_1 is sufficient for the later applications in the paper.

Corollary 3.13. *Suppose $g \geq 2$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$. Then $\overline{S}$ intersects $\Delta_1[\overline{\mathcal{H}}_g]^f$. Furthermore:*

(a) *if $f \leq g - 1$, then $\overline{S}$ contains an irreducible component of $\kappa_{1,g-1}(\widetilde{\mathcal{H}}_1^0 \times \widetilde{\mathcal{H}}_{g-1}^f)$; and*

(b) *if $f \geq 1$, then $\overline{S}$ contains an irreducible component of $\kappa_{1,g-1}(\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_{g-1}^{f-1})$.*

Proof. By Theorem 3.11(c), $\overline{S}$ contains the moduli point of a tree of elliptic curves, of which f are ordinary and $g-f$ are supersingular. Every tree has a leaf; by Theorem 3.11(b), that leaf can be chosen to be ordinary or supersingular if the obvious necessary constraint is satisfied. The result follows by Lemma 3.6. □

The ℓ-adic and p-adic monodromy proofs in Section 4 rely on degeneration to $\Delta_{1,1}$. One can label the four possibilities for (f_1, f_2, f_3) such that $f_1+f_2+f_3 = f$ and $0 \leq f_1, f_3 \leq 1$ as follows: (A) $(1, f-2, 1)$; (B) $(0, f-1, 1)$; (B′) $(1, f-1, 0)$; and (C) $(0, f, 0)$.

Corollary 3.14. *Suppose $g \geq 3$ and $0 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$.*

(a) *Then $\overline{S}$ intersects $\Delta_{1,1}[\overline{\mathcal{H}}_g]^f$.*

(b) *There is an irreducible component $\widetilde{S}$ of $\overline{S} \times \widetilde{\mathcal{H}}_g$, and a choice of (f_1, f_2, f_3) from cases (A)-(C); and there are irreducible components S_1 of $\widetilde{\mathcal{H}}_1^{f_1}$ and S_2 of $\widetilde{\mathcal{H}}_{g-2}^{f_2}$ and S_3 of $\widetilde{\mathcal{H}}_1^{f_3}$; and there are irreducible components S_R of $\widetilde{\mathcal{H}}_{g-1}^{f_2+f_3}$ and S_L of $\widetilde{\mathcal{H}}_{g-1}^{f_1+f_2}$; such that the restriction of the clutching maps of 2.4.7 yields a commutative diagram*

(3.4.1)
$$\begin{array}{ccc} S_1 \times S_2 \times S_3 & \longrightarrow & S_1 \times S_R \\ \downarrow & \searrow^{\widetilde{\kappa}_{1,g-2,1}} & \downarrow \\ S_L \times S_3 & \longrightarrow & \widetilde{S} \cap \Delta_{1,1}[\widetilde{\mathcal{H}}_g]. \end{array}$$

(c) *Furthermore, case (A) occurs as long as $f \geq 2$, case (B) or (B′) occurs as long as $1 \leq f \leq g-1$, and case (C) occurs as long as $f \leq g-2$.*

Proof. By Theorem 3.11(a), there is a clutching tree of elliptic curves Λ such that $\overline{S}$ contains a component of $\Delta_\Lambda[\overline{\mathcal{H}}_g]^f$. Let v_1 and v_3 be two leaves of Λ; using Theorem 3.11(b), one can assume that $f_{v_i} = f_i$ where (f_1, f_2, f_3) is chosen as in part (c). Let Λ' be the tree obtained by coalescing all vertices of Λ except for v_1 and v_3. Let v_2 denote this new vertex and let $f_{v_2} = f_2$. Since Λ refines Λ', then $\overline{S}$ intersects $\Delta_{\Lambda}[\overline{\mathcal{H}}_g]^f$ which completes part (a). Moreover, there is an irreducible component $\widetilde{S}$ of $\overline{S} \times_{\overline{\mathcal{H}}_g} \widetilde{\mathcal{H}}_g$ such that $\widetilde{S}$ intersects $\Delta_{1,1}[\widetilde{\mathcal{H}}_g]^f$. Part (b) follows from the definition of $\Delta_{1,1}[\widetilde{\mathcal{H}}_g]^f$ and Lemma 3.6. □

Corollary 3.15. *Let S be an irreducible component of $\mathcal{H}_g^f$. For each $0 \leq f' < f$, there exists an irreducible component T of $\mathcal{H}_g^{f'}$ such that $\overline{S}$ contains T.*

Proof. It suffices to prove the result for $f' = f-1$. Let S^* be the closure of S in $\overline{\mathcal{H}}_g - \Delta_0[\overline{\mathcal{H}}_g]$. A purity result [Oor74, Lemma 1.6] shows that $S^* - (S^*)^f$, if nonempty, is pure of dimension $\dim S^* - 1$. In particular, let $Z = (S^*)^{f-1}$; then Z, if nonempty, is pure of dimension $g-2+f$.

By Corollary 3.13(b), S^* contains an irreducible component of $\kappa_{1,1}(\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_{g-1}^{f-1})$. Since $\widetilde{\mathcal{H}}_1^1$ is dense in $\widetilde{\mathcal{H}}_1$, its closure contains the moduli points of supersingular elliptic curves (with labeled smooth ramification locus). Therefore, S^* contains an irreducible component of $\kappa_{1,1}(\widetilde{\mathcal{H}}_1^0 \times \widetilde{\mathcal{H}}_{g-1}^{f-1})$, and Z is nonempty. Then $\dim Z = g-2+f = \dim \overline{\mathcal{H}}_g^{f-1}$, and so Z contains a component $\overline{T}$ of $\overline{\mathcal{H}}_g^{f-1}$. By Lemma 3.2(a) $\overline{S}$ contains a component T of $\mathcal{H}_g^{f-1}$. □

3.5. **Application to Newton polygons.** Recall that a stable curve C of compact type is supersingular if all the slopes of the Newton polygon of its Jacobian equal $1/2$. This is equivalent to the condition that the Jacobian of C is isogenous to a product of supersingular elliptic curves. Note that a supersingular curve necessarily has p-rank zero. An abelian variety of p-rank zero is necessarily supersingular only when the dimension satisfies $g \leq 2$.

In this section, we prove that the Newton polygon of a generic hyperelliptic curve of p-rank 0 is not supersingular when $g \geq 3$. The result generalizes [Oor91, Thm. 1.12] which is the case $g = 3$.

Newton polygons have the following semicontinuity property: let $S = \mathrm{Spec}(R)$ be the spectrum of a local ring, with generic point η and geometric closed point s; if G is a p-divisible group over S, then $\nu(G_\eta)$ either equals or lies below $\nu(G_s)$. (The latter condition means that $\nu(G_\eta)$ and $\nu(G_s)$ have the same endpoints and all points of $\nu(G_\eta)$ lie below $\nu(G_s)$.)

Corollary 3.16. *Suppose p is an odd prime and $g \geq 3$. Let η be a generic point of $\mathcal{H}_g^0$. Then $\mathcal{C}_{g,\eta}$ is not supersingular. In particular, there exists a smooth hyperelliptic curve of genus g and p-rank 0 which is not supersingular.*

Proof. When $g = 3$, this follows from [Oor91, Thm. 1.12]. For $g \geq 4$, the proof proceeds by induction. Let S be the closure of η in $\mathcal{H}_g^0$. By Corollary 3.13, $\overline{S}$ contains a component of $\Delta_1[\overline{\mathcal{H}}_g]^0$. Thus there are components $\widetilde{V}_1$ of $\widetilde{\mathcal{H}}_1^0$ and $\widetilde{V}_2$ of $\widetilde{\mathcal{H}}_{g-1}^0$ such that $\overline{S}$ contains $\kappa_{1,g-1}(\widetilde{V}_1 \times \widetilde{V}_2)$. By the inductive hypothesis, the Newton polygon of the generic point of $\widetilde{V}_2$ is not supersingular; in particular, it has a slope λ such that $0 < \lambda < 1/2$. The same is then true of the generic point of $\kappa_{1,g-1}(\widetilde{V}_1 \times \widetilde{V}_2)$. By semicontinuity [Kat79, Thm. 2.3.1], the generic Newton polygon of $\overline{S}$ (and thus of S) either equals or lies below that of $\kappa_{1,g-1}(\widetilde{V}_1 \times \widetilde{V}_2)$. In particular, it has a slope $\lambda' < 1/2$. Thus $\mathcal{C}_{g,\eta}$ is not supersingular. □

Remark 3.17. When $p = 2$, there are some results about the slopes of Newton polygons of hyperelliptic curves of p-rank 0, see e.g. [SZ02].

3.6. **Open questions about the geometry of the hyperelliptic p-rank strata.**

Question 3.18. Does the closure of each component of $\mathcal{H}_g^f$ contain the moduli point of a chain of elliptic curves with p-rank f?

If the answer to Question 3.18 is affirmative then Lemma 3.10 implies that every ordering of f ordinary and $g - f$ supersingular elliptic curves occurs for such a chain. In [AP08, Cor. 3.6], the authors show the analogous question has a positive answer for every component of $\mathcal{M}_g^f$. The difference for $\mathcal{M}_g^f$ is that the clutching morphism identifies two curves at an arbitrary point of each, rather than a ramification point of each. The location of these points can then be changed using an analogue of Lemma 3.7.

Question 3.19. For $2 \leq i \leq g-2$, does the closure of each component of $\mathcal{H}_g^f$ intersect $\Delta_i[\mathcal{H}_g]^f$?

In [AP08, Prop. 3.4], the authors show that the analogous question has a positive answer for every component of $\mathcal{M}_g^f$, also with control over the arrangement of p-ranks. An affirmative answer to Question 3.18 would imply an affirmative answer to Question 3.19.

Question 3.20. How many irreducible components does $\mathcal{H}_g^f$ have?

One knows that $\mathcal{H}_g^f$ is irreducible for all p when $f = g$ or when $g = 2$ and $f = 1$. If $g \geq 3$, then $\mathcal{A}_g^f$ is irreducible by [Cha05, Remark 4.7]. If $\mathcal{H}_g^f$ is irreducible, then there is a very short proof that Questions 3.18 and 3.19 have affirmative answers.

4. Monodromy

In this section, we determine the ℓ-adic monodromy of components of $\mathcal{H}_g^f$ for odd primes ℓ using an inductive process. The first base case, when $g = 2$ and $1 \leq f \leq 2$, relies on a special case of [Cha05, Prop. 4.4]. Unfortunately, the monodromy group of $\mathcal{H}_2^0$ is small, since supersingular families of abelian varieties have finite ℓ-adic monodromy groups; and the methods of [Cha05] do not apply to $\mathcal{H}_g^0$ for $g \geq 3$, because the hyperelliptic Torelli locus is not Hecke-stable. Thus the proof requires another base case when $f = 0$. For lack of a strategy to calculate the ℓ-adic monodromy of $\mathcal{H}_3^0$, we analyze the case when $g = 5$ and $f = 0$ in Section 4.2 by studying endomorphism algebras.

4.1. **Integral monodromy.** We summarize the discussion in [AP07, Sec. 3.1] about $\mathbb{Z}/\ell$- and $\mathbb{Z}_\ell$-monodromy. Let S be a connected k-scheme on which the prime ℓ is invertible. Let $\pi : C \to S$ be a relative curve of compact type whose fibres have genus g. Then $R^1\pi_*(\boldsymbol{\mu}_\ell)$, or equivalently $\mathrm{Pic}^0(C)[\ell]$, is an étale sheaf of $\mathbb{Z}/\ell$-modules. If s is a geometric point of S, then $R^1\pi_*(\boldsymbol{\mu}_\ell)$ is equivalent to a linear representation

$$\rho_{C\to S,\mathbb{Z}/\ell} : \pi_1(S, s) \to \mathrm{Aut}((R^1\pi_*(\boldsymbol{\mu}_\ell))_s) \cong \mathrm{GL}_{2g}(\mathbb{Z}/\ell).$$

Let $\mathsf{M}_\ell(C \to S, s)$ be the image of $\rho_{C\to S,\mathbb{Z}/\ell}$ and let $\mathsf{M}_\ell(C \to S)$ be the isomorphism class of this image as an abstract group. If the family $C \to S$ is clear from context, these will be denoted $\mathsf{M}_\ell(S, s)$ and $\mathsf{M}_\ell(S)$, respectively. There is a canonical polarization on $\mathrm{Pic}^0(C)$, and thus (after a choice of ℓ^{th} root of unity on S) there is a symplectic pairing on $\mathrm{Pic}^0(C)[\ell]$. Therefore, there is an inclusion of groups $\mathsf{M}_\ell(S) \subseteq \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. Similarly, for each natural number n there is a representation

$$\rho_{C\to S,\mathbb{Z}/\ell^n} : \pi_1(S, s) \to \mathrm{Aut}((R^1\pi_*(\boldsymbol{\mu}_{\ell^n}))_s).$$

Let $\mathsf{M}_{\mathbb{Z}_\ell}(C \to S, s) = \varprojlim_n \rho_{C\to S,\mathbb{Z}/\ell^n}(\pi_1(S, s))$. One can employ an analogous formalism to define the monodromy group of a relative curve over a stack [Noo04].

4.2. **A base case for p-rank zero.** In this section we show that if S is an irreducible component of $\mathcal{H}_5^0$, then $\mathsf{M}_\ell(S) \cong \mathrm{Sp}_{10}(\mathbb{Z}/\ell)$ for all primes ℓ outside some finite set. The set of exceptional primes may depend on the characteristic p of the base field, so that our results are valid for $\ell \gg_p 0$.

Let $\mathcal{X}_g = \mathrm{Pic}^0_{\mathcal{C}_g/\overline{\mathcal{H}}_g}$ be the neutral component of the relative Picard functor of $\mathcal{C}_g$ over $\overline{\mathcal{H}}_g$; then $\mathcal{X}_g \to \overline{\mathcal{H}}_g$ is a semiabelian scheme. To ease notation, if X is an abelian variety, let $\mathbf{E}(X) = \mathrm{End}(X) \otimes \mathbb{Q}$, and let $\mathbf{E}_\ell(X) = \mathrm{End}(X) \otimes \mathbb{Z}_\ell$; then $\mathbf{E}_\ell(X)$ acts on the Tate module $T_\ell(X)$. If X is simple, then the center of $\mathbf{E}(X)$ is either a totally real or totally imaginary number field.

Lemma 4.1. *Let ξ be a geometric generic point of $\mathcal{H}_3^0$. Then $\mathcal{X}_{3,\xi}$ is simple and either $\mathbf{E}(\mathcal{X}_{3,\xi}) \cong \mathbb{Q}$ or $\mathbf{E}(\mathcal{X}_{3,\xi})$ is a totally real cubic field.*

Proof. Suppose there is an isogeny $\mathcal{X}_{3,\xi} \sim A_1 \oplus A_2$ for abelian varieties A_1 and A_2 of dimensions 1 and 2. Then A_1 and A_2 each have p-rank 0 and are thus supersingular. Then $\mathcal{X}_{3,\xi}$ is supersingular, which contradicts the fact that the Newton polygon of $\mathcal{X}_{3,\xi}$ has slopes $1/3$ and $2/3$ [Oor91, Thm. 1.12]. Thus $\mathcal{X}_{3,\xi}$ is simple.

By the classification of endomorphism algebras of simple abelian varieties of prime dimension (e.g., [Oor88, 7.2]), to complete the proof it suffices to show that neither a complex multiplication field of degree six nor a quadratic imaginary field acts on $\mathcal{X}_{3,\xi}$. Let S be the closure of ξ in $\mathcal{H}_3^0$. Since $\dim S = 2 > 0$ but abelian varieties with complex multiplication are rigid, $\mathbf{E}(\mathcal{X}_{3,\xi})$ is not a complex multiplication field of degree 6.

To address the possibility of an action by a quadratic imaginary field K, suppose to the contrary that there is a subring of $\mathrm{End}(\mathcal{X}_{3,\xi})$ isomorphic to an order $\mathcal{O}_K$ in K. Then $\mathcal{O}_K \otimes \mathbb{Z}_p$ acts on the p-divisible group $\mathcal{X}_{3,\xi}[p^\infty]$. There is an inclusion $K \otimes \mathbb{Q}_p \hookrightarrow \mathrm{End}(\mathcal{X}_{3,\xi}[p^\infty]) \otimes \mathbb{Q}_p \cong D_{1/3} \oplus D_{2/3}$. (Here, D_λ denotes the central simple $\mathbb{Q}_p$-algebra with Brauer invariant λ.) Every maximal subfield of $D_{1/3}$ or $D_{2/3}$ is a cubic extension of $\mathbb{Q}_p$, but $K \otimes \mathbb{Q}_p$ is a $\mathbb{Q}_p$-algebra of degree two, so $K \otimes \mathbb{Q}_p$ cannot be a field. In particular, $\mathcal{X}_{3,\xi}$ does not admit an action by a quadratic imaginary field inert or ramified at p.

Finally, suppose $\mathcal{X}_{3,\xi}$ admits an action by a quadratic imaginary field K which splits at p. Let (r,s) be the signature of the action of $\mathcal{O}_K$ on $\mathrm{Lie}(\mathcal{X}_{3,\xi})$; the dimensions r and s are nonnegative and $r+s=3$. Consider the moduli space $\mathcal{S}h_{\mathcal{O}_K;(r,s)}$ of abelian threefolds with an action by $\mathcal{O}_K$ of signature (r,s). The Torelli morphism τ restricts to a finite morphism from S to a component of $\mathcal{S}h_{\mathcal{O}_K;(r,s)}$. Since $\dim S = 2$ and $\dim \mathcal{S}h_{\mathcal{O}_K;(r,s)} = r \cdot s$, then (r,s) is either $(1,2)$ or $(2,1)$. Thus $\tau(S)$ is dense in $\mathcal{S}h_{\mathcal{O}_K;(r,s)}$. This gives a contradiction since $\mathcal{X}_{3,\xi}$ has p-rank zero but the generic member of $\mathcal{S}h_{\mathcal{O}_K;(r,s)}$ is ordinary [Wed99, Thm. 1.6.2]. □

Remark 4.2. In the situation of Lemma 4.1, using the description of the p-rank strata of Hilbert modular threefolds, one can show that if $\mathbf{E}(\mathcal{X}_{3,\xi})$ is a cubic field, then it is either (totally) inert or ramified at p. This will not be used in the sequel.

Lemma 4.3. *Let Y be a simple abelian variety whose dimension g is relatively prime to 3. If there exists a geometric generic point ξ_3 of $\mathcal{H}_3^0$ for which there is a nontrivial homomorphism $\mathrm{End}(Y) \to \mathbf{E}(\mathcal{X}_{3,\xi_3})$, then $\mathrm{End}(Y) \cong \mathbb{Z}$.*

Proof. If Z is a simple abelian variety, let $\mathbf{E}_0(Z)$ be the subfield of $\mathbf{E}(Z)$ fixed by the Rosati involution, and let $e_0(Z) = [\mathbf{E}_0(Z) : \mathbb{Q}]$. Then $e_0(Z) | \dim(Z)$.

By Lemma 4.1, $\mathbf{E}(\mathcal{X}_{3,\xi_3})$ is a totally real field of dimension 1 or 3 over $\mathbb{Q}$. On one hand, the existence of a nontrivial homomorphism $\mathrm{End}(Y) \to \mathbf{E}(\mathcal{X}_{3,\xi_3})$ forces $e_0(Y)$ to divide $e_0(\mathcal{X}_{3,\xi_3})$, and thus $e_0(Y)|3$. On the other hand, $e_0(Y)|g$. Therefore, $e_0(Y) = 1$ and $E_0(Y) \cong \mathbb{Q}$. Neither a noncommutative algebra nor a totally imaginary field admits a nontrivial homomorphism to $\mathbf{E}(\mathcal{X}_{3,\xi_3})$, and thus $\mathbf{E}(Y) = \mathbf{E}_0(Y)$ and $\mathrm{End}(Y) \cong \mathbb{Z}$. □

Lemma 4.4. *Let $X \to S$ be a polarized abelian scheme over a reduced irreducible Noetherian stack. Let η be the generic point of S, and let $s \in S$ be any point. Then there exists an inclusion $\mathrm{End}(X_{\overline{\eta}}) \hookrightarrow \mathrm{End}(X_{\overline{s}})$.*

Proof. By introducing a rigidifying structure on $X \to S$, such as coordinates on the space of sections of the third power of the ample line bundle given by the polarization, one can assume S is a reduced irreducible Noetherian scheme. Since the absolute endomorphism ring of an abelian variety is defined over a finite extension of the base field, it suffices to show the existence of an inclusion $\mathrm{End}(X_\eta) \hookrightarrow \mathrm{End}(X_s)$. If S is normal, then $\mathrm{End}(X_\eta)$ extends uniquely to $\mathrm{End}(X_S)$, and in particular to $\mathrm{End}(X_s)$ [FC90, I.2.7]. In general, let $\nu : S' \to S$ be the normalization map; let η' be the generic point of S', and let s' be a point of S' over s'. The desired result follows from the canonical map $\mathrm{End}((\nu^* X)_{\eta'}) \hookrightarrow \mathrm{End}((\nu^* X)_{s'})$ and the isomorphisms of abelian varieties $(\nu^* X)_{\eta'} \cong X_\eta$ and $(\nu^* X)_{s'} \cong X_s$. □

Proposition 4.5. *If ξ_4 is a geometric generic point of $\mathcal{H}_4^0$, then $\mathcal{X}_{4,\xi_4}$ is simple and* $\mathrm{End}(\mathcal{X}_{4,\xi_4}) \cong \mathbb{Z}$.

Proof. Suppose there is an isogeny $\mathcal{X}_{4,\xi_4} \sim A_1 \oplus A_2$ for two abelian varieties A_1 and A_2. If A_1 and A_2 each have dimension 2, then they are supersingular since they have p-rank 0. Then $\mathcal{X}_{4,\xi_4}$ is supersingular, which contradicts Corollary 3.16. If A_1 has dimension 1 and A_2 has dimension 3, then there is a curve W of genus 3 such that $\mathrm{Jac}(W) \cong A_2$. The inclusion of A_2 into $\mathcal{X}_{4,\xi_4}$ yields a cover $\psi : \mathcal{C}_{4,\xi_4} \to W$. By the Riemann-Hurwitz formula $6 \geq 4\deg(\psi)$ which is impossible since $\deg(\psi) \geq 2$. Thus $\mathcal{X}_{4,\xi_4}$ is simple.

Let S_4 be the closure of ξ_4 in $\mathcal{H}_4^0$. By Corollary 3.13(a) there exist components $\widetilde{V}_1 \subset \widetilde{\mathcal{H}}_1^0$ and $\widetilde{V}_2 \subset \widetilde{\mathcal{H}}_3^0$ such that $\overline{S}_4$ contains $\kappa_{1,3}(\widetilde{V}_1 \times \widetilde{V}_2)$. Let ξ_1 and ξ_3 be geometric generic points of $\widetilde{V}_1$ and $\widetilde{V}_2$, respectively, and let $\eta = \kappa_{1,3}(\xi_1, \xi_3)$. Since $\mathcal{X}_{3,\xi_3}$ is simple by Lemma 4.1, there are no nontrivial homomorphisms between $\mathcal{X}_{3,\xi_3}$ and $\mathcal{X}_{1,\xi_1}$. This yields an isomorphism

$$\mathbf{E}(\mathcal{X}_{4,\eta}) \cong \mathbf{E}(\mathcal{X}_{1,\xi_1}) \oplus \mathbf{E}(\mathcal{X}_{3,\xi_3}).$$

By Lemma 4.4, there is an inclusion $\mathbf{E}(\mathcal{X}_{4,\xi_4}) \hookrightarrow \mathbf{E}(\mathcal{X}_{4,\eta})$ and thus an inclusion $\mathbf{E}(\mathcal{X}_{4,\xi_4}) \hookrightarrow \mathbf{E}(\mathcal{X}_{3,\xi_3})$. Since $\mathcal{X}_{4,\xi_4}$ is simple, Lemma 4.3 implies that $\mathrm{End}(\mathcal{X}_{4,\xi_4}) \cong \mathbb{Z}$. □

Proposition 4.6. *If ξ_5 is a geometric generic point of $\mathcal{H}_5^0$, then* $\mathrm{End}(\mathcal{X}_{5,\xi_5}) \cong \mathbb{Z}$.

Proof. Let S_5 be the closure of ξ_5 in $\mathcal{H}_5^0$. By Corollary 3.14, $\overline{S}_5$ intersects $\Delta_{1,1}[\overline{\mathcal{H}}_5]^0$ and there is an irreducible component $\widetilde{S}$ of $\overline{S}_5 \times \widetilde{\mathcal{H}}_5$, and there are irreducible components S_1 of $\widetilde{\mathcal{H}}_1^0$ and S_2 of $\widetilde{\mathcal{H}}_3^0$ and S_3 of $\widetilde{\mathcal{H}}_1^0$; and there are irreducible components S_R of $\widetilde{\mathcal{H}}_4^0$ and S_L of $\widetilde{\mathcal{H}}_4^0$; such that the restriction of the clutching maps yields a commutative diagram

(4.2.1)
$$\begin{array}{ccc} S_1 \times S_2 \times S_3 & \longrightarrow & S_1 \times S_R \\ \downarrow & \searrow^{\widetilde{\kappa}_{1,3,1}} & \downarrow \\ S_L \times S_3 & \longrightarrow & \widetilde{S} \cap \Delta_{1,1}[\widetilde{\mathcal{H}}_5]. \end{array}$$

Let η_i be the generic point of S_i for $1 \leq i \leq 3$; similarly, let η_L be the generic point of S_L, and η_R that of S_R. Let $s = \widetilde{\kappa}_{1,3,1}(\eta_1, \eta_2, \eta_3)$. By Lemma 4.4, there are inclusions

(4.2.2)
$$\begin{array}{ccc} \mathrm{End}(\mathcal{X}_{5,s}) & \longleftarrow & \mathrm{End}(\mathcal{X}_{5,\widetilde{\kappa}_{1,4}(\eta_1 \times \eta_R)}) \\ \uparrow & & \uparrow \\ \mathrm{End}(\mathcal{X}_{5,\widetilde{\kappa}_{4,1}(\eta_L \times \eta_3)}) & \longleftarrow & \mathrm{End}(\mathcal{X}_{5,\xi_5}). \end{array}$$

Let ℓ be a prime such that $\mathbf{E}_\ell(E) \cong \mathrm{Mat}_2(\mathbb{Z}_\ell)$ for any supersingular elliptic curve E/k. There is a canonical isomorphism $T_\ell(\mathcal{X}_{5,s}) \cong T_\ell(\mathcal{X}_{1,\eta_1}) \times T_\ell(\mathcal{X}_{3,\eta_2}) \times T_\ell(\mathcal{X}_{1,\eta_3})$ of $\mathbb{Z}_\ell$-modules. Choose coordinates on $T_\ell(\mathcal{X}_{5,s})$ compatible with this decomposition. By Proposition 4.5, $\mathbf{E}(\mathcal{X}_{5,\widetilde{\kappa}_{4,1}(\eta_L \times \eta_3)}) \cong$

$\mathbf{E}(\mathcal{X}_{4,\eta_L}) \oplus \mathbf{E}(\mathcal{X}_{1,\eta_3}) \cong \mathbb{Q} \oplus \mathbf{E}(\mathcal{X}_{1,\eta_3})$, and so $\mathbf{E}_\ell(\mathcal{X}_{5,\widetilde{\kappa}_{4,1}(\eta_L\times\eta_3)})$ acts on $T_\ell(\mathcal{X}_{5,s})$ as $\mathrm{diag}_8(\mathbb{Z}_\ell) \oplus \mathrm{Mat}_2(\mathbb{Z}_\ell)$. Similarly, $\mathbf{E}_\ell(\mathcal{X}_{5,\widetilde{\kappa}_{1,4}(\eta_1\times\eta_R)})$ acts as $\mathrm{Mat}_2(\mathbb{Z}_\ell) \oplus \mathrm{diag}_8(\mathbb{Z}_\ell)$. Then $\mathbf{E}_\ell(\mathcal{X}_{5,\xi_5}) \subseteq \mathbf{E}_\ell(\mathcal{X}_{5,\widetilde{\kappa}_{4,1}(\eta_L\times\eta_3)}) \cap \mathbf{E}_\ell(\mathcal{X}_{5,\widetilde{\kappa}_{1,4}(\eta_1\times\eta_R)})$ acts on $T_\ell(\mathcal{X}_{5,s})$ as $\mathrm{diag}_{10}(\mathbb{Z}_\ell)$. Thus, $\mathbf{E}_\ell(\mathcal{X}_{5,\xi_5}) \cong \mathbb{Z}_\ell$ and $\mathrm{End}(\mathcal{X}_{5,\xi_5}) \cong \mathbb{Z}$. □

Remark 4.7. For $g \geq 5$, one can use the argument of 4.6to show inductively that $\mathrm{End}(\mathcal{X}_{g,\xi}) \cong \mathbb{Z}$ for any generic point ξ of $\mathcal{H}_g^0$. Instead, we will deduce this in Application 4.15 as a consequence of a monodromy result.

Corollary 4.8. *For $\ell \gg_p 0$, if S is an irreducible component of $\mathcal{H}_5^0$ then $\mathsf{M}_\ell(S) \cong \mathrm{Sp}_{10}(\mathbb{Z}/\ell)$.*

Proof. Since $\mathcal{H}_5^0$ has only finitely many irreducible components, it suffices to prove the statement for $\ell \gg_p 0$ for each component S. This follows from Proposition 4.6 and Serre's open image theorem [Ser00, Thm. 3]. (In fact, Serre's result is stated for abelian varieties of odd dimension over number fields; but his argument, which involves only the fundamental group of the base and the structure of symplectic groups, works in this generality [Ser00, 8.2].) □

4.3. **Monodromy of the hyperelliptic p-rank strata.** In this section, let p and ℓ be distinct odd primes. We find the integral monodromy of the p-rank strata $\mathcal{H}_g^f$ when $1 \leq f \leq g$. We also find the integral monodromy of $\mathcal{H}_g^0$ for $g \geq 5$ for all but finitely many ℓ. The integral monodromy of $\mathcal{H}_g$, which is the same as the case $f = g$, already appears in [AP07, Thm. 3.4] (see also unpublished work of J.-K. Yu, and [Hal08, Thm. 5.1]).

The following argument shows that to determine the monodromy of families of hyperelliptic curves, one may work with either $\overline{\mathcal{H}}_g$ or $\widetilde{\mathcal{H}}_g$.

Lemma 4.9. *Let p and ℓ be distinct odd primes and suppose $g \geq 2$. Let $S \subset \mathcal{H}_g$ be irreducible and let $\widetilde{S}$ be an irreducible component of $\overline{S} \times_{\overline{\mathcal{H}}_g} \widetilde{\mathcal{H}}_g$. Then $\mathsf{M}_\ell(\widetilde{S}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ if and only if $\mathsf{M}_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$.*

Proof. Since $\widetilde{S} \to S$ is finite, $\mathsf{M}_\ell(\widetilde{S})$ is a subgroup of $\mathsf{M}_\ell(\overline{S})$. If $\mathsf{M}_\ell(\widetilde{S}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ then $\mathsf{M}_\ell(\widetilde{S})$ is maximal, and thus so are $\mathsf{M}_\ell(\overline{S})$ and $\mathsf{M}_\ell(S)$.

Conversely, suppose $\mathsf{M}_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$. Since ϖ_g is étale with Galois group $\Gamma_g \cong \mathrm{Sym}(2g+2)$, the cover $\widetilde{S} \to \overline{S}$ is Galois with Galois group $G \subseteq \Gamma_g$. To show $\mathsf{M}_\ell(\widetilde{S}) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$, it suffices by the argument of [AP07, Lemma 3.3] to show that G and $\mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ have no common nontrivial quotient. This holds since the smallest integer N for which there exists an embedding of the finite simple group $\mathrm{PSp}_{2g}(\mathbb{Z}/\ell)$ into $\mathrm{Sym}(N)$ is $N = (\ell^{2g}-1)/(\ell-1) > 2g+2$ [Gre03, Thm. 3]. □

Theorem 4.10. *Let p and ℓ be distinct odd primes. Suppose $g \geq 1$ and $1 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$, the p-rank f stratum in $\mathcal{H}_g$. Then $\mathsf{M}_\ell(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$ and $\mathsf{M}_{\mathbb{Z}_\ell}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$.*

Proof. The proof is by induction on g. The base cases involve the monodromy of $\mathcal{H}_2^2$ and $\mathcal{H}_2^1$, which follow from [Cha05, Prop. 4.4]; see [AP08, Thm. 4.5].

Now suppose $g \geq 3$ and $1 \leq f \leq g$. As an inductive hypothesis assume, for all pairs (g', f') where $1 \leq f' \leq g' < g$, that $\mathsf{M}_\ell(S') \cong \mathrm{Sp}_{2g'}(\mathbb{Z}/\ell)$ for every irreducible component S' of $\mathcal{H}_{g'}^{f'}$.

Let S be an irreducible component of $\mathcal{H}_g^f$. Recall the degeneration types identified immediately before Corollary 3.14. If $f = g$, let $(f_1, f_2, f_3) = (1, g-2, 1)$ as in case (A); if $f = g-1$, let

$(f_1, f_2, f_3) = (0, g-2, 1)$ as in case (B); and if $1 \leq f \leq g-2$, let $(f_1, f_2, f_3) = (0, f, 0)$ as in case (C). By Corollary 3.14, there are irreducible components $\widetilde{S}$ of $\overline{S} \times_{\overline{\mathcal{H}}_g} \widetilde{\mathcal{H}}_g$, $\widetilde{S}_1$ of $\widetilde{\mathcal{H}}_1^{f_1}$, $\widetilde{S}_2$ of $\widetilde{\mathcal{H}}_{g-2}^{f_2}$ and $\widetilde{S}_3$ of $\widetilde{\mathcal{H}}_1^{f_3}$; and there are irreducible components $\widetilde{S}_R$ of $\widetilde{\mathcal{H}}_{g-1}^{f_2+f_3}$ and $\widetilde{S}_L$ of $\widetilde{\mathcal{H}}_{g-1}^{f_1+f_2}$; such that the restriction of the clutching maps yields a commutative diagram

(4.3.1)
$$\begin{array}{ccc} \widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3 & \longrightarrow & \widetilde{S}_1 \times \widetilde{S}_R \\ \downarrow & \searrow^{\widetilde{\kappa}_{1,g-2,1}} & \downarrow \\ \widetilde{S}_L \times \widetilde{S}_3 & \longrightarrow & \widetilde{S} \cap \Delta_{1,1}[\widetilde{\mathcal{H}}_g]. \end{array}$$

In all cases $f_1 + f_2$ and $f_2 + f_3$ are positive, and so $\widetilde{S}_L$ and $\widetilde{S}_R$ have monodromy $\mathsf{Sp}_{2(g-1)}(\mathbb{Z}/\ell)$, by induction and Lemma 4.9.

The rest of the proof is identical to that of [AP07, Thm. 3.4]. Briefly, one calculates the monodromy group of $\widetilde{S}$ at a point s in the image of $\widetilde{S}_1 \times \widetilde{S}_2 \times \widetilde{S}_3$ under $\widetilde{\kappa}_{1,g-2,1}$. On one hand, there is an a priori inclusion $\mathsf{M}_\ell(\widetilde{S}, s) \subseteq \mathsf{Sp}_{2g}(\mathbb{Z}/\ell)$. On the other hand, the previous paragraph shows that $\mathsf{M}_\ell(\widetilde{S}, s)$ contains two distinct subgroups isomorphic to $\mathsf{Sp}_{2(g-1)}(\mathbb{Z}/\ell)$. A group-theoretic result shows that $\mathsf{M}_\ell(\widetilde{S}, s) \cong \mathsf{Sp}_{2g}(\mathbb{Z}/\ell)$. The result then follows from Lemma 4.9.

The proof that $\mathsf{M}_{\mathbb{Z}_\ell}(S) \cong \mathsf{Sp}_{2g}(\mathbb{Z}_\ell)$ is identical. □

In the following result, note that the exceptional set of primes depends on p, but not on g.

Theorem 4.11. *If $\ell \gg_p 0$, if $g \geq 5$ and if S is an irreducible component of $\mathcal{H}_g^0$, then $\mathsf{M}_\ell(S) \cong \mathsf{Sp}_{2g}(\mathbb{Z}/\ell)$ and $\mathsf{M}_{\mathbb{Z}_\ell}(S) \cong \mathsf{Sp}_{2g}(\mathbb{Z}_\ell)$.*

Proof. Suppose ℓ is not in the finite set of exceptional primes from Corollary 4.8. The proof is by induction on g with the case $g = 5$ supplied by Corollary 4.8. The rest of the proof is identical to that of Theorem 4.10, with Corollary 3.14 being used to degenerate to an irreducible component of $\widetilde{\kappa}_{1,g-2,1}(\widetilde{\mathcal{H}}_1^0 \times \widetilde{\mathcal{H}}_{g-2}^0 \times \widetilde{\mathcal{H}}_1^0)$. □

Remark 4.12. The assertion of Theorem 4.11 is false for $\mathcal{H}_1^0$ and $\mathcal{H}_2^0$ if $\ell \geq 5$. Indeed, a hyperelliptic curve of genus $g \leq 2$ and p-rank 0 is supersingular. Since a supersingular p-divisible group over a scheme S becomes trivial after a finite pullback $\widetilde{S} \to S$, the monodromy group $\mathsf{M}_{\mathbb{Z}_\ell}(\mathcal{H}_g^0)$ is finite for $g \leq 2$. We do not know whether the assertion of Theorem 4.11 is true for $\mathcal{H}_3^0$ and $\mathcal{H}_4^0$.

4.4. **A p-adic complement.** In this section we determine the p-adic monodromy of components of the p-rank strata $\mathcal{H}_g^f$.

Let S be a connected scheme of characteristic p with geometric point s, and let $X \to S$ be an abelian scheme with constant p-rank f. The group scheme $X[p]$ and p-divisible group $X[p^\infty]$ admit largest étale quotients, $X[p]^{\text{ét}}$ and $X[p^\infty]^{\text{ét}}$. These are respectively classified by homomorphisms $\pi_1(S, s) \to \mathrm{Aut}(X[p]^{\text{ét}})_s \cong \mathrm{GL}_f(\mathbb{Z}/p)$ and $\pi_1(S, s) \to \mathrm{Aut}(X[p^\infty]^{\text{ét}})_s \cong \mathrm{GL}_f(\mathbb{Z}_p)$, whose images are denoted $\mathsf{M}_p(X \to S)$ and $\mathsf{M}_{\mathbb{Z}_p}(X \to S)$, or simply $\mathsf{M}_p(S)$ and $\mathsf{M}_{\mathbb{Z}_p}(S)$.

Lemma 4.13. *Suppose $g \geq 1$ and $1 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$, and let $\widetilde{S}$ be an irreducible component of $S \times_{\overline{\mathcal{H}}_g} \widetilde{\mathcal{H}}_g$. Then $\mathsf{M}_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ if and only if $\mathsf{M}_p(\widetilde{S}) \cong \mathrm{GL}_f(\mathbb{Z}/p)$.*

Proof. Since $\widetilde{S} \to \overline{S}$ is finite, $\mathsf{M}_p(\widetilde{S})$ is a subgroup of $\mathsf{M}_p(\overline{S})$. If $\mathsf{M}_p(\widetilde{S}) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ then $\mathsf{M}_p(\widetilde{S})$ is maximal, and thus so are $\mathsf{M}_p(\overline{S})$ and $\mathsf{M}_p(S)$.

Conversely, suppose $\mathsf{M}_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$. Let S^* be the closure of S in $\overline{\mathcal{H}}_g - \Delta_0[\overline{\mathcal{H}}_g]$ and let $\widetilde{S}^*$ be the closure of $\widetilde{S}$ in $\widetilde{\mathcal{H}}_g - \Delta_0[\widetilde{\mathcal{H}}_g]$. Let $T = S^* - (S^*)^f$ be the locus with p-rank smaller than f. Then T is nonempty by Corollary 3.15. On one hand, $\widetilde{S}^* \to S^*$ is étale since p is odd and the cover $\widetilde{S}^* \to S^*$ is tantamount to a partial level-two structure. On the other hand, the $\mathrm{GL}_f(\mathbb{Z}/p)$-cover $I_f := \mathrm{Isom}_S((\mathbb{Z}/p)^f, \mathrm{Jac}(\mathcal{C}_{g,S})[p]^{\text{ét}}) \to S$ is ramified along T. Therefore, the covers $I_f \to S$ and $\widetilde{S} \to S$ are disjoint, and $\mathsf{M}_p(\widetilde{S}) = \mathsf{M}_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$. □

Proposition 4.14. *Suppose $g \geq 2$ and $1 \leq f \leq g$. Let S be an irreducible component of $\mathcal{H}_g^f$. Then $\mathsf{M}_p(S) \cong \mathrm{GL}_f(\mathbb{Z}/p)$ and $\mathsf{M}_{\mathbb{Z}_p}(S) \cong \mathrm{GL}_f(\mathbb{Z}_p)$.*

Proof. First suppose $f = g$. When $g = 2$, the result for $\mathcal{H}_2^2$, or equivalently $\mathcal{M}_2^2$, is a special case of [Eke91, Thm. 2.1]. For $g \geq 3$, suppose as an inductive hypothesis that $\mathsf{M}_p(\mathcal{H}_{g-1}^{g-1}) \cong \mathrm{GL}_{g-1}(\mathbb{Z}/p)$. By Lemma 4.13, $\mathsf{M}_p(\widetilde{\mathcal{H}}_{g-1}^{g-1}) \cong \mathrm{GL}_{g-1}(\mathbb{Z}/p)$.

Recall the diagram (2.4.7), and consider a geometric point $s \in \widetilde{\mathcal{H}}_g^g$ in the image of $\widetilde{\mathcal{H}}_1^1 \times \widetilde{\mathcal{H}}_{g-2}^{g-2} \times \widetilde{\mathcal{H}}_1^1$ under $\widetilde{\kappa}_{1,g-2,1}$. By the inductive hypothesis and Lemma 4.13, $\mathsf{M}_p(\widetilde{\mathcal{H}}_g^g, s)$ contains two distinct copies of $\mathrm{GL}_{g-1}(\mathbb{Z}/p)$ and thus equals $\mathrm{GL}_g(\mathbb{Z}/p)$ by the argument of [AP07, Lemma 3.2].

Now suppose $1 \leq f \leq g - 1$. By Corollary 3.13(a), there are irreducible components $\widetilde{V}_1 \subset \widetilde{\mathcal{H}}_1^0$ and $\widetilde{V}_2 \subset \widetilde{\mathcal{H}}_{g-1}^f$ such that $\overline{S}$ contains $\widetilde{\kappa}_{1,g-1}(\widetilde{V}_1 \times \widetilde{V}_2)$. By the inductive hypothesis and Lemma 4.13, $\mathsf{M}_p(\widetilde{V}_2) \cong \mathrm{GL}_f(\mathbb{Z}/p)$, and thus $\mathsf{M}_p \cong \mathrm{GL}_f(\mathbb{Z}/p)$ as well.

The proof that $\mathsf{M}_{\mathbb{Z}_p}(S) \cong \mathrm{GL}_f(\mathbb{Z}_p)$ is identical. □

4.5. **Arithmetic applications.** The results of the previous section about the monodromy of components of $\mathcal{H}_g^f$ have arithmetic applications involving hyperelliptic curves over finite fields. For example, they imply that there exist hyperelliptic curves of given genus and p-rank with absolutely simple Jacobian (Application 4.15). Moreover, they give estimates for the proportion of hyperelliptic curves with a given genus and p-rank which have a rational point of order ℓ on the Jacobian (Application 4.17) or for which the numerator of the zeta function has large splitting field (Application 4.19).

Throughout this section, $\mathbb{F}$ denotes a finite extension of $\mathbb{F}_p$.

4.5.1. *Technical context.* We do not include proofs in this section, since they are very similar to those found in [AP08, Section 5]. Here is a brief description of the main ideas involved. One first defines $\mathcal{H}_g$ over the category of $\mathbb{F}_p$-schemes and defines the arithmetic monodromy group of a substack of $\mathcal{H}_g$. For a relative curve $\pi : C \to S/\mathbb{F}$ of genus $g \geq 2$ defined over a finite field, one shows that if $\mathsf{M}_\ell^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}/\ell)$, then $\mathsf{M}_{\mathbb{Z}_\ell}^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$; and $\mathsf{M}_{\mathbb{Z}_\ell}^{\mathrm{arith}}(S)$ has finite index in $\mathrm{GSp}_{2g}(\mathbb{Z}_\ell)$; and $\mathsf{M}_{\mathbb{Q}_\ell}^{\mathrm{geom}}(S) \cong \mathrm{Sp}_{2g}(\mathbb{Q}_\ell)$ [AP08, Lemma 5.1].

Secondly, in order to use Chebotarev arguments for curves over finite fields, it is necessary to add rigidifying data, such as the data of a tricanonical structure, so that the corresponding moduli problems are representable by schemes. Recall that $\Omega_{C/S}^{\otimes 3}$ is very ample, that $\pi_*(\Omega_{C/S}^{\otimes 3})$ is a locally free $\mathcal{O}_S$-module of rank $5g-5$, and that sections of this bundle define a closed embedding $C \hookrightarrow \mathbb{P}_S^{5g-5}$. A tricanonical ($3K$) structure on $\pi : C \to S$ is a choice of isomorphism $\mathcal{O}_S^{\oplus 5g-5} \cong \pi_*(\Omega_{C/S}^{\otimes 3})$, and the only automorphisms of a hyperelliptic curve with $3K$-structure are the identity and the hyperelliptic involution. The moduli space $\mathcal{H}_{g,3K}$ of smooth hyperelliptic curves of genus g equipped with a $3K$-structure is representable by a scheme [KS99, 10.6.5], [MFK94, Prop. 5.1].

Third, since $\mathcal{H}_g$ may be constructed as the quotient of $\mathcal{H}_{g,3K}$ by GL_{5g-5}, the forgetful functor $\psi_g : \mathcal{H}_{g,3K} \to \mathcal{H}_g$ is open [MFK94, p. 6] and a fibration with connected fibers [Noo04, Thm. A.12]. Thus, if $S \subset \mathcal{H}_g$ is a connected substack and $S_{3K} = S \times_{\mathcal{H}_g} \mathcal{H}_{g,3K}$, then $\mathsf{M}_\ell(S_{3K}) \cong \mathsf{M}_\ell(S)$ [AP08, Lemma 5.2]. Since the data of a tricanonical structure exists Zariski-locally on the base, one can relate point counts on $\mathcal{H}_{g,3K}(\mathbb{F})$ to those on $\mathcal{H}_g(\mathbb{F})$. Specifically, if $s \in \mathcal{H}_g^f(\mathbb{F})$ is such that $\mathrm{Aut}(\mathcal{C}_{g,s}) \cong \{\pm 1\}$, then the fiber of $\mathcal{H}_{g,3K}(\mathbb{F})$ over s consists of $|\mathrm{GL}_{2g}(\mathbb{F})|/2$ points [KS99, 10.6.8].

4.5.2. *Application to simple Jacobians.* Using the $\mathbb{Q}_\ell$-monodromy of $\mathcal{H}_g^f$, we deduce that there exist hyperelliptic curves of genus g and p-rank f with absolutely simple Jacobian.

Application 4.15. *Suppose $g \geq 1$ and $0 \leq f \leq g$ with $f \neq 0$ if $g \leq 2$. Let S be an irreducible component of $\mathcal{H}_g^f$.*

(a) *Then there exists $s \in S(\overline{\mathbb{F}})$ such that the Jacobian of $\mathcal{C}_{g,s}$ is absolutely simple.*
(b) *If $g \geq 4$ and η is a geometric generic point of S, then $\mathrm{End}(\mathrm{Jac}(\mathcal{C}_{g,\eta})) \cong \mathbb{Z}$.*

Proof. When $g = 3$ and $f = 0$, part (a) follows from Lemma 4.1. When $g = 4$ and $f = 0$, parts (a) and (b) follow from Proposition 4.5. If $g \geq 5$ or if $f \geq 1$, the proof of both parts is very similar to that of [AP08, Application 5.7] and uses Theorems 4.10-4.11. □

Remark 4.16. Under the hypotheses of Application 4.15, one can deduce that $\mathrm{Aut}(\mathcal{C}_{g,\eta}) = \{\pm 1\}$. This yields a new proof of [AGP08, Thm. 3.7].

4.5.3. *Application to class groups.* Recall that if $s \in \mathcal{H}_g(\mathbb{F})$, then $\mathrm{Pic}^0(\mathcal{C}_{g,s})(\mathbb{F})$ is isomorphic to the class group of the function field $\mathbb{F}(\mathcal{C}_{g,s})$. The size of the class group is divisible by ℓ exactly when there is a point of order ℓ on the Jacobian. Roughly speaking, Application 4.17 shows that among all curves over $\mathbb{F}$ of specified genus and p-rank, slightly more than $1/\ell$ of them have an $\mathbb{F}$-rational point of order ℓ on their Jacobian.

Application 4.17. *Suppose ℓ and p are distinct odd primes, $g \geq 1$ and $1 \leq f \leq g$. Suppose S is an irreducible component of $\mathcal{H}_g^f$ such that $S(\mathbb{F}) \neq \emptyset$. Let m be the image of $|\mathbb{F}|$ in $(\mathbb{Z}/\ell)^\times$. There exists a rational function $\alpha_{g,m}(T) \in \mathbb{Q}(T)$ and a constant $B = B(p,g,\ell)$ such that*

$$\left| \frac{\#\{s \in S(\mathbb{F}) : \ell | \# \mathrm{Pic}^0(\mathcal{C}_{g,s})(\mathbb{F})\}}{\# S(\mathbb{F})} - \alpha_{g,m}(\ell) \right| < \frac{B}{\sqrt{q}}. \tag{4.5.1}$$

If $f = 0$ and $g \geq 5$, the same result is true for all $\ell \gg_p 0$.

Proof. The proof is very similar to that of [AP08, Application 5.9] and uses Theorems 4.10-4.11 and [KS99, Thm. 9.7.13]. □

Remark 4.18. For ℓ odd, one knows that $\alpha_{g,1}(\ell) = \frac{\ell}{\ell^2-1} + \mathcal{O}(1/\ell^3)$, while $\alpha_{g,m}(\ell) = \frac{1}{\ell-1} + \mathcal{O}(1/\ell^3)$ if $m \neq 1$. A formula for $\alpha_{g,1}(\ell)$ is given in [Ach06].

4.5.4. *Application to zeta functions.* If $C/\mathbb{F}$ is a smooth projective curve of genus g, its zeta function has the form $L_{C/\mathbb{F}}(T)/(1-T)(1-qT)$, where $L_{C/\mathbb{F}}(T) \in \mathbb{Z}[T]$ is a polynomial of degree $2g$. The principal polarization on the Jacobian of C forces a symmetry among the roots of $L_{C/\mathbb{F}}(T)$; the largest possible Galois group for the splitting field over $\mathbb{Q}$ of $L_{C/\mathbb{F}}(T)$ is the Weyl group of Sp_{2g} which is a group of size $g!2^g$.

Application 4.19. *Suppose $g \geq 1$ and $1 \leq f \leq g$, or that $g \geq 5$ and $f = 0$. Suppose $p > 2g+1$ and that S is an irreducible component of $\mathcal{H}_g^f$ such that $S(\mathbb{F}) \neq \emptyset$. There exists a constant $\gamma = \gamma(g) > 0$ and a constant $E = E(p,g)$ such that*

$$\frac{\#\{s \in S(\mathbb{F}) : L_{\mathcal{C}_{g,s}/\mathbb{F}}(T) \text{ is reducible, or has splitting field with degree } < 2^g g!\}}{\#S(\mathbb{F})} < Eq^{-\gamma}. \tag{4.5.2}$$

Proof. The proof is very similar to that of [AP08, Application 5.11] and uses Theorems 4.10-4.11 and [Kow06, Thm. 6.1 and Remark 3.2.(4)]. ☐

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