A short guide to p-torsion of abelian varieties in characteristic p

Rachel Pries *

Abstract

There are many equivalent ways to describe the p-torsion of a principally polarized abelian variety in characteristic p. We briefly explain these methods and then illustrate them for abelian varieties A of arbitrary dimension g in several important cases, including when A has p-rank f and a-number 1 and when A has p-rank f and a-number $g-f$. We provide complete tables for abelian varieties of dimension up to four.

1 Introduction

In recent years, there have been many important results about the p-torsion of a principally polarized abelian variety in characteristic p. This p-torsion can be described in terms of a group scheme or a Dieudonné module. It can be classified using its final type or its Young type. It can be identified with an element in the Weyl group of the sympletic group or with a cycle class in the tautological ring of $\mathcal{A}_g$.

In this paper, we briefly summarize the main types of classification. We give a thorough description of the p-torsion of a principally polarized abelian variety A of arbitrary dimension g in several important cases, including when A has p-rank f and a-number 1, and when A has p-rank f and a-number $g-f$. We provide complete tables for the p-torsion types that occur for $g \leq 4$, including the sixteen types of p-torsion that occur for abelian varieties of dimension four. We hope that this paper will inspire the reader to learn more about the outstanding research in this area.

2 Methods to classify the p-torsion

Let k be an algebraically closed field of characteristic p. Let $\mathcal{A}_g := \mathcal{A}_g \otimes \mathbb{F}_p$ be the moduli space of principally polarized abelian varieties of dimension g defined over k. For an abelian variety $A \in \mathcal{A}_g(k)$, let $A[p]$ denote its p-torsion. We summarize several different ways of describing $A[p]$.

2.1 Group schemes

Let A be an abelian variety of dimension g defined over k. The p-torsion $A[p]$ is a finite commutative group scheme annihilated by p with rank p^{2g} having homomorphisms F (Frobenius) and V (Vershiebung). If A is principally polarized, then $\mathrm{im}(F) = \ker(V)$ and $\mathrm{im}(V) = \ker(F)$. Then $A[p]$ is called a quasi-polarized BT_1 k-group scheme (short for truncated Barsotti-Tate group of level 1). The quasi-polarization implies that $A[p]$ is symmetric. These group schemes were classified independently by Kraft (unpublished) [Kra] and by Oort [Oor01]. A complete description of this topic can be found in [Oor01] or [Moo01].

*The author was partially supported by NSF grant DMS-04-00461.

Example 2.1. Let $\mathbb{Z}/p$ be the constant group scheme and let μ_p be the kernel of Frobenius on $\mathbb{G}_m$. As a k-scheme, $\mu_p \simeq \mathrm{Spec}(k[x]/(x^p - 1))$. If E is an ordinary elliptic curve then $E[p] \simeq \mathbb{Z}/p \oplus \mu_p$. We denote this group scheme by L.

Example 2.2. Let α_p be the kernel of Frobenius on $\mathbb{G}_a$. As a k-scheme, $\alpha_p \simeq \mathrm{Spec}(k[x]/x^p)$. The isomorphism type of the p-torsion of any two supersingular elliptic curves is the same. If E is a supersingular elliptic curve, we denote the isomorphism type of its p-torsion by $I_{1,1}$. By [Gor02, Ex. A.3.14], $I_{1,1}$ fits into a non-split exact sequence of the form $0 \to \alpha_p \to I_{1,1} \to \alpha_p \to 0$. The image of the embedded α_p is unique and is the kernel of both Frobenius and Verschiebung.

Example 2.3. Let A be a supersingular non-superspecial abelian surface. In other words, A is isogenous, but not isomorphic, to the direct sum of two supersingular elliptic curves. Let $I_{2,1}$ denote the isomorphism class of the group scheme $A[p]$. By [Gor02, Ex. A.3.15], there is a filtration $H_1 \subset H_2 \subset I_{2,1}$ where $H_1 \simeq \alpha_p$, $H_2/H_1 \simeq \alpha_p \oplus \alpha_p$, and $I_{2,1}/H_2 \simeq \alpha_p$. If G_1 (resp. G_2) is the kernel of Frobenius (resp. Verschiebung) then $G_1 \subset H_2$ and $G_2 \subset H_2$. There is an exact sequence $0 \to H_1 \to G_1 \oplus G_2 \to H_2 \to 0$.

Two invariants of (the p-torsion of) an abelian variety are the p-rank and a-number. The *p-rank* of A is $f = \dim_{\mathbb{F}_p} \mathrm{Hom}(\mu_p, A[p])$. Then p^f is the cardinality of $A[p](k)$. The *a-number* of A is $a = \dim_k \mathrm{Hom}(\alpha_p, A[p])$. It is well-known that $0 \le f \le g$ and $0 \le a \le g - f$. In Example 2.1, $f = 1$ and $a = 0$. In Example 2.2, $f = 0$ and $a = 1$. The group scheme $I_{2,1}$ in Example 2.3 has p-rank 0 since it is an iterated extension of copies of α_p and has a-number 1 since $\ker(V^2) = G_1 \oplus G_2$ has rank p^3.

2.2 Covariant Dieudonné modules

One can describe the p-torsion $A[p]$ using the theory of covariant Dieudonné modules. This is the dual of the contravariant theory found in [Dem86]; see also [Gor02, A.5]. Briefly, let σ denote the Frobenius automorphism of k. Consider the non-commutative ring $\mathbb{E} = k[F,V]$ with the relations $FV = VF = 0$ and $F\lambda = \lambda^\sigma F$ and $\lambda V = V\lambda^\sigma$ for all $\lambda \in k$. Let $(A,B)_\ell$ denote the left ideal $\mathbb{E}A + \mathbb{E}B$ of $\mathbb{E}$ generated by A and B. The Dieudonné functor D gives an equivalence of categories between BT_1 group schemes $\mathbb{G}$ (with rank p^{2g}) and finite left $\mathbb{E}$-modules $D(\mathbb{G})$ (having dimension $2g$ as a k-vector space). If $\mathbb{G}$ is quasi-polarized, then there is a sympletic form on $D(\mathbb{G})$.

For example, $D(\mathbb{Z}/p \oplus \mu_p) \simeq \mathbb{E}/(F, 1-V)_\ell \oplus \mathbb{E}/(V, 1-F)_\ell$, [Gor02, Ex. A.5.1 & 5.3]. The Dieudonné module for $I_{1,1}$ and $I_{2,1}$ can be found in Lemma 3.1.

The p-rank of $A[p]$ is the dimension of $V^g D(\mathbb{G})$. The a-number of $A[p]$ equals $g - \dim(V^2 D(\mathbb{G}))$.

2.3 Final types

The isomorphism type of a symmetric BT_1 group scheme $\mathbb{G}$ over k can be encapsulated into combinatorial data. This topic can be found in [Oor01]. If $\mathbb{G}$ has rank p^{2g}, then there is a *final filtration* $N_1 \subset N_2 \subset \cdots \subset N_{2g}$ of $D(\mathbb{G})$ as a k-vector space which is stable under the action of V and F^{-1} so that $i = \dim(N_i)$. If $\mathbb{G}$ is quasi-polarized, then N_{2g-i} and N_i are orthogonal under the sympletic pairing.

The *final type* of $\mathbb{G}$ is $\nu = [\nu_1, \ldots, \nu_r]$ where $\nu_i = \dim(V(N_i))$. The final type of $\mathbb{G}$ is canonical, even if the final filtration is not. There is a restriction $\nu_i \le \nu_{i+1} \le \nu_i + 1$ on the final type. All sequences satisfying this restriction occur. This implies that there are 2^g isomorphism types of symmetric BT_1 group schemes of rank p^{2g}. The p-rank is $\max\{i \mid \nu_i = i\}$ and the a-number is $g - \nu_g$.

Together with Ekedahl, Oort used this classification by final type to stratify $\mathcal{A}_g$. The stratum of $\mathcal{A}_g$ whose points have final type ν is locally closed and quasi-affine with dimension $\sum_{i=1}^{g} \nu_i$, [Oor01, Thm. 1.2].

2.4 Young types

Another combinatorial method to describe the isomorphism type of $\mathbb{G}$ uses a Young diagram. This method was introduced by Van der Geer [vdG99] as a means of describing the Ekedahl-Oort strata in terms of degeneration loci for maps between flag varieties.

Given a final type ν, let $\mu_j = \#\{i \mid 1 \leq i \leq g \mid j \leq i - \nu_i\}$. Consider the Young diagram with μ_j squares in the jth row. The *Young type* of $\mathbb{G}$ is $\mu = \{\mu_1, \mu_2, \ldots\}$. The p-rank is $g - \mu_1$ and the a-number is $a = \max\{j \mid \mu_j \neq 0\}$. The codimension in $\mathcal{A}_g$ of the stratum whose points have Young type μ is $\sum_{j=1}^{a} \mu_j$.

2.5 Elements of the Weyl group

One can associate to μ an element ω of the Weyl group W_g of the sympletic group Sp_{2g}, [vdG99]. Here W_g is identified with the subgroup of all $\omega \in S_{2g}$ so that $\omega(i) + \omega(2g+1-i) = 2g+1$ for $1 \leq i \leq g$. This subgroup is generated by the following involutions: $s_i = (i, i+1)(2g-i, 2g+1-i)$ for $1 \leq i < g$; and $s_g = (g, g+1)$.

Given a Young type μ, one defines ω as follows. For $1 \leq i \leq g$, let $\omega(i) = c$ (respectively $\omega(i) = g + c$) if i is the cth number such that $\mu_i = \mu_{i+1}$ (respectively $\mu_i \neq \mu_{i+1}$). For $1 \leq i \leq g$, let $\omega(2g+1-i) = 2g+1-\omega(i)$. This yields an element of W_g. One can express ω as a word in the involutions $s_1, \ldots, s_g$ of S_{2g}, although this expression is not unique.

For example, in the ordinary case where $\mu = \emptyset$, then ω is given by $\langle 1, \ldots, 2g\rangle \xrightarrow{\omega} \langle g+1, \ldots, 2g, 1, \ldots, g\rangle$. In the superspecial case where $\mu = \{g, \ldots, 1\}$, then $\omega = \text{id}$. Further examples with $g \leq 4$ are in Section 4.

We briefly explain the importance of the Weyl group characterization. There is a second filtration of $D(\mathbb{G})$ which is stable under the action of F and V^{-1}, which we denote by $N'_1 \subset N'_2 \subset \cdots \subset N'_{2g}$. Then ω measures the interaction between these two filtrations.

For example, when $\mathbb{G}$ is ordinary ($f = g$) then $N_g \cap N'_g = 0$. Informally speaking, this means that the intersection of $\text{Im}(V)$ and (a twist under σ of) $\text{Im}(F)$ is trivial. When $\mathbb{G}$ is superspecial ($a = g$), then $\dim(N_i \cap N'_g) = i$ for $1 \leq i \leq g$. Informally speaking, this implies that N_i is contained in (a twist under σ of) $\text{Im}(F)$. In general, $\dim(N_i \cap N'_g) \geq i - \nu_i$. The a-number is $\dim(VD(\mathbb{G}) \cap FD(\mathbb{G})) = \dim(N_g \cap N'_g)$.

One can identify the closures of the Ekedahl-Oort strata with cycle classes in the tautological ring of $\mathcal{A}_g$. Let λ_i for $1 \leq i \leq g$ be the Chern classes of the Hodge bundle of $\mathcal{A}_g$. These classes generate the tautological subring of $CH^*_{\mathbb{Q}}(\mathcal{A}_g)$ and satisfy $(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g) = 1$, [vdG99, Thm. 1.1].

3 Important examples

3.1 Abelian varieties with p-rank f

Given g and f such that $0 \leq f \leq g$, let $V_{g,f}$ denote the stratum of $\mathcal{A}_g$ whose points correspond to principally polarized abelian varieties A of dimension g with $f_A \leq f$. Every component of $V_{g,f}$ has codimension $g - f$, [NO80]. In this section, we describe the p-torsion that occurs for the generic point(s) of $V_{g,f}$. The generic point of $V_{g,g} = \mathcal{A}_g$ has p-rank g, a-number 0, and $A[p] \cong L^g$. Using a dimension count, one can show that the generic point of every component of $V_{g,f}$ has a-number 1 when $f < g$.

Lemma 3.1. *Let $r \in \mathbb{N}$. There is a unique symmetric BT_1 group scheme of rank p^{2r} with p-rank 0 and a-number 1, which we denote $I_{r,1}$. The covariant Dieudonné module of $I_{r,1}$ is $\mathbb{E}/(F^r + V^r)_\ell$.*

Proof. Let $I_{r,1}$ be a symmetric BT_1 group scheme of rank p^{2r} with p-rank 0 and a-number 1. It is sufficient to show that the final type of $I_{r,1}$ is uniquely determined. The p-rank 0 condition implies that V acts nilpotently on $D(I_{r,1})$, so $\nu_1 = 0$. The a-number 1 condition implies that $r-1$ is the dimension of $V^2D(I_{r,1})$, so $\nu_r = r-1$. The restrictions on ν_i imply that there is a unique final type possible for $I_{r,1}$, namely $\nu = [0,1,\dots,r-1]$.

Consider $D = \mathbb{E}/(F^r + V^r)_\ell$. Note that $F^{r+1} = 0$ and $V^{r+1} = 0$ on D. Then D is an $\mathbb{E}$-module with dimension $2r$ as a k-vector space. It has basis $\{F,\dots,F^r,1,V,\dots,V^{r-1}\}$. Then VD has basis $\{V,\dots,V^{r-1},F^r\}$ and V^2D has basis $\{V^2,\dots,V^{r-1},F^r\}$. Thus D has a-number 1. Continuing, one sees that V is nilpotent on D and thus the p-rank of D is 0. Thus D must be the covariant Dieudonné module corresponding to $I_{r,1}$. □

Proposition 3.2. *Let $A \in \mathcal{A}_g(k)$ be a principally polarized abelian variety of dimension g with p-rank f and a-number* 1. *Then $A[p] \simeq L^f \oplus I_{g-f,1}$. The covariant Dieudonné module of $A[p]$ is*

$$D \simeq (\mathbb{E}/(F,1-V)_\ell \oplus \mathbb{E}/(V,1-F)_\ell)^f \oplus \mathbb{E}/(F^{g-f} - V^{g-f})_\ell.$$

The final type of $A[p]$ is $\nu = [1,\dots,f,f,\dots,g-1]$. The Young type is $\mu = \{g-f\}$.

Proof. The decomposition of $A[p]$ must include f copies of L along with a group scheme of rank $p^{2(g-f)}$ with p-rank 0 and a-number 1. By Lemma 3.1, the only possibility for the latter is $I_{g-f,1}$. The statement about the Dieudonné module follows immediately. For the final type, note that $\nu_g = g-1$ since $A[p]$ has a-number 1 and $\nu_f = f$ since $A[p]$ has p-rank f. The numerical restrictions on ν_i then imply that $\nu = [1,\dots,f,f,\dots,g-1]$. The Young type follows by direct calculation. □

If $f < g$, one can show that the group scheme $L^f \oplus I_{g-f,1}$ corresponds to the element ω of the Weyl group so that $\omega(f+1) = 1$ and $\omega : \{1,\dots,g\} - \{f+1\} \mapsto \{g+1,\dots,2g-1\}$ is increasing. The cycle class of the (reduced) stratum $V_{g,f}$ in the tautological ring of $\mathcal{A}_g$ is given by $(p-1)(p^2-1)\dots(p^{g-f}-1)\lambda_{g-f}$, [vdG99, Thm. 2.4].

3.2 Abelian varieties with a-number a

Given g and f such that $0 \le f \le g$, let $T_{g,a}$ denote the stratum of $\mathcal{A}_g$ whose points correspond to principally polarized abelian varieties of dimension g with $a_A \ge a$. Then $T_{g,a}$ is irreducible unless $a = g$, [vdG99, Thm. 2.11]. In this section, we describe the p-torsion that occurs for the generic point(s) of $T_{g,a}$. It is well-known that $T_{g,a}$ has codimension $a(a+1)/2$. The generic point(s) of $T_{g,a}$ have a-number a and p-rank $g-a$.

Proposition 3.3. *Let $A \in \mathcal{A}_g(k)$ be an abelian variety of dimension g with p-rank f and a-number $g-f$. Then $A[p] \simeq L^f \oplus (I_{1,1})^{g-f}$. The covariant Dieudonné module of $A[p]$ is*

$$D \simeq (\mathbb{E}/(F,1-V)_\ell \oplus \mathbb{E}/(V,1-F)_\ell)^f \oplus (\mathbb{E}/(F+V)_\ell)^{g-f}.$$

The final type is $\nu = [1,\dots,f,\dots,f]$. The Young type is $\mu = \{g-f, g-f-1,\dots,1\}$ or $\emptyset$ if $g = f$.

Proof. The decomposition of $A[p]$ must include f copies of L along with a group scheme of rank $p^{2(g-f)}$ with p-rank 0 and a-number $g-f$. The only possibility for the latter is $g-f$ copies of $D(I_{1,1})$. The statement about the Dieudonné module follows immediately. For the final type, note that $\nu_g = f$ since $A[p]$ has a-number $g-f$ and $\nu_f = f$ since $A[p]$ has p-rank f. The numerical restrictions on ν_i then imply that $\nu = [1,\dots,f,\dots,f]$. The Young type follows by direct calculation. □

If $f > 0$, one can show that the group scheme $L^f \oplus (I_{1,1})^{g-f}$ corresponds to the element ω of the Weyl group $\langle 1,\dots,2g\rangle \xrightarrow{\omega} \langle g+1,\dots,g+f,1,\dots a, g+f+1,\dots 2g, a+1,\dots,g\rangle$. In [vdG99, Thm. 2.6], one finds a result on the cycle class of the (reduced) stratum $T_{g,a}$ in the tautological ring of $\mathcal{A}_g$.

3.3 Some indecomposable group schemes with p-rank 0 and a-number 2

A symmetric BT_1 group scheme $\mathbb{G}$ is *indecomposable* if $\mathbb{G} \neq \mathbb{G}_1 \oplus \mathbb{G}_2$ where $\mathbb{G}_1$ and $\mathbb{G}_2$ are nontrivial symmetric BT_1 group schemes. The group schemes $I_{r,1}$ are indecomposable. We now describe an indecomposable group scheme $I_{r,2}$ of rank p^{2r}, p-rank 0, and a-number 2.

Lemma 3.4. *Let $r \in \mathbb{N}$ with $r \geq 3$. Let $D = \mathbb{E}/(F^{r-1} - V)_\ell \oplus \mathbb{E}/(V^{r-1} - F)_\ell$. Then D is the covariant Dieudonné module of an indecomposable symmetric BT_1 group scheme with rank p^{2r}, p-rank 0 and a-number 2, which we denote by $I_{r,2}$. It has final type $[0, 1, \ldots, r-3, r-2, r-2]$ and Young type $\{r, 1\}$.*

Proof. The given decomposition of D is the only possible decomposition of D into covariant Dieudonné modules, but neither of the factors in this decomposition is symmetric. Thus $I_{r,2}$ is indecomposable.

Note that $F^r = 0$ (resp. $V^r = 0$) on the first (resp. second) factor of D. Then $D = N_{2r} = \langle 1, F, \ldots, F^{r-1} \rangle \oplus \langle 1, V, \ldots, V^{r-1} \rangle$. Thus D has dimension $2r$ as a k-vector space and $I_{r,2}$ has rank p^{2r}. Then $VD = \langle F^{r-1} \rangle \oplus \langle V, \ldots, V^{r-1} \rangle = N_r$. Also $V^2 D = 0 \oplus \langle V^2, \ldots, V^{r-1} \rangle = N_{r-2}$. Thus D has a-number 2. Continuing, one sees that $\nu_i = i - 1$ for $1 \leq i \leq r-2$. In particular, V is nilpotent on D and thus the p-rank of D is 0.

More information on the final filtration is necessary to determine the final type of $I_{r,2}$. First, $F^{-1}(N_r) = \langle F^{r-2}, F^{r-1} \rangle \oplus \langle 1, V, \ldots, V^{r-1} \rangle = N_{r+2}$. Second, $F^{-r+2}(N_r) = \langle F, \ldots, F^{r-1} \rangle \oplus \langle 1, V, \ldots, V^{r-1} \rangle = N_{2r-1}$. Then $VN_{2r-1} = 0 \oplus \langle V, \ldots, V^{r-1} \rangle = N_{r-1}$ and $VN_{r-1} = \langle V^2, \ldots, V^{r-1} \rangle = N_{r-2}$. Thus $\nu_{r-1} = r-2$. Then $I_{r,2}$ has final type $[0, 1, \ldots, r-3, r-2, r-2]$ and Young type $\{r, 1\}$. □

Lemma 3.5. *When $g = 3$ or $g = 4$, there is a unique indecomposable symmetric BT_1 group scheme $I_{g,2}$ of rank p^{2g} with p-rank 0 and a-number 2.*

Proof. Let $\mathbb{G}$ be a symmetric BT_1 group scheme with rank p^{2g}, p-rank 0 and a-number 2. Its Young type is $\{g, i\}$ for some $i \in \{1, \ldots, g-1\}$. There are exactly $\lfloor g/2 \rfloor$ such group schemes which are decomposable, namely $I_{r,1} \oplus I_{g-r,1}$ for $1 \leq r \leq g/2$. Thus there is a unique such $\mathbb{G}$ which is indecomposable when $g = 3$ of $g = 4$. By Lemma 3.4, it is $I_{g,2}$. □

3.4 One more indecomposable group scheme of dimension four

There is one more indecomposable group scheme which occurs for dimension $g \leq 4$, which we denote by $I_{4,3}$. It has covariant Dieudonné module $D(I_{4,3}) = \mathbb{E}/(F^2 - V)_\ell \oplus \mathbb{E}/(F - V)_\ell \oplus \mathbb{E}/(V^2 - F)_\ell$. Then $D(I_{4,3})$ has basis $\langle 1, F, F^2 \rangle \oplus \langle 1, F \rangle \oplus \langle 1, V, V^2 \rangle$. One can check that $V^2 D$ has basis $0 \oplus 0 \oplus \langle V^2 \rangle$ and thus $I_{4,3}$ has a-number 3. Also $I_{4,3}$ has p-rank 0 since V acts nilpotently on $D(I_{4,3})$. By the process of elimination, $I_{4,3}$ has final type $[0, 0, 1, 1]$ and Young type $\{4, 3, 1\}$.

4 Complete tables for dimension up to four

For convenience, we provide tables for dimension $g \leq 4$. Some parts of these tables can be found in [EvdG].

4.1 The case $g = 1$:

Name	codim	f	a	ν	μ	ω	cycle class (reduced)
L	0	1	0	[1]	$\emptyset$	s_1	λ_0
$I_{1,1}$	1	0	1	[0]	$\{1\}$	1	$(p-1)\lambda_1$

4.2 The case $g = 2$:

Name	codim	f	a	ν	μ	ω	cycle class (reduced)
L^2	0	2	0	$[1,2]$	$\emptyset$	$s_2 s_1 s_2$	λ_0
$L \oplus I_{1,1}$	1	1	1	$[1,1]$	$\{1\}$	$s_1 s_2$	$(p-1)\lambda_1$
$I_{2,1}$	2	0	1	$[0,1]$	$\{2\}$	s_2	$(p-1)(p^2-1)\lambda_2$
$I_{1,1}^2$	3	0	2	$[0,0]$	$\{2,1\}$	1	$(p-1)(p^2+1)\lambda_1\lambda_2$

This is the smallest dimension for which the Newton polygon of A does not determine the group scheme $A[p]$. The Newton polygon $2G_{1,1}$ (supersingular, with four slopes of $1/2$) occurs for both $(I_{1,1})^2$ and $I_{2,1}$.

4.3 The case $g = 3$:

Name	codim	f	a	ν	μ	ω	cycle class (reduced)
L^3	0	3	0	$[1,2,3]$	$\emptyset$	$s_3 s_2 s_3 s_1 s_2 s_3$	λ_0
$L^2 \oplus I_{1,1}$	1	2	1	$[1,2,2]$	$\{1\}$	$s_2 s_3 s_1 s_2 s_3$	$(p-1)\lambda_1$
$L \oplus I_{2,1}$	2	1	1	$[1,1,2]$	$\{2\}$	$s_3 s_1 s_2 s_3$	$(p-1)(p^2-1)\lambda_2$
$L \oplus I_{1,1}^2$	3	1	2	$[1,1,1]$	$\{2,1\}$	$s_1 s_2 s_3$	$-(p-1)(p^2+1)\lambda_1\lambda_2 - 2(p^3-1)\lambda_3$
$I_{3,1}$	3	0	1	$[0,1,2]$	$\{3\}$	$s_3 s_2 s_3$	$(p-1)(p^2-1)(p^3-1)\lambda_3$
$I_{3,2}$	4	0	2	$[0,1,1]$	$\{3,1\}$	$s_2 s_3$	$(p-1)^2(p^2-p+1)\lambda_1\lambda_3$
$I_{1,1} \oplus I_{2,1}$	5	0	2	$[0,0,1]$	$\{3,2\}$	s_3	$-(p-1)^3(p+1)(p^2-p+1)(p^2+p+1)\lambda_2\lambda_3$
$I_{1,1}^3$	6	0	3	$[0,0,0]$	$\{3,2,1\}$	1	$(p-1)(p^2+1)(p^3-1)\lambda_1\lambda_2\lambda_3$

This is the smallest dimension for which the group scheme $A[p]$ does not determine the Newton polygon of A. If $A[p] \simeq I_{3,1}$, then the Newton polygon of A is usually $G_{1,2} + G_{2,1}$ (three slopes of $1/3$ and of $2/3$) but by [Oor91, Thm. 5.12] it can also be $3G_{1,1}$ (supersingular, with six slopes of $1/2$).

4.4 The case $g = 4$:

Name	codim	f	a	ν	μ	ω
L^4	0	4	0	$[1,2,3,4]$	$\emptyset$	$s_4s_3s_4s_2s_3s_4s_1s_2s_3s_4$
$L^3 \oplus I_{1,1}$	1	3	1	$[1,2,3,3]$	$\{1\}$	$s_3s_4s_2s_3s_4s_1s_2s_3s_4$
$L^2 \oplus I_{2,1}$	2	2	1	$[1,2,2,3]$	$\{2\}$	$s_4s_2s_3s_4s_1s_2s_3s_4$
$L^2 \oplus I_{1,1}^2$	3	2	2	$[1,2,2,2]$	$\{2,1\}$	$s_2s_3s_4s_1s_2s_3s_4$
$L \oplus I_{3,1}$	3	1	1	$[1,1,2,3]$	$\{3\}$	$s_4s_3s_4s_1s_2s_3s_4$
$L \oplus I_{3,2}$	4	1	2	$[1,1,2,2]$	$\{3,1\}$	$s_3s_4s_1s_2s_3s_4$
$I_{4,1}$	4	0	1	$[0,1,2,3]$	$\{4\}$	$s_4s_3s_4s_2s_3s_4$
$L \oplus I_{1,1} \oplus I_{2,1}$	5	1	2	$[1,1,1,2]$	$\{3,2\}$	$s_4s_1s_2s_3s_4$
$I_{4,2}$	5	0	2	$[0,1,2,2]$	$\{4,1\}$	$s_3s_4s_2s_3s_4$
$L \oplus I_{1,1}^3$	6	1	3	$[1,1,1,1]$	$\{3,2,1\}$	$s_1s_2s_3s_4$
$I_{1,1} \oplus I_{3,1}$	6	0	2	$[0,1,1,2]$	$\{4,2\}$	$s_4s_2s_3s_4$
$I_{1,1} \oplus I_{3,2}$	7	0	3	$[0,1,1,1]$	$\{4,2,1\}$	$s_2s_3s_4$
$I_{2,1} \oplus I_{2,1}$	7	0	2	$[0,0,1,2]$	$\{4,3\}$	$s_4s_3s_4$
$I_{4,3}$	8	0	3	$[0,0,1,1]$	$\{4,3,1\}$	s_3s_4
$I_{1,1}^2 \oplus I_{2,1}$	9	0	3	$[0,0,0,1]$	$\{4,3,2\}$	s_4
$I_{1,1}^4$	10	0	4	$[0,0,0,0]$	$\{4,3,2,1\}$	1

The cycle classes for this table can be found in [EvdG, 15.3].

It is not straight-forward to determine which Ekedahl-Oort strata lie in the boundary of which others. When $g = 4$, the answer to this question is given by the natural partial ordering on the Young type, which matches the Bruhat-Chevalley order on the elements of the Weyl group.

$\{\emptyset\}$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$			
		$\{2,1\}$	$\{3,1\}$	$\{4,1\}$			
			$\{3,2\}$	$\{4,2\}$	$\{4,3\}$		
			$\{3,2,1\}$	$\{4,2,1\}$	$\{4,3,1\}$	$\{4,3,2\}$	$\{4,3,2,1\}$

References

[Dem86] M. Demazure. *Lectures on p-divisible groups*, volume 302 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. Reprint of the 1972 original.

[EvdG] E. Ekedahl and G. van der Geer. Cycle classes of the E-O stratification on the moduli of abelian varieties. arXiv:math.AG/0412272.

[Gor02] E. Goren. *Lectures on Hilbert modular varieties and modular forms*, volume 14 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2002. With M.-H. Nicole.

[Kra] H. Kraft. Kommutative algebraische p-gruppen (mit anwendungen auf p-divisible gruppen und abelsche varietäten). manuscript, University of Bonn, September 1975, 86 pp.

[Moo01] B. Moonen. Group schemes with additional structures and Weyl group cosets. In *Moduli of abelian varieties (Texel Island, 1999)*, volume 195 of *Progr. Math.*, pages 255–298. Birkhäuser, Basel, 2001.

[NO80] P. Norman and F. Oort. Moduli of abelian varieties. *Ann. of Math. (2)*, 112(3):413–439, 1980.

[Oor91] F. Oort. Hyperelliptic supersingular curves. In *Arithmetic algebraic geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 247–284. Birkhäuser Boston, Boston, MA, 1991.

[Oor01] F. Oort. A stratification of a moduli space of abelian varieties. In *Moduli of abelian varieties (Texel Island, 1999)*, volume 195 of *Progr. Math.*, pages 345–416. Birkhäuser, Basel, 2001.

[vdG99] G. van der Geer. Cycles on the moduli space of abelian varieties. In *Moduli of curves and abelian varieties*, Aspects Math., E33, pages 65–89. Vieweg, Braunschweig, 1999. arXiv:alg-geom/9605011.